$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \Sigma$ $\quad (\Omega, \Sigma, P)$

now X_i is measurable in $\mathcal{F}_i$ $\quad$ (X is adapted to $\mathcal{F}$).

and sps. $E[X_i | \mathcal{F}_{i-1}] = X_{i-1}$

take X_i and average over regions of $\mathcal{F}_{i-1}$

then $(X_0, X_1, \ldots)$ is a martingale.

---

by induction $E[X_t] = \cancel{\ldots} = E[X_0]$. typically set this to 0.

E.g. random walk.
n lire.

Example: $X_{i+1} = X_i + Y_{i+1}$ ← indpt random var. (0 mean).

but could also be $X_{i+1} = X_i + Y_{i+1}(X_i)$ ← depends on past, but $E[Y_{i+1} | X_1 \ldots X_i] = 0$.
Conditionally still 0 mean.

E.g. random walk with absorbing borders.

Stopping time measurable wrt filtration.

Sps. $ET < \infty$ and all step sizes are bounded a.s.

then $E[X_T] = E[X_0]$. (Doob's Optional Stopping thm)

E.g: $E[S_T] = E[S_0] = 0$ in random walk. $\quad S_t = X_1 + X_2 + \cdots + X_t$

So. $p_a(-a) + p_b(b) = 0. \Rightarrow p_a = \frac{b}{a+b}$.

but also. $S_T^2 - t$ is a martingale for the movements being 0 mean var 1.

$\Rightarrow E[S_T^2] = E[t] = \frac{b}{a+b} \cdot a^2 + \frac{a}{a+b} b^2 = ab$, $\Rightarrow n^2$ time to hit barriers @ $\pm n$.

Ville's thm: if S_n is a supermartingale, nonnegative then $\Pr(\sup_n S_n > x) \le \frac{E[S_0]}{x}$.

Like Markov but now for supermartingales.

And get supremum for free.

Concentration bounds (Azuma-Hoeffding)

$0 = X_0, X_1, X_2 \dots$ martingale with $|X_i - X_{i-1}| \le c_i$.

then $\Pr[|X_t| \ge \lambda] \le \exp(-\lambda^2/2\sigma^2)$ where $\sigma^2 = \sum_i c_i^2$.

And using tricks like Ville's

$$\Pr\left[\max_{s \le t} |X_s| \ge \lambda\right] \le \exp(-\lambda^2/2\sigma^2) \text{ as well.}$$

Example: sps. $f(\bar{x}) \in \mathbb{R}$ $\bar{X} = (X_1, X_2 \dots X_n)$. X_i indep r.v.s.

then def $Y_0 = E[f]$

$Y_1 = E[f \mid X_1]$

$\vdots$

$Y_n = E[f \mid X_1, X_2 \dots X_n] = f$.

Verify!!

Moreover: sps. changing coordinate i changes f by $\le c_i$ $\Rightarrow |Y_i - Y_{i-1}| \le c_i$

Now apply Azuma: $\Pr[|f - Ef| \ge \lambda] \le \exp(-\lambda^2/2\sigma^2)$. [McDiarmid]

Example: chromatic number of random graph $G_{n,p}$ is concentrated on $\le O(\sqrt{n})$ values. [Shamir – Spencer]

[Bollobas: at most 4 values.

[Alon Krivilevich, Achlioptas and Naor: 2 values]

$p \le n^{1/2-\varepsilon}$ $p = d/n$.

Btw: instead of requiring that $|X_i - X_{i-1}| \le c_i$ a.s.

we could use that $[X_i \mid X_{i-1} = z]$ has ~~exponential tails~~. $\boxed{\text{subgaussian tails.}}$

and hence can extend Azuma-Hoeffding to sums of subgaussian r.v.s.

3

Discrepancy

$\mathcal{S} = S_1, S_2 \ldots S_m \subseteq U$

$\chi : U \to \{-1, 1\}$.

- $\chi(S) = \sum_{i \in S} \chi(i)$
- $\mathrm{disc}_\chi(\mathcal{S}) = \max_{S \in \mathcal{S}} \chi(S)$.
- Want χ to get low discrepancy $\mathrm{disc}_\chi(S)$.

Thm: $\exists$ set systems such that $\mathrm{disc}_\chi(\mathcal{S}) \geq \Omega(\sqrt{n})$.
$m = O(n)$ sets n elts

take Hadamard matrix $H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $H_k = \begin{pmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{pmatrix}$.

now $m = 2.2^k$ sets for each row i; $A_i = +$ entries $\bar{A}_i = -$ entries.

then $\chi(A_i) - \chi(\bar{A}_i) = \langle \chi, h_i \rangle$

$$\Rightarrow |\langle \chi, h_i \rangle| \leq |\chi(A_i)| + |\chi(\bar{A}_i)| \qquad (*)$$

But h's form an orthogonal basis, with vector length $\|h_i\|^2 = 2^k$.

$$\Rightarrow \sum_i (\langle \chi, \underbrace{\bar{h}_i}_{\text{scaled}} \rangle)^2 = \frac{\sum \langle \chi, h_i \rangle^2}{2^k} = \|\chi\|^2 = 2^k.$$

$$\Rightarrow \exists i \text{ st } \langle \chi, h_i \rangle^2 \geq 2^k$$

$$\Rightarrow |\langle \chi, h_i \rangle| \geq 2^{k/2}$$

using $(*)$ get $\exists i$, A_i or $\bar{A}_i$ satisfy $\chi(\mathcal{S}) \geq \frac{1}{2}\sqrt{n}$.

Spencer Showed: $\forall$ set-system with $m = O(n)$ sets,
$\exists$ coloring with $disc_\chi(\mathcal{S}) \le O(\sqrt{n})$.

Recent proofs using CS-type ideas (based on making the results algorithmic)

Bansal, Lovett-Meka, Rothvoss, Eldan-Singh, Bansal-Laddha-Vempala, ...

Give the Lovett-Meka proof

Want $\quad \chi(S_i) \le c\sqrt{n \log(m/n)} \quad \forall i$

$x_i \in \{-1, 1\}$.

relax to $x_i \in [-1, 1]$.

☹ $x_i = 0$ is a solution with discrepancy zero!!

So want to round.

try 1: independent rounding:- $\Pr[disc(S_i) \ge c\sqrt{n \lg m}] \le 1/poly(m)$
then union bound..

But: get disc $\sqrt{n \lg m}$ for m sets, and hence $\sqrt{n \log n}$ for n sets
"Penalty for union bound"

Smarter Idea: small steps, and fix problems as they arise

let's make this geometric:-

$\chi(S_j) \le c\sqrt{n \lg(m/n)} \quad \rightarrow \quad \langle a_j, x \rangle \le \|a_i\| . c_i \quad \forall i \in [m].$

scale $a_i \in \mathbb{R}^n$ s.t. $\|a_i\| = 1$ (wlog)

think of this as $\Theta(\sqrt{\log(m/n)})$.

$x_j \in [-1, 1]$.
$\forall j \in [n]$.

start @ x_0 (think as 0).

Call constraint a_i dangerous if $\langle a_i, x\rangle \ge c_i - \delta$
variable x_j frozen if $|x_j| \ge 1-\delta$.
} apx satisfied

Each time t: Find subspace orthogonal to all dangerous constraints
& all frozen variables.

until subspace V_t dimension smaller than $n/4$.
(i.e. will have $3n/4$ of these constraints/vars).

Idea: want to ensure that many of these are frozen vars.
not ~~high~~ dangerous constraints.

[Pick $g_t \sim N(V_t)$ gaussian from that subspace.
$X^0 = x_0$, $X^t \leftarrow X^{t-1} + \gamma g_t$]

tiny

martingale.

$\delta \ge \gamma \cdot \sqrt{\log \frac{mn}{\gamma}}$

Claim 1: $\dim(V_{t+1}) \le \dim(V_t)$.

Claim 2: if we are not dangerous / frozen, it will not be violated whp.

Why? For a gaussian $\Pr[|g| \ge \lambda] \le 2\exp(-\lambda^2/2)$

now $|\gamma g| > \delta \le 2\exp\left(-(\delta/\gamma)^2/2\right) = 2\exp\left(-\frac{\lg(mn/\gamma)}{2}\right)$
$= \left(\frac{\gamma}{mn}\right)^{O(1)}$.

⇒ union bound over all steps.

Claim 3: $E[\|x^t\|^2] = E[\|x^{t-1}\|^2] + \underbrace{2E[\langle x^t, \gamma g^t\rangle]}_{0 \text{ indep}} + \gamma^2 \underbrace{E[\|g^t\|^2]}_{\gamma^2 \dim(V_t)}$

$= \gamma^2 \sum_{s \le t} \dim(V_s)$ ⇒ as long as $\dim(V_s) \ge n/4$, progress.