Beyond Alhazen's Problem: Analytical Projection Model for Non-Central Catadioptric Cameras with Quadric Mirrors

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Supplementary Material: Derivation of Forward Projection Equation

Here we present the details of the derivation of Forward Projection Equation. We start with the rotated coordinate system in which $\texttt{COP}_R = [0, d_y, d_z]$, $\mathbf{P}_R = [u, v, w]$, and $\mathbf{M}_R = [x, y, z]$ as described in the paper.

The normal at $\mathbf{M}_R$ can be computed from the mirror equation to be

$$\mathbf{n} = [x, y, Az + B/2]^T. \tag{1}$$

Since the mirror is rotationally symmetric around the z axis, the normal intersects the z axis at point $\mathbf{K} = [0, 0, z - Az - B/2]^T$, which also lies on π. Thus, the equation of π can be obtained using the points $\mathbf{K}$, $\texttt{COP}_R$ and $\mathbf{P}_R$ and is given by

$$c_1(z)x + c_2(z)y + c_3(z) = 0, \tag{2}$$

where

$$\begin{aligned} c_1(z) &= (B + 2Az)(d_y - v) + 2d_y(w - z) + 2v(z - d_z), \\ c_2(z) &= u(B + 2d_z - 2z + 2Az), \\ c_3(z) &= ud_y(B + 2Az). \end{aligned} \tag{3}$$

Using this equation, we can easily compute x in terms of y and z as

$$\begin{aligned} x &= \frac{-c_2(z)y - c_3(z)}{c_1(z)} \\ &= u\frac{d_y(B + 2Az) - y(B + 2d_z - 2z + 2Az)}{d_y(B + 2w - 2z + 2Az) - v(B + 2d_z - 2z + 2Az)} \end{aligned}$$

Substituting x in the mirror equation gives us our first intermediate equation $\mathbf{IE}_1$

$$\mathbf{IE}_1 : (c_1^2(z) + c_2^2(z))y^2 + 2c_2(z)c_3(z)y + c_3^2(z) + c_1^2(z)(Az^2 + Bz - C) = 0. \tag{4}$$

Rewriting it as

$$\mathbf{IE}_1 : k_{41}(z)y^2 + k_{42}(z)y + k_{43}(z) = 0, \tag{5}$$

we have

$$\begin{aligned} k_{41}(z) &= c_1^2(z) + c_2^2(z) \\ &= 4A^2d_y^2z^2 - 8A^2d_yvz^2 + 4A^2u^2z^2 + 4A^2v^2z^2 + 4ABd_y^2z - 8ABd_yvz + 4ABu^2z + 4ABv^2z + 8Ad_y^2wz \\ &\quad -8Ad_y^2z^2 - 8Ad_yd_zvz - 8Ad_yvwz + 16Ad_yvz^2 + 8Ad_zu^2z + 8Ad_zv^2z - 8Au^2z^2 - 8Av^2z^2 + B^2d_y^2 \\ &\quad -2B^2d_yv + B^2u^2 + B^2v^2 + 4Bd_y^2w - 4Bd_y^2z - 4Bd_yd_zv - 4Bd_yvw + 8Bd_yvz + 4Bd_zu^2 + 4Bd_zv^2 \\ &\quad -4Bu^2z - 4Bv^2z + 4d_y^2w^2 - 8d_y^2wz + 4d_y^2z^2 - 8d_yd_zvw + 8d_yd_zvz + 8d_yvwz - 8d_yvz^2 + 4d_z^2u^2 \\ &\quad +4d_z^2v^2 - 8d_zu^2z - 8d_zv^2z + 4u^2z^2 + 4v^2z^2 \end{aligned}$$

$$
\begin{aligned}
k_{42}(z) &= 2c_2(z)c_3(z) \\
&= -2d_y u^2(B+2Az)(B+2d_z-2z+2Az)
\end{aligned}
$$

$$
\begin{aligned}
k_{43}(z) &= c_3^2(z) + c_1^2(z)(Az^2+Bz-C) \\
&= 4A^3d_y^2z^4 - 8A^3d_yvz^4 + 4A^3v^2z^4 + 8A^2Bd_y^2z^3 - 16A^2Bd_yvz^3 + 8A^2Bv^2z^3 \\
&\quad +4A^2d_y^2u^2z^2 + 8A^2d_y^2wz^3 - 8A^2d_y^2z^4 - 4CA^2d_y^2z^2 - 8A^2d_yd_zvz^3 - 8A^2d_yvwz^3 \\
&\quad +16A^2d_yvz^4 + 8CA^2d_yvz^2 + 8A^2d_zv^2z^3 - 8A^2v^2z^4 - 4CA^2v^2z^2 + 5AB^2d_y^2z^2 \\
&\quad -10AB^2d_yvz^2 + 5AB^2v^2z^2 + 4ABd_y^2u^2z + 12ABd_y^2wz^2 - 12ABd_y^2z^3 - 4CABd_y^2z \\
&\quad -12ABd_yd_zvz^2 - 12ABd_yvwz^2 + 24ABd_yvz^3 + 8CABd_yvz + 12ABd_zv^2z^2 - 12ABv^2z^3 \\
&\quad -4CABv^2z + 4Ad_y^2w^2z^2 - 8Ad_y^2wz^3 - 8CAd_y^2wz + 4Ad_y^2z^4 + 8CAd_y^2z^2 - 8Ad_yd_zvwz^2 \\
&\quad +8Ad_yd_zvz^3 + 8CAd_yd_zvz + 8Ad_yvwz^3 + 8CAd_yvwz - 8Ad_yvz^4 - 16CAd_yvz^2 + 4Ad_z^2v^2z^2 \\
&\quad -8Ad_zv^2z^3 - 8CAd_zv^2z + 4Av^2z^4 + 8CAv^2z^2 + B^3d_y^2z - 2B^3d_yvz + B^3v^2z + B^2d_y^2u^2 \\
&\quad +4B^2d_y^2wz - 4B^2d_y^2z^2 - CB^2d_y^2 - 4B^2d_yd_zvz - 4B^2d_yvwz + 8B^2d_yvz^2 + 2CB^2d_yv \\
&\quad +4B^2d_zv^2z - 4B^2v^2z^2 - CB^2v^2 + 4Bd_y^2w^2z - 8Bd_y^2wz^2 - 4CBd_y^2w + 4Bd_y^2z^3 \\
&\quad +4CBd_y^2z - 8Bd_yd_zvwz + 8Bd_yd_zvz^2 + 4CBd_yd_zv + 8Bd_yvwz^2 + 4CBd_yvw - 8Bd_yvz^3 \\
&\quad -8CBd_yvz + 4Bd_z^2v^2z - 8Bd_zv^2z^2 - 4CBd_zv^2 + 4Bv^2z^3 + 4CBv^2z - 4Cd_y^2w^2 \\
&\quad +8Cd_y^2wz - 4Cd_y^2z^2 + 8Cd_yd_zvw - 8Cd_yd_zvz - 8Cd_yvwz + 8Cd_yvz^2 - 4Cd_z^2v^2 + 8Cd_zv^2z - 4Cv^2z^2
\end{aligned}
$$

Obtaining $\mathbf{IE}_2$ using Law of Reflection

To obtain $\mathbf{IE}_2$, we use the constraint that the reflected ray $\mathbf{v}_r$ should pass through the given point $\mathbf{P}_R$, yielding,

$$\mathbf{v}_r \times (\mathbf{P}_R - \mathbf{M}_R) = 0, \tag{6}$$

where $\times$ denotes the cross product. The incoming ray $\mathbf{v}_i$ is given by

$$\mathbf{v}_i = \mathbf{M}_R - \mathbf{COP}_R = [x, y-d_y, z-d_z]^T. \tag{7}$$

The reflection equation is given by

$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{v}_i^T\mathbf{n})/(\mathbf{n}^T\mathbf{n}). \tag{8}$$

By substituting $\mathbf{v}_i$ and $\mathbf{n}$ in the reflection equation (8), we obtain the 3×1 vector $\mathbf{v}_r$. After simplification, its components are given by

$$
\begin{aligned}
\mathbf{v}_r[1] &= x(4A^2z^2 + 4ABz - 4Az^2 + 8d_zAz + B^2 + 4d_zB - 4C + 8d_yy) \\
\mathbf{v}_r[2] &= 4A^2yz^2 - 4d_yA^2z^2 + 4AByz - 4d_yABz - 4Ayz^2 + 8d_zAyz + 4d_yAz^2 \\
&\quad +B^2y - d_yB^2 + 4d_zBy + 4d_yBz + 8d_yy^2 - 4Cy - 4Cd_y \\
\mathbf{v}_r[3] &= 4Cz - 4Cd_z + B^2d_z - 4Az^3 - 4Bz^2 + 3B^2z + 4A^2z^3 - 4BC + 4A^2d_zz^2 \\
&\quad -8ACz + 4Bd_yy + 4Bd_zz + 8ABz^2 + 4Ad_zz^2 + 4ABd_zz + 8Ad_yyz
\end{aligned}
$$

Then $\mathbf{v}_r$ is substituted in (6) along with $\mathbf{P}_R$ and $\mathbf{M}_R$ to result in following three equations (since the cross product is in 3D):

$$
\begin{aligned}
&E_1 : k_{11}(z)x + k_{12}(z)y + k_{13}(z)xy + k_{14}(z)y^2 + k_{15}(z) = 0. \\
&E_2 : k_{21}(z)x + k_{22}(z)y + k_{23}(z)xy + k_{24}(z) = 0. \\
&E_3 : k_{31}(z)y^2 + k_{32}(z)y + k_{33}(z) = 0.
\end{aligned} \tag{9}
$$

The coefficients are given by

$$
\begin{aligned}
k_{11} &= 4Cd_y + 4Cv + B^2 d_y - B^2 v + 4A^2 d_y z^2 - 4A^2 v z^2 - 4Bd_z v - 4Bd_y z - 4Ad_y z^2 \\
&\quad + 4Avz^2 + 4ABd_y z - 4ABvz - 8Ad_z vz \\
k_{12} &= u(4A^2z^2 + 4ABz - 4Az^2 + 8d_z Az + B^2 + 4d_z B - 4C) \\
k_{13} &= -8d_y v \\
k_{14} &= 8d_y u \\
k_{15} &= -d_y u(4A^2z^2 + 4ABz - 4Az^2 + B^2 - 4Bz + 4C)
\end{aligned}
$$

$$
\begin{aligned}
k_{21} &= 4Cd_z + 4Cw - 8Cz - B^2 d_z - B^2 w + 4Bz^2 - 2B^2 z + 4BC - 4A^2 d_z z^2 \\
&\quad - 4A^2 wz^2 + 8ACz - 4Bd_z w - 4ABz^2 + 4Ad_z z^2 + 4Awz^2 - 4ABd_z z - 4ABwz - 8Ad_z wz \\
k_{22} &= 4d_y u(B + 2Az) \\
k_{23} &= -4d_y(B + 2w - 2z + 2Az) \\
k_{24} &= u(4A^2z^3 + 4d_z A^2 z^2 + 8ABz^2 + 4d_z ABz - 4Az^3 + 4d_z Az^2 - 8CAz \\
&\quad + 3B^2 z + d_z B^2 - 4Bz^2 + 4d_z Bz - 4CB + 4Cz - 4Cd_z)
\end{aligned}
$$

$$
\begin{aligned}
k_{31} &= -4d_y(B + 2w - 2z + 2Az) \\
k_{32} &= 4Cd_z + 4Cw - 8Cz - B^2 d_z - B^2 w + 4Bz^2 - 2B^2 z + 4BC - 4A^2 d_z z^2 \\
&\quad - 4A^2 wz^2 + 8ACz + 4Bd_y v - 4Bd_z w - 4ABz^2 + 4Ad_z z^2 + 4Awz^2 \\
&\quad - 4ABd_z z - 4ABwz + 8Ad_y vz - 8Ad_z wz \\
k_{33} &= 4A^2 vz^3 - 4A^2 d_y z^3 - 4BCv - 4Cd_z v + 4Cd_y w - 4Cd_y z + 4Cvz + B^2 d_z v \\
&\quad + B^2 d_y w + 4Ad_y z^3 + 4Bd_y z^2 - B^2 d_y z - 4Avz^3 - 4Bvz^2 + 3B^2 vz - 4ABd_y z^2 \\
&\quad + 8ABvz^2 + 4Ad_z vz^2 - 4Ad_y wz^2 + 4A^2 d_z vz^2 + 4A^2 d_y wz^2 - 8ACvz \\
&\quad + 4Bd_z vz - 4Bd_y wz + 4ABd_z vz + 4ABd_y wz
\end{aligned}
$$

The desired forward projection equation is given by (as derived in the paper)

$$
\begin{aligned}
\textbf{FP:}\quad & k_{41}(z)\left(k_{43}(z)k_{32}^2(z) - k_{42}(z)k_{32}(z)k_{33}(z) + k_{41}(z)k_{33}^2(z)\right) \\
& - k_{31}(z)(-k_{33}(z)k_{42}^2(z) + k_{43}(z)k_{32}(z)k_{42}(z) + 2k_{41}(z)k_{43}(z)k_{33}(z)) + k_{43}^2(z)k_{31}^2(z) = 0,
\end{aligned} \tag{10}
$$

where all $k_{ij}(z)$ are defined above in terms of known quantities.