OK, now to HDXs.

$$X = (\mathcal{U}, \mathcal{F})$$

- Simplicial Complex (SC) = downward closed set system ← e.g. matroid ind sets

- dim of a set S = $|S| - 1$. ← face
- dim of SC = dim of largest set in it. ← a "facet"
- pure SC = all facets have same size (also as in matroid).

- So matroids are pure SCs with exchange ppty

— x —

$X(i)$ = sets of size $i+1$. (dim i)

$$X = X(0) \uplus X(1) \cup \cdots \uplus X(d)$$

"vertices" "edges" "facets"

abc

ab bc ac ad

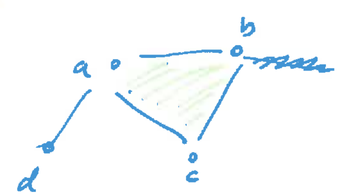
a b c d

ϕ ← dim -1.

— x —

- 1-skeleton of X = graph $(X(0), X(1))$

called $G(X)$



— x —

- Links of X.

given $\alpha \in \mathcal{F}$, $\quad X_\alpha = \{\beta \setminus \alpha \mid \beta \supseteq \alpha, \beta \in \mathcal{F}\}$. ← "contraction" of α

If X is a pure d-dim SC $\Rightarrow$ X_α = pure $d-|\alpha|$ dim SC.

Local Spectral Expanders: X is pure d-dim SC. X is γ-(local)-spectral-expander

if $\forall$ faces α (of dim $\le d-2$), $G(X_\alpha)$ has λ_2 of the random walk matrix $\le \gamma$.

Moreover we can define a probability distribution over facets, and over each layer, as follows:—

X be a pure SC of dim d. Let π_d be a prob distrib over $X(d)$.

Now can define a distrib over $X(d-1)$ by setting for each $\alpha \in X(d-1)$:—

$$\pi_{d-1}(\alpha) = \frac{1}{d+1} \sum_{\substack{\beta \in X(d) \\ \beta \supseteq \alpha}}' \pi_d(\beta).$$

size of sets β. in $X(d)$

this gives a distrib over $X(d-1)$. Now induct.

—— X ——

Given (X, π_d) ~~be~~ d-dim pure SC with weights. For α and $\tau \in X_\alpha$

get $\pi^\alpha(\tau) = \Pr_{\beta \sim \pi_{|\tau|+|\alpha|-1}}[\beta = \alpha \cup \tau \mid \beta \supseteq \alpha] = \dfrac{\pi(\alpha \cup \tau)}{\binom{|\alpha \cup \tau|}{|\alpha|} \cdot \pi(\alpha)}.$

just the natural thing.

—— X ——

Now: gives weights on the 1-skeleton (edges and vtxs) of any X.

Call G_α the ~~[illegible]~~ skeleton of the link, π_1^α be distrib over edges.

D_α = diagonal matrix with $D_\alpha(x,x) = \sum'_{y \in X_\alpha(0)} \pi_1^\alpha(\{x,y\}) = 2\pi_0^\alpha(\{x\})$.

Now the walk is not unweighted: the weights decide the prob of going places.

$W_\alpha = D_\alpha^{-1} A_\alpha$. has $\lambda_{max} = 1$, with $f_{max} = \mathbb{1}$.

[ALOV]

Thm: The SC of any matroid with the uniform distribution on facets is a 0-local expander.

Pf: Show that ① every link ~~is~~ has connected skeleton, and

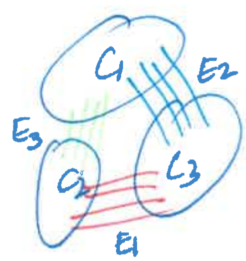
② every top link has skeleton which is 0-local-expander.

① Say contract α. So left with matroid. Can get from any vtx to any other b/c of exchange axiom.

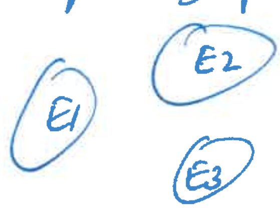
② Let's sketch the proof for graphic matroids, where $d = n-2$ (since bases are of size $n-1$). Now any link X_α corresponds to contracting the edges of a forest $F \subseteq E$ with $|F| = n-3$ edges). This gives

$\Rightarrow$ 3 components

the following



Hence the link has skeleton whose elements correspond to these colored edges (E_1, E_2, E_3 say), and whose pairs correspond to pairs of colored edges that can belong to some base. Note these are edges in a complete tripartite graph, since we can add any pair of elements of different colors.



It is a fact that the complete k-partite graphs have at most one positive evalue (and this is a characterization of complete k-partite graphs).

We discussed this on Piazza so lets give a proof (another to be added later).

<u>Pf 1:</u> Cauchy's Interlacing thm says that if

$$A = \underset{\uparrow \text{rank } 1 \text{ (PSD)}}{B + \omega\omega^T} \qquad \text{then} \qquad \lambda_1(A) \geq \lambda_1(B) \geq \lambda_2(A) \geq \cdots$$

<u>Now:</u> the adjacency matrix of a complete k-partite graph is

$$A = J + \underbrace{(-B_1 - B_2 \ldots - B_k)}_{B}. \quad \text{where} \quad B_i = \chi_{E_i}\chi_{E_i}^T \text{ and } J = \mathbb{1}\mathbb{1}^T$$

$$= B + \mathbb{1}\mathbb{1}^T.$$

Now B is negative semidefinite so $\lambda_{max}(B) \leq 0$.

Now $\lambda_2(A) \leq \lambda_1(B) \leq 0$.

$\blacksquare$

<u>Pf 2:</u>

OK: given a distribution π over facets (usually uniform), we have a ~~natural~~ natural way to get distributions over links, and over each 1-skeleton. (and so over random walks)

Each has a spectral gap (for the matrix $D_\alpha^{-1} A_\alpha$).

(X,π) is a γ-local-spectral expander if λ_2 for each of these link matrices is at most γ. (1st, largest eigenvalue $\lambda_1 = 1$ for each, with $f_1 \propto \mathbb{1}$).

Do we need to check the expansion for all links?

Oppenheimer gave a nice "trickle-down" theorem that says: if the top links are good expanders then the lower links are, too (with some loss).

Thm [Opp18]: (X,π) be pure SC of dim d (weighted by π).

Sps. $G_\phi = (X(0), X(1), \pi_1)$ is connected and $\lambda_2(D_v^{-1} A_v) \le \gamma$ for all $v \in X(0)$

$$\Rightarrow \lambda_2(D_\phi^{-1} A_\phi^{-1}) \le \frac{\gamma}{1-\gamma}$$

Corollary: if top links $\lambda_2(D_\alpha^{-1} A_\alpha) \le \gamma \quad \forall \alpha \in X(d-2)$

and G_β is connected for all $\beta \in X(k) \ \forall k \le d-2$

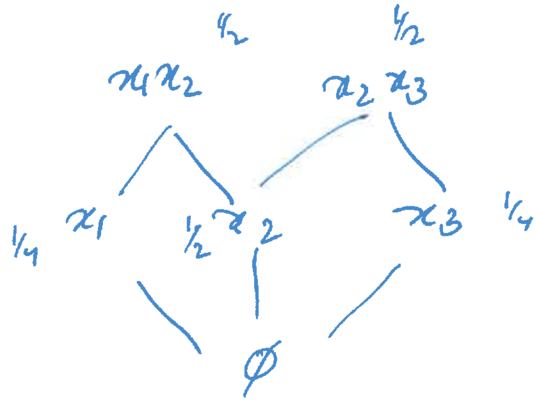
then $\lambda_2(D_\phi^{-1} A_\phi^{-1}) \le \frac{\gamma}{1-(d-1)\gamma}$

Pf: omitted

But note: if each G_α is a 0-local-expander

$\Rightarrow G_\phi$ is also a 0-local expander!

This is what we show for case for top links of matroidal SC.



P_0^Δ not symm

$$A = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{array}\left[\begin{array}{ccc|cc} 1/2 & 1/2 & 0 & & \\ 1/4 & 1/2 & 1/4 & & 0 \\ 0 & 1/2 & 1/2 & & \\ \hline & 0 & & 3/4 & 1/4 \\ & & & 1/4 & 3/4 \end{array}\right]$$

P_1^∇

eigenvalues of $P_0^\Delta = 1 = \lambda_0,\ 1/2,\ 0$ evecs. $(1\,1\,1)$, $(-1\,0\,1)$, $(1\,-1\,1)$

$P_1^\nabla = 1 = \lambda_1,\ 1/2.$ evs $= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

not orthonormal, since P_0^Δ & P_1^∇ are not necc. symmetric

<u>Lemma</u>: Eigenvalues of P_k^Δ and P_{k+1}^∇ are the same.

And so we can talk about the spectral gap of a pair of layers $X(k)$ $X(k+1)$
(one for the down-up walk on $X(k+1)$
another for up-down walk on $X(k)$.)

<u>Last time</u> we saw that if G was a d-regular n vtx graph with
$\max(\lambda_2, |\lambda_n|) \le \varepsilon \Rightarrow$ called (n,d,ε) graph.

then random walk "mixes" in $O\left(\frac{\log(n/\delta)}{\varepsilon}\right)$ time
(TV distance to uniform $\le \delta$)

If graph not regular $\Rightarrow$ cannot use same proof, b/c matrices not symmetric.
but still walks on undirected graphs (perhaps with weights).

Now the mixing time (to the stationary distribution π) $\le \frac{1}{\varepsilon}\log\left(\frac{n}{\delta}\cdot\frac{d_{max}}{d_{min}}\right)$ max/min (wtd) degree ratio

Hence still interesting to bound the spectral radius of, by ε.

$X(k+1)$

$X(k)$.

In fact we care about the spectral radius of P_d^∇

But that's the same as P_{d-1}^Δ.

Now if we could relate P_{d-1}^Δ to P_{d-1}^∇ etc.....

which is the proof of

non lazy walk.

Actually relate $P_{d-1}^\wedge$ to P_{d-1}^∇ lose γ

and then relate P_{d-1}^Δ to $P_{d-1}^\wedge$ lose a little more.

.....

$1-\frac{1}{d}$ to $1-\frac{1}{d+1}$

Thm [Kaufman-Oppenheim] if (X,Π) is a γ-local-spectral expander then $\forall\, k \le d$.

$$\lambda_2(P_k^\nabla) \le 1 - \frac{1}{k+1} + k\gamma$$

For matroidal sc $\gamma = 0 \Rightarrow \lambda_2(P_k^\nabla) \le 1 - \frac{1}{k+1} \Rightarrow \lambda_2(P_d^\nabla) \le \frac{d}{d+1}$.

$\Rightarrow$ spectral gap $\le \Theta(1/d)$

\# of bases

Now plugging into mixing bounds: #steps $\le O\left(\frac{1}{\varepsilon}\log\left(N/\delta\right)\right)$

but $\varepsilon = 1/d$

$N = n^d \leftarrow$ # of bases

$\Rightarrow O(d^2 \log(n/\delta))$ time!

d = rank of matroid.

Intuition:

$P^\wedge$

X

$\cdot \quad \cdot \quad , X(k)$ $\leftarrow$ "expander"

P^∇ $\leftarrow$ "clique"