15-455: UCT	**K. Sutner**
Assignment 1	Due: **Thursday 01/25/2024 24:00**.

1. Super Halting (30)

Background

We have seen that a number of versions of the Halting problem are undecidable, but semidecidable. Here is stronger version of Halting: we are interested in machines that halt on all inputs.

$$\mathsf{TOT} = \{ e \in \mathbb{N} \mid \forall x \, (\mathcal{M}_e(x) \downarrow) \}$$

Task

A. Explain intuitively why TOT is harder than plain Halting.

B. Prove that TOT is undecidable.

C. Prove that TOT is not even semidecidable.

Comment For the proofs, use reductions from Halting.

2. Partitioned Turing Machines (30)

Background

It is customary to define Turing machines via a transition function of the form

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \Delta$$

Here Q is the set of states, Γ the tape alphabet including a blank symbol, and $\Delta = \{-1, 0, +1\}$ indicates movement of the head. An instruction $\delta(p, a) = (q, b, d)$ indicates that the machine, when in state p and reading symbol a on the tape, will write symbol b, move the head by d and go into state q.

Instead of using these fairly complex instructions we can simplify matters a bit by distinguishing several types of states.

- Read: for a read state p the machine scans the current tape symbol a and makes a transition into state $s(p, a)$.
- Write: for a write state p the machine writes $w(p)$ into the current tape cell and makes a transition into state p'.
- Move: for a left move state p the machine moves the head one cell to the left and makes a transition into state p'. Likewise for right move states.

We call such a machine a partitioned Turing machine (PTM). So the state set in a PTM is partitioned into four blocks

$$Q = Q_R \cup Q_W \cup Q_l \cup Q_r$$

Task

1. Give a precise definition of what it means for a partitioned Turing machine to compute a function.
2. Show that every ordinary Turing machine can be simulated by a partitioned Turing machine.
3. How do the machines compare in size?

Comment

This is just the tip of an iceberg. In the case where $\Gamma = \{0, 1\}$ one can even insist that 1's are never overwritten by 0's (a so-called non-erasing TM), but the proof is rather complicated.

3. Graphs of Computable Functions (40)

Background

A set is decidable if its characteristic function is computable. Similarly, a set is semidecidable if its semi-characteristic function (returns 0 on elements, is undefined everywhere else) is computable. One can also go in the opposite direction.

Define the graph of a partial function $f : \Sigma^\star \nrightarrow \Sigma^\star$ to be the set

$$\mathrm{Gr}(f) = \{\, (x, y) \mid f(x) \simeq y \,\} \subseteq \Sigma^\star \times \Sigma^\star$$

Let I_x be the initial segment $\{\, z \mid z < x \,\} \subseteq \Sigma^\star$ where $<$ is the standard length-lex order on words. For $A \subseteq \Sigma^\star$, the principal function (aka Hauptfunktion) of A is the unique order-preserving bijection between some initial segment I and A. So I has the same cardinality as A.

For example, assuming an alphabet $\Sigma = \{a, b\}$, the function $f = \{(\varepsilon, aaa), (a, aab), (b, baa), (aa, aaaa)\}$ is the principal function of $A = \{aaa, aab, baa, aaaa\}$; the corresponding initial segment is I_{ab}.

Task

A. Show that a partial function $f : \Sigma^\star \nrightarrow \Sigma^\star$ is computable iff its graph is semidecidable.

B. What can you say about the graph of a total computable function?

C. Show that for any semidecidable set W and any partial computable function f the image $f(W) = \{\, f(x) \mid f(x)\downarrow \wedge x \in W \,\}$ of W under f is again semidecidable.

D. Show that a set is decidable iff its principal function is computable.

E. Show that for any partial computable function f there is a partial computable function g such that for all x in the domain of f: $f(g(f(x))) = f(x)$. If f were injective we could let $g = f^{-1}$, but the claim is that this works in general.

Comment We are using strings rather than natural numbers since that is the standard in complexity theory; arguably, this particular problem would be more natural when phrased in terms of $\mathbb{N}$ rather than $\Sigma^\star$.