15-451/651 Algorithm Design & Analysis, Spring 2024

Extra Review Problems

The Multiplicative Weights Algorithm

1. **(More expert analysis)** In lecture we saw that the simple procedure that multiplied the weight of each expert by $\frac{1}{2}$ whenever the expert made a mistake, resulted in

$$m = \text{ \#mistakes of algorithm } \leq 2.41(M + \log_2 n),$$

where M = #mistakes made by the best expert and n = # of experts. In Problem 1, we saw what happens if we replace 1/2 with 2/3. In general, it turns out that the closer this factor gets to 1, the better the bound. Show that if we reduce the weights by a factor of $(1-\epsilon)$ for $\epsilon \leq 1/2$, then the number of mistakes is

$$m \leq 2(1+\epsilon)M + O\left(\frac{\log n}{\epsilon}\right).$$

2. **(Fun with experts)** In class we saw the randomized weighted majority (RWM) algorithm, in which we were given n experts. Then over any sequence of T rounds, and any expert i, we had

$$\mathbb{E}[\text{number of mistakes by RWM}] \leq (1+\epsilon)m_i + \frac{\ln n}{\epsilon}.$$

Here m_i is the number of mistakes made by expert i until time T. In Section 4 of the notes, we observed that $m_i \leq T$, so dividing the above by T and choosing $\epsilon := \sqrt{\frac{\ln n}{T}}$ we get that for any i

$$\mathbb{E}[\text{rate of mistakes by RWM}] \leq \frac{m_i}{T} + 2\sqrt{\frac{\ln n}{T}}.$$

I.e., for any expert i (which includes the best expert at time T) the *average regret (versus that expert)*, which is our mistake rate minus that of the expert's mistake rate, goes to zero as $T \to \infty$. Above, we assumed we knew the time horizon T and hence could set $\epsilon = \sqrt{\frac{\ln n}{T}}$. What if we don't know T? Here's one algorithm: for $s = 1, 2, \ldots$, play 2^s rounds of RWM (starting from scratch) with $\epsilon = \sqrt{\frac{\ln n}{2^s}}$. Show that the average regret of this algorithm after time T is $O\left(\sqrt{\frac{\ln n}{T}}\right)$.

3. **(Experts with fractional loss – hard)** In lecture we saw the randomized weighted majority algorithm, which scales the weight by $(1-\epsilon)$ when an expert makes a mistake. We bound the number of mistakes we make with the number of mistakes the best expert makes. Here, we are interested in generalizing this framework.

First, we will allow more than binary outcomes, so the experts are predicting from a set of possible outcomes. Then, instead of just being right or wrong, an expert's prediction can be valued from 0 to 1 (where 0 could mean a perfect prediction and 1 the worst prediction, with other values in between). We call this value the "loss", which generalizes the "mistakes" from our original framework. Once again, this is the quantity that we want to minimize.

Let $\mathscr{P}$ be all the possible outcomes. We define a matrix $M_{i,j}$, with $i \in [n]$, $j \in \mathscr{P}$ to be the loss that expert i experiences when the outcome is j. For all i, j, we have $M_{i,j} \in [0,1]$. Similar to the algorithm in class, we initialize the weight of each expert to 1. To make a prediction, we randomly sample an expert with the weight. Our expected loss could be measured by summing over the expected loss of each round.

To use the loss matrix to update the weights, if the outcome of round t is j_t, for each expert, $w^{(t+1)} = w^{(t)}(1-\epsilon)^{M_{i,j_t}}$. Intuitively, expert i tends to make a decision that incurs M_{i,j_t} loss when the outcome is j_t.

Our expected loss each round, given that the outcome is j_t, is

$$\left(\sum_{i=1}^{n} w_i^{(t)} M_{i,j_t}\right) \Big/ \sum_{i=1}^{n} w_i^{(t)}$$

Let this be denoted by $M(E^t, j_t)$. We are interested in upper bounding $\sum_{t=1}^{T} M(E^t, j_t)$.

Let $\epsilon < \frac{1}{2}$. After T rounds, for any expert i, we want to show that

$$\sum_{t=1}^{T} M(E^t, j_t) \le \frac{\ln n}{\epsilon} + (1+\epsilon)\sum_{t} M_{i,j_t}$$

Use these inequalities:

$$\begin{aligned}
(1-\epsilon)^x &\le (1-\epsilon x) \qquad && \forall x \in [0,1] \\
(1+\epsilon)^{-x} &\le (1-\epsilon x) \qquad && \forall x \in [-1,0] \\
\ln\left(\frac{1}{1-\epsilon}\right) &\le \epsilon + \epsilon^2 \qquad && \forall \epsilon : 0 < \epsilon < \frac{1}{2} \\
\ln(1+\epsilon) &\ge \epsilon - \epsilon^2 \qquad && \forall \epsilon : 0 < \epsilon < \frac{1}{2}
\end{aligned}$$

and a similar potential function approach as in class to prove the bound above.

4. **(Fickle experts)** In class you saw the randomized weighted majority theorem, in which we were given n experts. Now suppose you don't just want to compare yourself to the best you could have done by choosing a single expert and sticking with them. Call an deterministic algorithm K-*fickle* if over the time horizon T, it follows the advice of some expert i_1 for the first t_1 steps, then i_2 for the next t_2 steps, etc, and then i_K for the last t_K steps, where each $i_j \in [n]$, $t_j \ge 0$ and $\sum_{j=1}^{K} t_j = T$. Give an algorithm such that for any K-fickle (deterministic) algorithm A,

$$\mathbb{E}[\text{\# mistakes by your algo}] \le (\text{\# mistakes by } A)(1+\varepsilon) + \frac{O(K\log(nT))}{\varepsilon}.$$

Your algorithm is allowed to run in time $(nT)^{O(K)}$.