15-455: UCT	**K. Sutner**
Assignment 7	Due: **March 29, 2024, 24:00**.

1. Σ_2^p (20)

Background

Recall that a Σ_2 Boolean formula looks like

$$\exists\, \boldsymbol{x}\, \forall\, \boldsymbol{y}\; \varphi(\boldsymbol{x}, \boldsymbol{y})$$

where $|\boldsymbol{x}| = k$, $|\boldsymbol{y}| = \ell$ and φ is a formula in propositional logic (no quantifiers) without any free variables. We will refer to the problem of testing such a formula for validity Σ_2-SAT. Clearly, this problem is at least as hard as SAT and TAUT. Intuitively, it should be harder than both.

Task

A. Show that the Maximum Independent Set problem from class is reducible to Σ_2-SAT.

B. Show that Σ_2-SAT lies at level Σ_2^p of the polynomial hierarchy for both the projection definition and the oracle definition.

C. Show that Σ_2-SAT is hard for level Σ_2^p of the polynomial hierarchy.

Comment You can use polynomial time reductions (though your arguments will probably automatically wind up in log-space). Don't get bogged down with technical details, just give a compelling proof sketch.

2. Uninspired Sets (30)

Background

Let $K(x \mid y)$ be the conditional Kolmogorov-Chaitin complexity of $x \in \mathbf{2}^\star$, given y. For any set $A \subseteq \mathbb{N}$ write $A_n = A \cap \{0, 1, \dots, n-1\}$ for the initial segment of A of length n. Think of A_n as bitvector of length n.

As we have seen, incompressibility with respect to Kolmogorov-Chaitin complexity is akin to randomness: there are no particular patterns one could exploit to obtain a shorter definition. How about the opposite notion? Call $A \subseteq \mathbb{N}$ uninspired if there is a constant c such that

$$K(A_n \mid n) \leq \log n + c.$$

So only some $\log n$ bits are needed to describe the corresponding bitvector of length n.

Task

A. Show that any decidable set A is uninspired.

B. How about the Halting Set H? State whether H is uninspired and explain your reasoning.

C. How about the complement of the Halting Set? Again, state whether this set is uninspired and explain your reasoning.

3. Kolmogorov versus Palindromes (30)

Background

Suppose M is a one-tape Turing machine recognizing palindromes over $\{0, 1\}$. We say that M crosses tape cell number i if either

- the head moves right from i to $i + 1$, or
- the head moves left from $i + 1$ to i.

We can construct of a crossing sequence $((p_1, s_1), (p_2, s_2), \dots)$ of all crossings of position i keeping track of the state p_i and the read symbol s_i at the moment of crossing (before the move). Note that right/left crossings must alternate.

Write $T(x)$ for the running time of M on input x, and assume that the machine always halts with the head on the right end of the string (it starts on the left). To streamline the argument a bit, it's best to consider input of the form $x = z\,0^n z^{\mathsf{op}}$ of length $3n$. The region $[n + 1, n + 2, \dots, 2n]$ is called the desert. Note that every position in the desert has at least one crossing.

Task

A. Give an intuitive explanation of why Kolmogorov complexity can help for this lower bound argument.

B. Show that some position in the desert must have a crossing sequence of length $m \leq T(x)/n$.

C. Show that a crossing sequence uniquely determines the string z.

D. Exploit this to conclude that we cannot have $T(x) = o(n^2)$.