15-853:Algorithms in the Real World

- Expander Graphs
- LDPC (Expander) codes

Examples of Codes

Hamming codes are binary (2^r-1–1, 2^r-1-r, 3) codes.

Basically (n, n – log n, 3)

Hadamard codes are binary (2^r-1, r, 2^{r-1}) codes.

Basically (n, log n, n/2)

Reed Solomon codes are $(n, k, n-k+1)_n$

Optimal but large alphabet

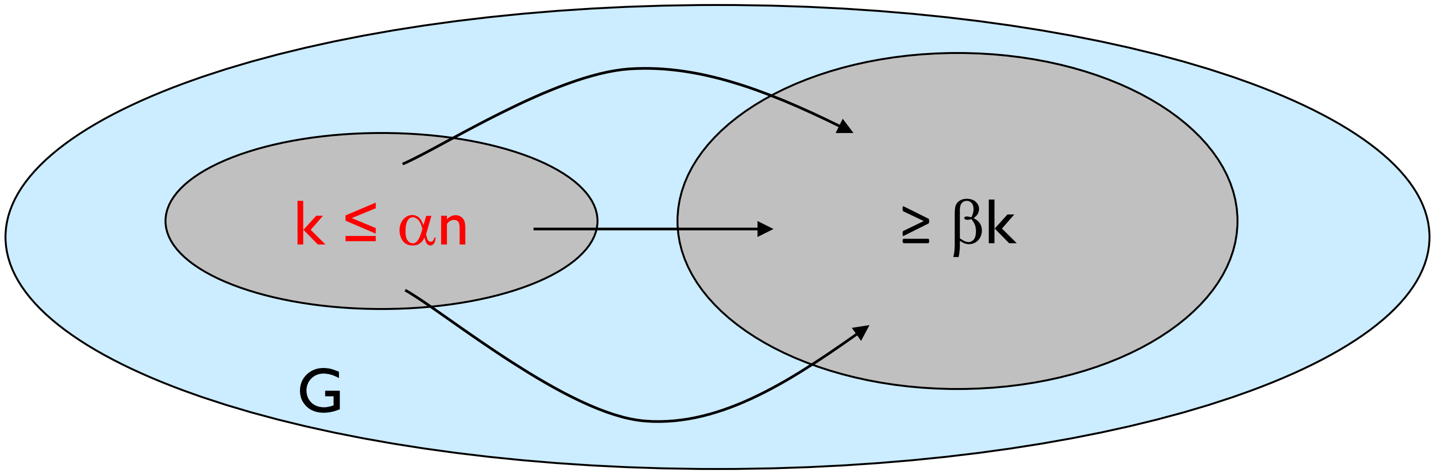
Concatenated codes can get best of both worlds

Next:

Another set of codes, **optimized for fast (de)coding.**

Based on **graphical constructions**.

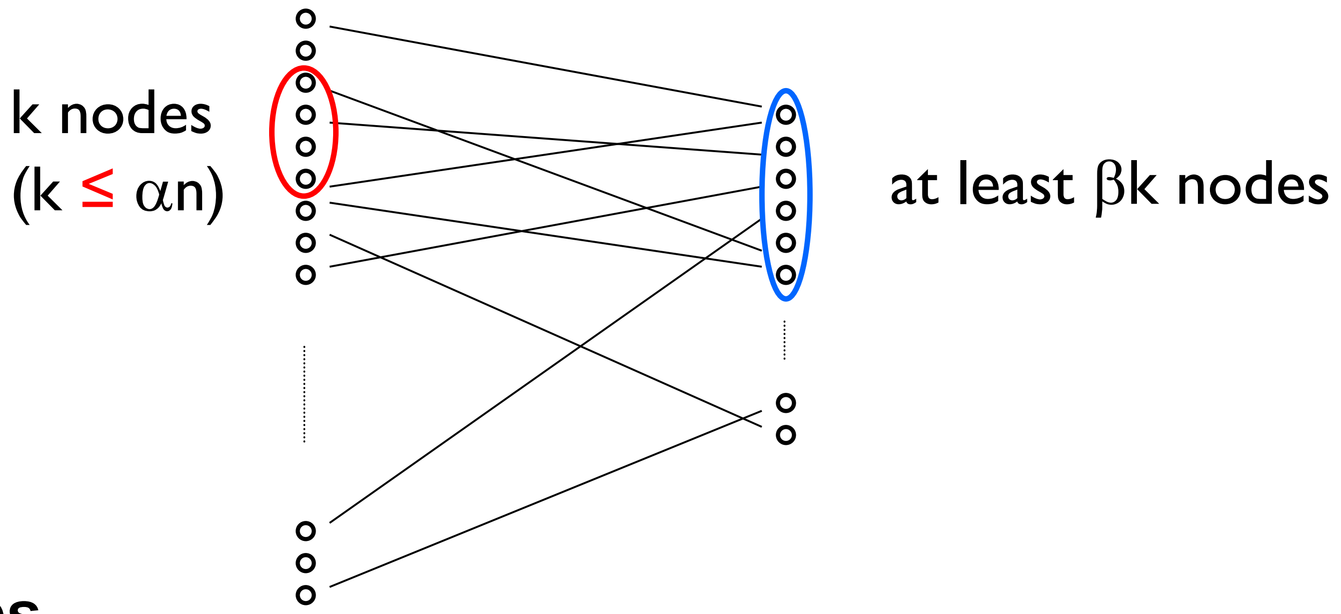
(α, β) Expander Graphs (non-bipartite)



Properties

- **Expansion:** every small subset ($k \le \alpha n$) has many ($\ge \beta k$) neighbors
- **Low degree –** not technically part of the definition, but typically assumed

(α, β) Expander Graphs (bipartite)



Properties

– **Expansion:** every small subset ($k \leq \alpha n$) on left has many ($\geq \beta k$) neighbors on right

– **Low degree –** not technically part of the definition, but typically assumed

Expander Graphs: Applications

Pseudo-randomness: implement randomized algorithms with few random bits

Cryptography: strong one-way functions from weak ones.

Hashing: efficient n-wise independent hash functions

Random walks: quickly spreading probability as you walk through a graph

Error Correcting Codes: several constructions

Communication networks: fault tolerance, gossip-based protocols, peer-to-peer networks

d-regular graphs

An undirected graph is **d-regular** if every vertex has d neighbors.

A **bipartite** graph is **d-left-regular** if every vertex on the left has d neighbors on the right.

We consider only d-left-regular constructions.

(And call it d-regular with abuse of notation.)

Expander Graphs: Constructions

Important parameters:size (n), degree (d), expansion (β)

Randomized constructions

- A random d-regular graph is an expander with a high probability
- Time consuming and cannot be stored compactly

Explicit constructions

- Cayley graphs, Ramanujan graphs etc
- Typical technique – start with a small expander, apply operations to increase its size

Expander Graphs: Constructions

Theorem: For every constant $0 < c < 1$, can construct bipartite graphs with

n nodes on left,
cn on right,
d-regular (left),

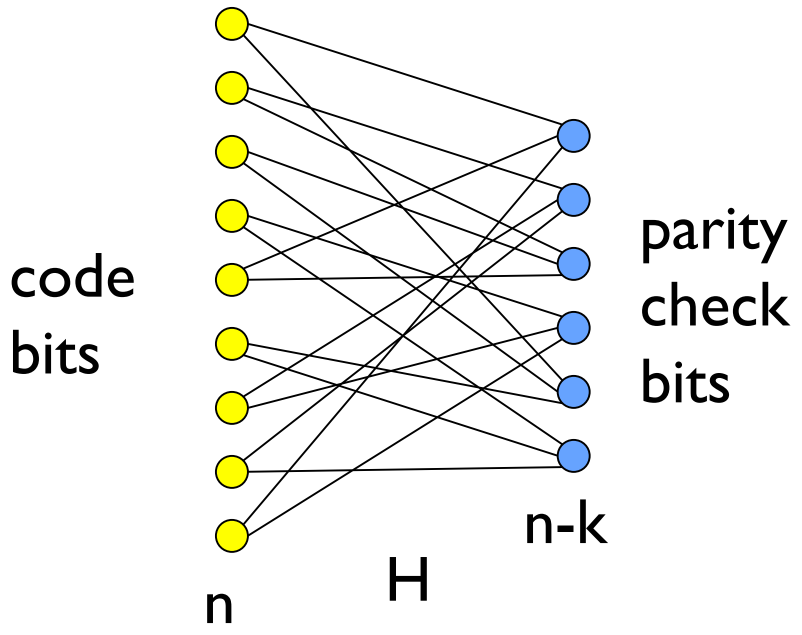
that are $(\alpha, 3d/4)$ expanders, for constants α and d that are functions of c alone.

"Any set containing at most alpha fraction of the left has (3d/4) times as many neighbors on the right"

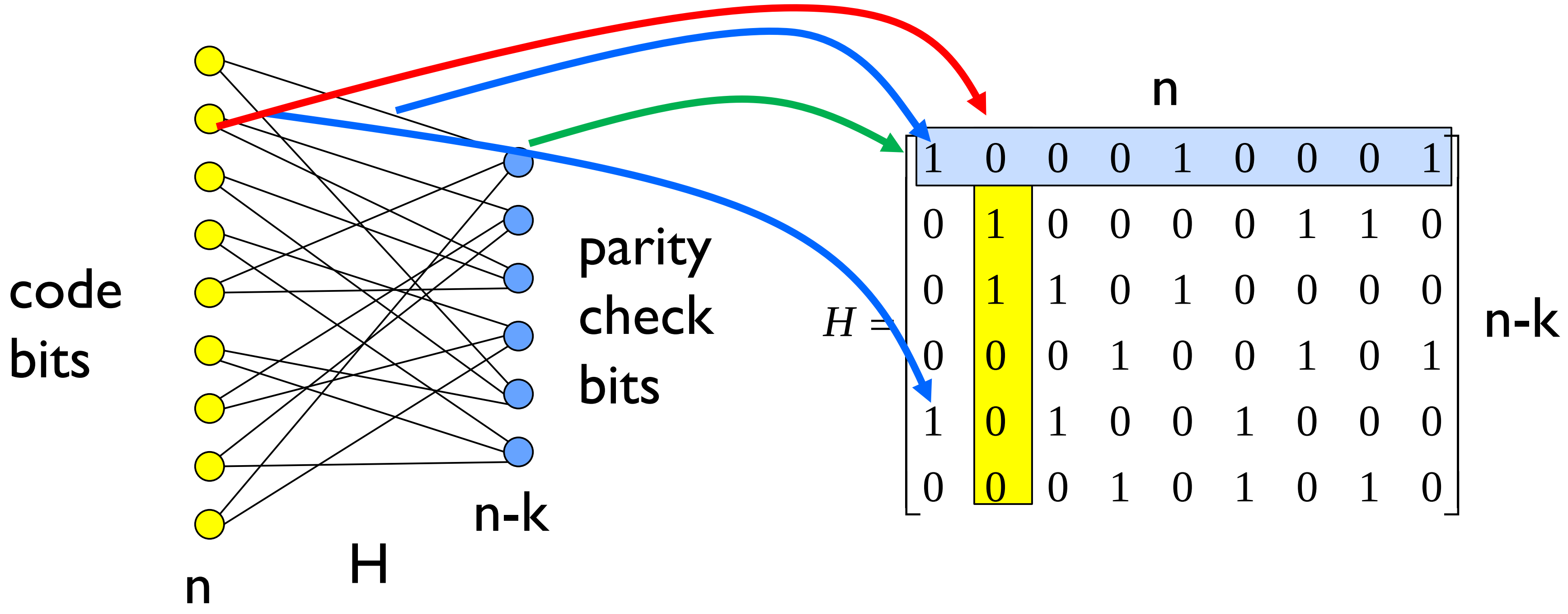
From expanders to codes

LDPC CODES

Low Density Parity Check (LDPC) Codes



Low Density Parity Check (LDPC) Codes



$$H = \begin{array}{c} n \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} \; n-k$$

Each row is a vertex on the right and each column is a vertex on the left.

A codeword on the left is valid if each right "parity check" vertex has parity 0.

The graph has O(n) edges (**low density**)

Applications in the real world

- 5G cellular technology (3GPP 5G NR, first release Dec. 2017)
- Solid State Drives (NAND Flash memory)
- NASA
  - Proposed for all their space data systems
- 10Gbase-T (IEEE 802.3an, 2006)
  - Standard for 10 Gbits/sec over copper wire
- WiMax (IEEE 802.16e, 2006)
  - Standard for medium-distance wireless. Approx 10Mbits/sec over 10 Kilometers.

History

Invented by Gallager in 1963 (his PhD thesis)

Generalized by Tanner in 1981 (instead of using parity and binary codes, use other codes for “check” nodes).

Mostly forgotten by community at large until the mid 90s when revisted by Spielman, MacKay and others.

Distance of LDPC codes

Consider a d-regular LPDC with (α,3d/4) expansion.

Theorem: Distance of code is greater than αn.

Proof. (by contradiction)

Linear code; distance= min weight of non-0 codeword.

Assume a codeword with weight w $\leq \alpha$n.

Let W be the set of 1 bits in codeword

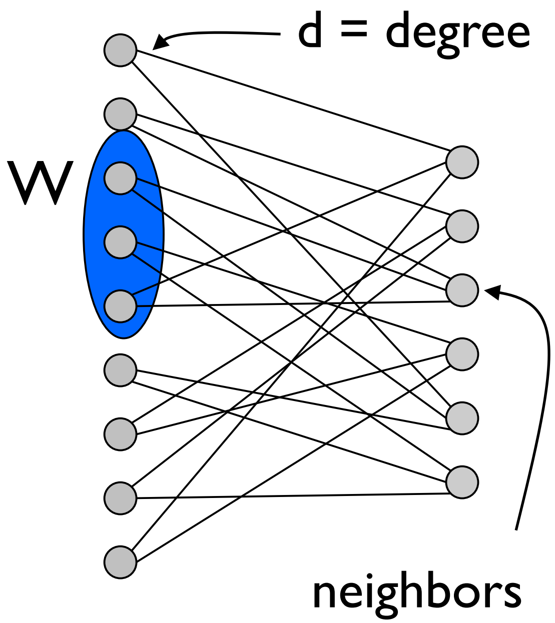
#edges = wd

#neighbors on right >3/4*wd

Max #neighbors with >1 edge from W?

#unique neighbors = wd/2

So at least one neighbor sees a single 1-bit. Parity check would fail!



Correcting Errors in LPDC codes

We say a vertex is *unsatisfied* if parity $\neq 0$

Algorithm:

While there are unsatisfied check bits

1. Find a bit on the left for which more than d/2 neighbors are unsatisfied
2. Flip that bit

Q: Converges?

Since every step reduces unsatisfied parity by at least 1.

Correcting Errors in LPDC codes

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Q: Running time?

Runs in linear time (for constant maximum degree on the right).

Why must there be a node with more than d/2 unsatisfied neighbors (if we're not at a codeword)?

Correcting Errors in LPDC codes

Theorem: Always exists a node > d/2 unsatisfied neighbors if we're not at a codeword.

Proof: (by contradiction)

Suppose not. (Let d be odd.) Let S be the corrupted bits.

Each such bit has majority of satisfied neighbors

(sat. neighbors see at least two corrupted bits on left)

(unsat. neighbors may see only one corrupted bit on left)

Each corrupt bit gives \$1 to each unsat nbr, \$½ to sat nbr.

Total money given < 3d/4 |S|.

Each node in N(S) collects \$1 at least.

Total money collected at least |N(S)|.

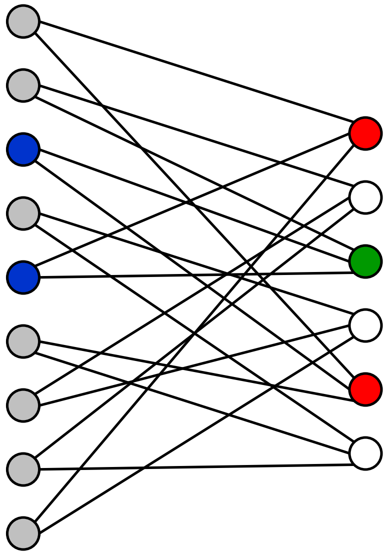
So |N(S)| < 3d/4 |S|.   Contradicts expansion.

Coverges to closest codeword

Theorem: Assume (α,3d/4) expansion. If # of error bits is less than αn/4 then simple decoding algorithm converges to closest codeword.

Proof: let:

- 🔴 u_i = # of unsatisfied check bits on step i
- 🔵 r_i = # corrupt code bits on step i
- 🟢 s_i = # satisfied check bits with corrupt neighbors on step i



Q: What do we have to show about r_i?

We know that u_i decrements on each step, but what about r_i?

Will be continued in the next lecture