15-455: UCT	**K. Sutner**
Assignment 2	Due: **Thursday 02/01/2024 24:00**.

1. Diophantine Solutions (30)

Background

According to a famous theorem by Matiyasevic, it is undecidable whether a multivariate polynomial with integer coefficients $P(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ has a solution over the integers. The same is true if we look for solutions over $\mathbb{N}^n$, for simplicity we'll use the second version.

Write $\#\mathsf{sol}(P)$ for the number of solutions of $P(\boldsymbol{x}) = 0$ over $\mathbb{N}^n$. If follows from Matiyasevic's theorem that "$\#\mathsf{sol}(P) = 0$" is undecidable.

Task

A. We are given a polynomial $P(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$. Show that one can easily construct a polynomial Q such that the number of solutions of P over $\mathbb{Z}$ is the same as the number of solutions of Q over $\mathbb{N}$.

B. Given an arbitrary $k \in \mathbb{N}$, show that it is undecidable whether $\#\mathsf{sol}(P) = k$.

C. Show that it is undecidable whether $\#\mathsf{sol}(P) = \infty$.

Comment

For part (B), given a polynomial Q, construct a polynomial Q' such that $\#\mathsf{sol}(Q') = \#\mathsf{sol}(Q) + 1$.

This is a good example of a reduction: the very difficult part here is to show Matiyasevic's theorem; from there to asking specific cardinality questions is a fairly small step.

2. Minimal Machines (40)

Background

All models of computation can be associated with a natural size function. This is particularly obvious for machine-based models: the machine is just a finite data structure, and has a canonical size. For example, we could define the size of a Turing machine M to be the product $|Q||\Sigma|$, or the number of bits needed to specify its transition function. Or we could think of the index $\widehat{M}$ as a natural number, and use that number.

Fix one such measure, and call M minimal if no smaller machine is equivalent to M. Here equivalent means that $\forall z\,(M(z) \simeq M'(z))$: the computations may unfold in a different way, but the final result has to be the same for all inputs.

Task

A. Explain intuitively why minimality of TMs should not be semidecidable. You might want to start with an argument that shows that the problem is not decidable.

B. Assume that minimality of TMs is semidecidable. Show that there is an effective enumeration (N_e) of all minimal TMs.

C. Show that minimality of TMs fails to be semidecidable using the recursion theorem and part (B).

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3. Classifying Index Sets (30)

Background

Consider the index sets

$$\mathsf{ONE} = \{\, e \mid |W_e| = 1 \,\}$$

$$\mathsf{EXT} = \{\, e \mid \{e\} \text{ is extendible to a total computable function} \,\}$$

Here a partial function $f : \mathbb{N} \nrightarrow \mathbb{N}$ is called extendible if there is a total function $F : \mathbb{N} \to \mathbb{N}$ such that $F \restriction D = f$ where $D \subseteq \mathbb{N}$ is the support of f.

Task

A. Find the location of ONE in the arithmetical hierarchy.

B. Find the location of EXT in the arithmetical hierarchy.

Lower bounds are not required, Extra Credit if you can prove a completeness result. But make sure your upper bounds are tight, a "solution" $\mathsf{EXT} \in \Sigma_{42}$ is useless.

Comment

It is known that $\mathsf{EXT} \neq \mathbb{N}$, there are partial computable functions that cannot be extended to a total computable function–which is really too bad, since otherwise we could just get rid of pesky partial functions.