1.   Spanning Trees (30)

Background

Recall that a spanning tree of a ugraph $G = \langle V, E \rangle$ is a subgraph $T = \langle V, E_0 \rangle$ such that T is a tree (connected, acyclic). Consider the following instances: the graph G together with either a bound k, $1 \le k \le n = |V|$ (for parts A, B, C), or a vertex set $L \subseteq V$ (for D, E). The questions are as follows:

Is there a spanning tree T of G such that

A. T has k leaves.

B. T has at most k leaves.

C. The nodes of T have degree at most k.

D. The set of leaves of T is L.

E. There are no leaves of T outside L.

Task

A. Show that the five spanning tree problems from above are $\mathbb{NP}$-complete.

B. Show that problem (C) is hard for any *fixed* $k \ge 2$.

Comment

The problems are clearly very similar, try to exploit that to avoid endless repetitions.

2.   Disjoint Unions (30)

Background

Assume that $\{0,1\} \subseteq \Sigma$ and define the disjoint union two languages $A, B \subseteq \Sigma^\star$ to be the language

$$A \oplus B = 0A \cup 1B$$

In the following, $\preceq$ means polynomial time many-one reducible. Argue in terms of projections, do not use nondeterministic Turing machines.

Task

A. Show that $A, B \preceq A \oplus B$.

B. Show that $A, B \preceq C$ implies $A \oplus B \preceq C$.

C. Show that $\mathbb{P}$, $\mathbb{NP}$ and co-$\mathbb{NP}$ are all closed under $\oplus$.

D. Let $A \in \mathbb{NP}$ and $B \in$ co-$\mathbb{NP}$. Where in polynomial hierarchy should we expect to find $A \oplus B$?

E. Assume all the mixed languages in part (D) are reducible to SAT.
What conclusions can you draw?

3. Cook-Levin, Again (40)

Background

The proof of Cook-Levin given in class is based on the characterization of $\mathbb{NP}$ in terms of projections. One can modify the argument to use nondeterministic Turing machines instead: we need a formula

$$M \text{ accepts } x \iff \Psi_x \text{ is satisfiable}$$

Task

A. What is the essential difference between the two approaches? Is there any reason why one might be better than the other?

B. Explain how to modify the construction of Φ_x from class to produce the right Ψ_x.

C. Show that your formula can be constructed in polynomial time and that it works as advertised.