Expanders and High Dimensional Expanders

$G = (V, E)$ undirected, unwtd ← could generalize to weighted

, often d-regular ← also can generalize, but math simpler

$A =$ adjacency matrix $\in \{0,1\}^{n \times n}$.

- the edge expansion of set $S \subseteq V$ is

$$\Phi(S) = \frac{|E(S, \bar{S})|}{|S|}.$$

and of graph: $\Phi(G) := \min_{|S| \le n/2} \Phi(S)$.

- closely related: the conductance

$$\phi(S) := \frac{|E(S,\bar{S})|}{\sum_{v \in S} \deg(v)}$$

called $\mathrm{vol}(S)$

subtle diff !!

and of graph. $\phi(G) := \min_{S:\, \mathrm{vol}(S) \le |E|} \phi(S)$

Note: $\Phi(G) \in [0, d_{max}]$

$\phi(G) \in [0,1]$. It's a ratio.

for d-regular graphs, $\phi = \Phi/d$.

Other notions: sparsity

A graph is an α-combinatorial expander if $\phi(S) \ge \alpha$.

the best we can hope for is α being an absolute constant, indep of n.

Also: spectral defs.

recall that $A =$ adjacency matrix, $L = D - A$ ← degree (diagonal) matrix

then
- $L \mathbb{1} = 0$.
- both are symmetric real matrices, have real eigenvalues
- $L = BB^T$ is psd and hence has non-neg eigenvalues.

$\boxed{\text{For the rest of today, let } G \text{ be } d\text{-regular}}$ also ammended!

then eigenvalues, eigenvectors of L and A are ~~just~~ related $\lambda_i(L) = d - \lambda_i(A)$.
with the same eigenvectors

We ~~lets~~ talk about them interchangeably.

For L : $\lambda_1(L) = 0 < \lambda_2(L) \le \lambda_3(L) \; \ldots \ldots \le \lambda_n(L)$.

$\lambda_1(A) = d > \lambda_2(A) \ge \; \ldots \ldots \ge \lambda_n(A)$.

eigenvectors $\frac{\mathbb{1}}{\sqrt{n}} = f_1 , f_2 \; \ldots \ldots \; f_n$.
form an orthonormal basis of $\mathbb{R}^n$.

Consider a scaled version $\bar{A} = \frac{1}{d} A$. ← eigenvalues $\frac{\lambda_1(A)}{d} = 1 > \bar{\lambda}_2(\bar{A}) \ldots \ge \lambda_n(\bar{A})$

then if we do a random walk on the graph.

$$p^t(u) = \sum_{v \in G} p^{t-1}(v) \, A_{vu} \cdot \frac{1}{d} = (\bar{A} p^{t-1})(u) . \quad \Rightarrow p^t = \bar{A} \, p^{t-1}.$$

So suppose $p^0 = \sum_i c_i f_i$
in the eigenbasis

then $p^t = \sum_i c_i (\bar{\lambda}_i^t f_i)$

now if $\bar{\lambda}_2, |\bar{\lambda}_n| \le 1 - g = \bar{\lambda}_1 - g$
gap.

~~then $c_1 f_1 + \sum_{i \ge 2} c_i (1-g)^t$~~

all terms except $c_1 f_1$ go to 0.
as $t \to \infty$

and left with $\lim_{t \to \infty} p^t = c_1 f_1$

But $f_1 = \mathbb{1}/\sqrt{n}$ and $p_0 = $ prob. so

$\langle f_1, p_0 \rangle = \langle \mathbb{1}/\sqrt{n}, p_0 \rangle = 1/\sqrt{n} = c_1$.

$\Rightarrow \lim_{t \to \infty} p^t = \frac{1}{\sqrt{n}} \cdot \mathbb{1}/\sqrt{n} = \mathbb{1}/n$ ☺

~~A graph is a λ^*-spectral expander if $\max(\bar{\lambda}_2, |\bar{\lambda}_n|) <$ ... there is a~~

- A graph has a (two sided) spectral gap g if
$$\min(\bar{\lambda}_1 - \bar{\lambda}_2 , \; \bar{\lambda}_1 - |\bar{\lambda}_n|) \ge g.$$

- A graph is a g spectral expander if
it has constant spectral gap.
$\quad = g$

Cheeger's Inequality G be d-regular, with normalized adjacency $\bar{A}$ having eigenvalues $\bar{\lambda}_1 = 1 \geq \bar{\lambda}_2 \geq \ldots$ and $\phi(G)$ be the conductance. Then

$$\frac{1}{2}(1-\bar{\lambda}_2) \leq \phi(G) \leq \sqrt{2(1-\bar{\lambda}_2)}$$

"easy" "hard"

So spectral expansion $\Leftrightarrow$ combinatorial expansion, at least qualitatively.

Note: $1-\bar{\lambda}_2$ = second eigenvalue of normalized Laplacian $= \frac{1}{d}(D-A)$.

$$= \min_{f \perp \mathbb{1}} \frac{\sum_{i \sim j}(f_i - f_j)^2}{d \cdot \sum_i f_i^2} \leq \text{same thing for vector } f_S = \begin{cases} 1/|S| & \text{for } i \in S \\ -1/|V-S| & \text{for } i \notin S. \end{cases}$$

where S is the minimizer of $\phi(G)$

$$= \frac{|E(S,\bar{S})| \cdot \left(\frac{1}{|S|} + \frac{1}{|V-S|}\right)^2}{d\left(|S| \cdot \frac{1}{|S|^2} + |V-S| \cdot \frac{1}{|V-S|^2}\right)} = \frac{|E(S,\bar{S})|}{d\,|S|} \cdot \frac{|V|}{|V \setminus S|}$$

$$\leq 2 \cdot \phi(S) = 2\phi(G)$$

$\therefore$ easy direction proved!!

Hard direction is not very hard; but another time.

Examples:

- for an n-cycle, $\bar{\lambda}_2 \cong 1 - \Theta\left(\frac{1}{n^2}\right)$ but $\phi(G) = \Theta\left(\frac{1}{n}\right)$.
  $\Rightarrow$ hard ineq tight.

- for complete graph, both are constant $\Rightarrow$ easy direction tight.

Can extend to weighted graphs, see notes by many people (Dan Spielman's book) (eg. Lap Chi Lau @ Waterloo) (survey by Hoory Linial Wigderson) (Luca Trevisan's book)

The calculation we did for showing that the random walk on regular graphs converges to the uniform distribution also gives quantitative bounds.

def: total variation distance b/w p, q on some set Ω (of size n say)

$$d_{TV}(p,q) = \max_{S \subseteq \Omega} p(S) - q(S) = \frac{1}{2} \| p - q \|_1$$

$$\leq O(\sqrt{n}) . \| p - q \|_2 .$$

But we wanted $d_{TV}(p^t, \mathbb{1}/n) \leq O(\sqrt{n}) . \| \sum_{i \geq 2} c_i \lambda_i^t f_i \|_2$

$\| \cdot \|_2^2$ note $\leq (1-g)^{2t} . \underbrace{\sum_{i \geq 2} c_i^2}_{\leq \|p_0\|_2^2 \leq \|p_0\|_1^2 = 1}$ gap $\qquad \leq O(\sqrt{n}) . (1-g)^t \leq \sqrt{n} . e^{-gt}$

$\leq \varepsilon$

if $t \geq \frac{\Omega(\log n/\varepsilon)}{g}$

So if gap $g = \Omega(1)$ then mix in $\Omega(\log n)$ steps! ← "rapid" mixing

Interesting — must mean that graph has small diam $O(\log n)$.

And indeed follows from combinatorial def^n.

— — X —

Markov Chain Monte Carlo

Another interesting angle. Sps. want to do MCMC for sampling matroid bases, or colorings, or independent sets, etc. Define a large graph $\mathcal{G}$ on space of all valid objects, and walk from one to the other. (random walk on $\mathcal{G}$). Now if $\lambda_2(\mathcal{G})$ is good (has good gap) then walk mixes rapidly in $\frac{\log(N/\varepsilon)}{g}$.

OK even if $N = \exp(n)$; which is basis of a lot of work on this topic.

A couple of other facts:

Expander Mixing Lemma, or "Expanders behave like random graphs"

- G be d-regular, let $1 = \bar\lambda_1 > \bar\lambda_2 \geq \cdots \quad \bar\lambda_n \geq -1$ be eigenvalues of normalized adj matrix.

  call G an (n, d, ε) graph if $\max\{\bar\lambda_2, |\bar\lambda_n|\} \leq \varepsilon$.

  "spectral radius"

- Expander Mixing Lemma: G be an (n,d,ε)-graph then $\forall S, T \subseteq V$

$$\left| E(S,T) - \frac{d}{n}|S||T| \right| \leq \varepsilon d \sqrt{|S||T|}.$$

"expected number from S to T"

small compared to Exp # if $|S| \cdot |T| \gg (n\varepsilon)^2$

Pf: ~~omitted, fairly simple Cauchy-Schwarz after using eigendecomp.~~

$$|E(S,T)| = \chi_S^T A \chi_T = d \sum_i a_i b_i \bar\lambda_i = \frac{d}{n}|S||T| + d\sum_{i \geq 2} a_i b_i \bar\lambda_i$$

Let $\chi_S = \sum_i a_i f_i \quad \chi_T = \sum_i b_i f_i \quad A = d\sum_i \bar\lambda_i f_i f_i^T$

$\Rightarrow a_1 = \langle \chi_S, f_1 \rangle = |S|/\sqrt{n} \qquad b_1 = |T|/\sqrt{n}$

abs value $\leq \varepsilon d \sum_i |a_i||b_i| \overset{(C\text{-}S)}{\leq} \varepsilon d \|a\| \|b\| = \varepsilon d \sqrt{|S|}\sqrt{|T|}$ ∎

Applications

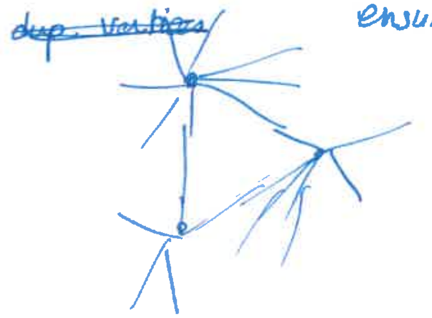
- Derandomization. Sps. have an expander graph on 2^r vertices so that each node (H) represents an r-bit random string. Suppose have an algo that fails with prob $\leq 1/4$ on any ~~single~~ random r-bit string (drawn u.a.r.). Then if we want to amplify:
  - either choose t diff r-bit strings, prob of error $\leq (1/4)^t$.
  - or pick a random node in H, and take a t-length random walk.
    - requires $r + t \log d$ bits
    - prob of error on all tries $\leq (1/4 + O(\varepsilon))^t$ if H is an $(2^r, d, \varepsilon)$ graph.

Intuition: cannot stay in the "bad" region most of the time.

- Error Correcting Codes: getting longer codes from small ones.
  - take a (linear) code on d-bits $\mathcal{C}$. (asymptotically good, constant rel. dist, constant rate) say a $[d, rd, \delta d]$ code
  - take an (n, d, ε)-graph G.

[Tanner Code]

~~dup. vertices~~



ensure that each vertex sees a code word around it

Gives a: $\left[\frac{nd}{2}, (2r-1)\frac{nd}{2}, \delta(\delta-\varepsilon)\cdot\frac{nd}{2}\right]$ code

original rate better be $> 1/2$.

rel distance $\to \delta^2$ if $\varepsilon \ll \delta$.

Proof uses the Expander Mixing Lemma.

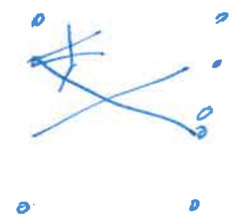
Pf: Sps Linear base code $\mathcal{C}$, ~~then~~ lets use a "double cover" construction instead

then

① Linear ~~exp~~ code.

of nd bits

L = V R = V



nd edges

d regular, bip

Both sides check if nbrhood in $\mathcal{C}$.

So just suffices to verify the weight of least weight codeword (non-zero)

Say w^* is least Hamming wt. uses edges F for 1s.

then let S, T be vtxs on L, R incident to edges in F.

(a) $\forall v \in S \cup T$ $|F \cap \partial v| \geq \delta d$.

$$\Rightarrow |F| \geq \delta d|S|,\ \delta d|T| \Rightarrow |F| \geq \delta d\sqrt{|S||T|}. \quad (*)$$

(b) $F \subseteq E(S,T)$ and

$$\text{so } \delta d\sqrt{|S||T|} \le |E(S,T)| \le \underbrace{\tfrac{d}{n}|S||T| + \varepsilon d\sqrt{|S||T|}}_{\text{(EML)}}$$

$$\Rightarrow (\delta-\varepsilon)n \le \sqrt{|S||T|}$$

by (*) $\Rightarrow |F| \geq \delta(\delta-\varepsilon)\cdot dn. \Rightarrow$ rel. dist $\geq \delta(\delta-\varepsilon)$

Can do linear time (and log round) decoding as well,

the proofs are similar, show that # edges that disagree with nearest codeword decrease geometrically!

[Zemor decoding]

OK, now to HDXs.

$X = (U, \mathcal{F})$

- Simplicial Complex (SC) = downward closed set system ← eg. matroid ind sets
- dim of a set S (face) $= |S| - 1$.
- dim of SC = ~~[illegible]~~ dim of largest set in it (a "facet")

pure SC = all facets have same size (also as in matroid).

- So matroids are pure SCs with exchange ppty

——— x ———

$X(i)$ = sets of size $i+1$. (dim i)

$X = X(0) \cup X(1) \cup \cdots \cup X(d)$

"vertices" "edges" "facets"

abc

ab bc ac ad

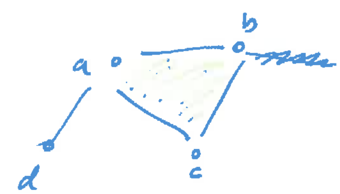
a b c d

ϕ ← dim -1.

——— x ———

- 1-skeleton of X = graph $(X(0), X(1))$

called $G(X)$



——— x ———

- Links of X.

given $\alpha \in \mathcal{F}$, $X_\alpha = \{ \beta \setminus \alpha \mid \beta \supseteq \alpha, \beta \in \mathcal{F} \}$. ← "contraction" of α

If X is a pure d-dim SC $\Rightarrow$ X_α = pure $d - |\alpha|$ dim SC.

Local Spectral Expanders: X is pure d-dim SC. X is γ-(local)-spectral-expander

if $\forall$ faces α (of dim $\le d-2$), $G(X_\alpha)$ has λ_2 of the random walk matrix $\le \gamma$.