Statistical Techniques in Robotics (16-831, F09) Lecture #18 (Thursday October 22)

Gauss Markov Filter

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Gauss Markov Filter

Consider $X_1, X_2, ...X_t, X_{t+1}$ to be the state variables and $Y_1, Y_2, ...Y_t, Y_{t+1}$ be the sequence of corresponding observations. As in Hidden Markov models, conditional independencies (see Figure 1) dictate that past and future states are decorrelated given the current state, X_t at time t. For example, if we know what X_2 is, then no information about X_1 can possibly help us to reason about what X_3 should be.

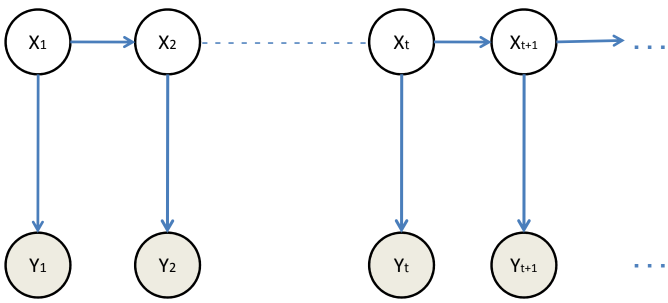

Figure 1: The Independence Diagram of a Gauss-Markov model

The update for state variable X_{t+1} is given by:

$$X_{t+1} = AX_t + \epsilon$$

where,

$$\epsilon \sim N(0, Q)$$

$$\Rightarrow X_{t+1}|X_t \sim N(AX_t, Q)$$

The corresponding observation Y_{t+1} is given by equation:

$$Y_{t+1} = CX_{t+1} + \delta$$

where,

$$\delta \sim N(0, R)$$

$$\Rightarrow Y_0 \sim N(\mu_0, \epsilon_0)$$

Lazy Gauss Markov Filter

Motion Model:

Before the observation is taken:

$$X_{t+1} \sim \mu^{-}_{t+1} = A\mu_t$$

Proof:

$$E[X_{t+1}] = E[AX_t + \epsilon]$$

$$\Rightarrow E[X_{t+1}] = E[AX_t] + E[\epsilon]$$

since variance of ϵ is 0,

$$\Rightarrow E[X_{t+1}] = AE[X_t] = A\mu_t$$

Variance,

$$\Sigma^{-}_{t+1} = E[X_{t+1} * {X_{t+1}}^T]$$

$$\Rightarrow \Sigma^{-}_{t+1} = E[(AX_t + \epsilon)(AX_t + \epsilon)^T]$$

$$= E[(AX_t)(AX_t)^T] + E[\epsilon_{terms}]$$

$$= AE[(X_t)(X_t)^T]A^T + E[\epsilon_{terms}]$$

$$\Rightarrow \Sigma^{-}_{t+1} = A\Sigma_t A^T + E[\epsilon_{terms}]$$

E$[\epsilon_{terms}]$ is equal to the variance of ϵ which is Q.
Therefore Motion Update becomes:

$$\mu^{-}_{t+1} = A\mu_t$$

$$\Sigma^{-}_{t+1} = A\Sigma_t A^T + Q$$

Observation Model:

For the observation model Natural parameterization is more suitable as it involves multiplication of terms. When, Y is the corresponding observation for state variable X, the model equation in terms of Natural Parameters J and P is given by,

$$e^{(J^{-T}X - \frac{1}{2}X^T PX)} * e^{-\frac{1}{2}(Y-CX)^T R^{-1}(Y-CX)}$$

$$\Rightarrow e^{-\frac{1}{2}[-2Y^T R^{-1}CX + X^T C^T R^{-1}CX + Y^T R^{-1}Y]}$$

The last term is a constant with respect to X, so it goes into the marginalization term.

$$\Rightarrow e^{-\frac{1}{2}[-2Y^T R^{-1}CX + X^T C^T R^{-1}CX]}$$

Therefore the Observation Update is:

$$J^+ = J^- + (Y^T R^{-1} C)^T$$
$$P^+ = P^- + C^{-1} R^{-1} C$$

This form is useful when there are large number of motion and observation updates.
Lazy Gauss Markov can be expressed in two forms:

- When expressed in terms of moment parameters μ and Σ acts as Kalman Filter.
- When expressed in terms of natural parameters J and P acts as Information Filter.

Observation Update in terms of moment parameters μ and Σ:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{X_t} \\ \mu_{Y_t} \end{pmatrix} \begin{pmatrix} \Sigma_{XX} & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{YY} \end{pmatrix} \right)$$

Observation Update:

$$\mu_{X|Y} = \mu_X + \Sigma_{XY} \Sigma_{YY}^{-1} (Y - \mu_Y) \leftarrow (1)$$
$$\Sigma_{X|Y} = \Sigma_{XX} - \Sigma_{XY} \Sigma_{YY}^{-1} \Sigma_{YX} \leftarrow (2)$$

$(Y - \mu_Y)$ is called the Innovation Term.

We know,

$$\mu_Y = C \mu_X$$
$$\Sigma_{YY} = R + C \Sigma_{XX} C^T$$

Therefore, we have to find out Σ_{XY} to calculate the remaining terms in equation 1 and 2.
By definition,

$$\Sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)^T]$$
$$= E[(X - \mu_X)(Y - C\mu_X)^T]$$

However, $Y = CX + \delta$ with δ having 0 mean and independent of all other observations.

$$\Sigma_{XY} = E[(X - \mu_X)(X - \mu_X)^T] C^T$$

$$\Rightarrow \Sigma_{XY} = \Sigma_{XX} C^T$$

Putting, these values in equations 1 and 2,

$$\mu_{X|Y} = \mu_X + \Sigma_{XX} C^T (R + C \Sigma_{XX} C^T)^{-1} (Y - C\mu_X)$$

$$\Sigma_{X|Y} = \Sigma_{XX} - \Sigma_{XX} C^T (R + C \Sigma_{XX} C^T)^{-1} C \Sigma_{XX}$$

KF to BLR transformation:

The corresponding parameters for converting the above KF form to BLR is given by:

$$BLR \begin{pmatrix} \Theta = X \\ \sigma^2 = R \\ 0 = Q \\ I = A \\ Y = Y \\ X = C_t \end{pmatrix} KF$$