A Theory of Multi-Layer Flat Refractive Geometry*
Supplementary Materials

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For refraction between i^{th} and $(i+1)^{th}$ layer, we use the following vector form as shown in the paper

$$\mathbf{v}_{i+1} = a_{i+1}\mathbf{v}_i + b_{i+1}\mathbf{n}, \tag{1}$$

where $a_{i+1} = \mu_i/\mu_{i+1}$ and

$$b_{i+1} = \frac{-\mu_i \mathbf{v}_i^T\mathbf{n} - \sqrt{\mu_i^2(\mathbf{v}_i^T\mathbf{n})^2 - (\mu_i^2 - \mu_{i+1}^2)\mathbf{v}_i^T\mathbf{v}_i}}{\mu_{i+1}}. \tag{2}$$

This equation also ensures that $\mathbf{v}_{i+1}^T\mathbf{v}_{i+1} = \mathbf{v}_i^T\mathbf{v}_i$. In addition, since Snell's law only depends on the ratio of the refractive indices, we assume $\mu_0 = 1$ without loss of generality.

1. Unknown Refractive Indices

In this section, we describe in detail the analytical solutions to compute the layer thicknesses and translation along the axis when refractive indices are unknown. As shown in the paper, the axis can be computed independently of the layer thicknesses and refractive indices. We assume that axis A, rotation R and translation orthogonal to the axis, $t_A^{\perp}$, has been computed as described in Section 3 of the paper. Our goal is to compute the translation t_A along the axis, layer thicknesses and refractive indices, using the given 2D-3D correspondences. Let $t_A = \alpha A$, where α is the unknown translation magnitude along the axis.

We first apply the computed R and $t_{A^\perp}$ to the 3D points $\mathbf{P}$. Let $\mathbf{P}_c = R\mathbf{P} + t_{A^\perp}$. The plane of refraction is obtained by the estimated axis A and the given camera ray $\mathbf{v}_0$. Let $[\mathbf{z}_2, \mathbf{z}_1]$ denote an orthogonal coordinate system on the plane of refraction (POR). We choose $\mathbf{z}_1$ along the axis. For a given camera ray $\mathbf{v}_0$, let $\mathbf{z}_2 = \mathbf{z}_1 \times (\mathbf{z}_1 \times \mathbf{v}_0)$ be the orthogonal direction.

The projection of $\mathbf{P}_c$ on POR is given by $\mathbf{u} = [u^x, u^y]$, where $u^x = \mathbf{z}_2^T P_c$ and $u^y = \mathbf{z}_1^T P_c$. Similarly, each ray $\mathbf{v}_i$ on the light-path of $\mathbf{v}_0$ can be represented by a 2D vector $\mathbf{vp}_i$ on POR, whose components are given by $\mathbf{z}_2^T\mathbf{v}_i$ and $\mathbf{z}_1^T\mathbf{v}_i$. Let $\mathbf{z}_p = [0; 1]$ be a unit 2D vector and $c_i = \mathbf{vp}_i^T\mathbf{zp}$. On the plane of refraction, the normal $\mathbf{n}$ of the refracting layers is given by $\mathbf{n} = [0; -1]$.

1.1. Case 1: Single Refraction

In this case, we have three unknowns d_0, μ_1 and α. When $\mu_i's$ are unknown, ray directions cannot be pre-computed and flat refraction constraint needs to be written in terms of camera rays. For Case 1, the flat refraction constraint is given by

$$\begin{aligned} 0 &= \mathbf{vp}_1 \times (\mathbf{u} + \alpha\mathbf{z}_p - \mathbf{q}_1) & (3)\\ &= \mathbf{vp}_1 \times (\mathbf{u} + \alpha\mathbf{z}_p + d_0\mathbf{vp}_0/c_0). & (4) \end{aligned}$$

$\mathbf{vp}_1$ is given by

$$\mathbf{vp}_1 = a_1\mathbf{vp}_0 + b_1\mathbf{n}, \tag{5}$$

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where $a_1 = 1/\mu_1$. Since the camera ray $\mathbf{vp}_0$ is known, we can normalize it. Let $\mathbf{vp}_0 = [v^x; v^y]$. From (2),

$$b_1 = \frac{v^y - \sqrt{{\mu_1}^2 + (v^y)^2 - 1}}{\mu_1} \tag{6}$$

Using a_1 and b_1, $\mathbf{vp}_1$ can be obtained. Substituting $\mathbf{vp}_1$ and $\mathbf{vp}_0$ in the FRC equation (4)

$$(d_0 v^x - v^y u^x)\sqrt{\mu_1^2 + (v^y)^2 - 1} + v^x v^y(\alpha - d_0 - u^y) = 0 \tag{7}$$

Removing the square root term, we get

$$(d_0 v^x - v^y u^x)(\gamma + (v^y)^2 - 1) = (v^x v^y(\alpha - d_0 - u^y))^2, \tag{8}$$

where $\gamma = \mu_1^2$. γ can be obtained as a function of d_0 and α.

$$\gamma = \frac{(v^x v^y(\alpha - d_0 - u^y))^2}{(d_0 v^x - v^y u^x)} - (v^y)^2 + 1 \tag{9}$$

Let $[EQ_i]_{i=1}^3$ be the 3 equations for 3 correspondences. Using EQ_1, γ can be obtained as a function of d_0 and α as above. Substituting γ in EQ_2 and EQ_3 makes them cubic in d_0 and quadratic in α. We get the following form for EQ_2 and EQ_3

$$EQ_2: \quad k_{11}\alpha^2(k_{12}d_0^2 + k_{13}d_0 + k_{14}) + k_{15}\alpha(k_{16}d_0^3 + k_{17}d_0^2 + k_{18}d_0 + k_{19}) + (k_{31}d_0^3 + k_{32}d_0^2 + k_{33}d_0 + k_{34}) = 0 \tag{10}$$
$$EQ_3: \quad k_{21}\alpha^2(k_{22}d_0^2 + k_{23}d_0 + k_{24}) + k_{25}\alpha(k_{26}d_0^3 + k_{27}d_0^2 + k_{28}d_0 + k_{29}) + (k_{41}d_0^3 + k_{42}d_0^2 + k_{43}d_0 + k_{44}) = 0 \tag{11}$$

where k_{ij} depend on known quantities. α^2 can be eliminated between EQ_2 and EQ_3 by

$$EQ_2 = k_{21}k_{22}EQ_2 - k_{11}k_{12}EQ_3. \tag{12}$$

The resulting EQ_2 is linear in α and cubic in d_0, using which α can be obtained as a cubic function of d_0. Substituting α in EQ_3 and simplifying results in a 6^{th} degree equation in single unknown d_0. Matlab code is provided which gives this equation.

1.2. Case 2: Two Refractions, $\mu_2 = \mu_0$

In this case, we have four unknowns d_0, d_1, μ_1 and α. However, as shown in the paper, d_0 cannot be estimated. The resulting FRC turns out to be independent of d_0. For Case 2, the flat refraction constraint is given by

$$0 = \mathbf{vp}_0 \times (\mathbf{u} + \alpha\mathbf{z}_p - \mathbf{q}_2) \tag{13}$$

since $\mathbf{vp}_2$ is parallel to $\mathbf{vp}_0$. The refraction point $\mathbf{q}_2$ is given by

$$\mathbf{q}_2 \quad = \quad \mathbf{q}_1 - d_1\mathbf{vp}_1/(\mathbf{vp}_1^T\mathbf{n}). \tag{14}$$

$\mathbf{vp}_1$ is given by

$$\mathbf{vp}_1 = a_1\mathbf{vp}_0 + b_1\mathbf{n}, \tag{15}$$

where $a_1 = 1/\mu_1$. Since the camera ray $\mathbf{vp}_0$ is known, we can normalize it. Let $\mathbf{vp}_0 = [v^x; v^y]$. From (2),

$$b_1 = \frac{v^y - \sqrt{{\mu_1}^2 + (v^y)^2 - 1}}{\mu_1} \tag{16}$$

Using a_1 and b_1, $\mathbf{vp}_1$ and $\mathbf{q}_2$ can be obtained. Substituting $\mathbf{vp}_1$ and $\mathbf{vp}_0$ in the FRC equation (13)

$$(d_1 v^x - \alpha v^x + v^y u^x - v^x u^y)\sqrt{\mu_1^2 + (v^y)^2 - 1} + d_1 v^x v^y = 0 \tag{17}$$

Removing the square root term, we get

$$(d_1 v^x - \alpha v^x + v^y u^x - v^x u^y)^2(\gamma + (v^y)^2 - 1) = (d_1 v^x v^y)^2 \tag{18}$$

where $\gamma = \mu_1^2$. Once again, γ can be obtained as a function of d_1 and α.

$$\gamma = \frac{(d_1 v^x v^y)^2}{(d_1 v^x - \alpha v^x + v^y u^x - v^x u^y)^2} - (v^y)^2 + 1 \tag{19}$$

Similar to Case 1, let $[EQ_i]_{i=1}^3$ be the 3 equations for 3 correspondences. Using EQ_1, γ can be obtained as a function of d_1 and α as above. Substituting γ in EQ_2 and EQ_3 makes them cubic in d_1 and fourth degree in α. We found it difficult to solve in Matlab, due to large number of terms. Therefore, we used an automatic generator of Grobner basis solver [4] to obtain the final equation. It results in a 6^{th} degree equation.

Note that if we don't make the substitution of $\gamma = \mu_1^2$, then the automatic solver will result a 12^{th} degree equation instead of a 6^{th} degree equation. Thus, it is important to do correct parametrization by carefully analyzing the equations.

1.3. Case 3: Two Refractions, $\mu_2 \neq \mu_0$

In this case, we have five unknowns d_0, d_1, μ_1, μ_2 and α. However, this case is extremely difficult to solve and we were unable to get an analytical equation. As shown, in this case the FRC will result in an equation in above five unknowns, with fourth degree terms of each unknown. Thus, it is clear that more than two layers or multi-layer systems are quite difficult to solve for analytically and require a good initial guess for non-linear refinement, when refractive indices are unknown.

For Case 3, the flat refraction constraint is given by

$$0 = \mathbf{vp}_2 \times (\mathbf{u} + \alpha \mathbf{z}_p - \mathbf{q}_2) \tag{20}$$

since $\mathbf{vp}_2$ is *not* parallel to $\mathbf{vp}_0$. $\mathbf{vp}_2$ is given by

$$\mathbf{vp}_2 = a_2 \mathbf{vp}_1 + b_2 \mathbf{n} = a_2 a_1 \mathbf{vp}_0 + (a_2 b_1 + b_2)\mathbf{n} \tag{21}$$

where $a_2 = \mu_1/\mu_2$ and

$$b_2 = -\frac{\sqrt{{\mu_1}^2\left(\frac{\mathrm{v^y}}{\mu_1} - \frac{\mathrm{v^y} - \sqrt{{\mu_1}^2 + {\mathrm{v^y}}^2 - 1}}{\mu_1}\right)^2 - {\mu_1}^2 + {\mu_2}^2} - \mu_1\left(\frac{\mathrm{v^y}}{\mu_1} - \frac{\mathrm{v^y} - \sqrt{{\mu_1}^2 + {\mathrm{v^y}}^2 - 1}}{\mu_1}\right)}{\mu_2} \tag{22}$$

Using a_1, b_1, a_2, b_2, we can obtain $\mathbf{vp}_2$ and $\mathbf{q}_2$. Substituting in FRC equation (20), we get

$$k_1\sqrt{D_1} + k_2\sqrt{D_1 D_2} + k_3\sqrt{D_2} = 0, \tag{23}$$

where

$$\begin{aligned}
k_1 &= v^x v^y (d_0 - \alpha + d_1 - u^y) && (24)\\
k_2 &= u^x v^y - d_0 v^x && (25)\\
k_3 &= -d_1 v^x v^y && (26)\\
D_1 &= \mu_1^2 + (v^y)^2 - 1 && (27)\\
D_2 &= \mu_2^2 + (v^y)^2 - 1 && (28)\\
& && (29)
\end{aligned}$$

Removing the square root terms, we get

$$(k_1^2 D_1 + k_3^2 D_2 - k_2^2 D_1 D_2)^2 - 4k_1^2 k_3^2 D_1 D_2 = 0 \tag{30}$$

which is a fourth degree equation in five unknowns d_0, d_1, μ_1 and α. The above equation has up to fourth degree terms of each of the unknowns d_0, d_1, μ_1 and α. We were not able to get a polynomial equation in a single unknown using 5 correspondences.

2. Calibration using a Planar Grid

Now we describe in detail the 8pt algorithm for calibration using a planar grid (Section 6.1 of the paper). Starting from the coplanarity constraints we have

$$0 = \mathbf{v}_0^T(\mathbf{A} \times (R\mathbf{P} + t)) = \mathbf{v}_0^T E\mathbf{P} + \mathbf{v}_0^T\mathbf{s}, \tag{31}$$

where $E = [A]_\times R$ and $\mathbf{s} = \mathbf{A} \times t$. Stacking equations for 8 correspondences, we get a linear system

$$\underbrace{\begin{bmatrix} (\mathbf{P}(1)^T \otimes \mathbf{v}_0(1)^T) & \mathbf{v}_0(1)^T \\ \vdots & \vdots \\ (\mathbf{P}(8)^T \otimes \mathbf{v}_0(8)^T) & \mathbf{v}_0(8)^T \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} E(:) \\ \mathbf{s} \end{bmatrix} = 0, \tag{32}$$

where B is a 8×12 matrix.

For plane based calibration, assume that the plane is aligned with xy plane ($\mathbf{P}^z(i) = 0$). Substituting in above, the columns 7, 8, 9 of B matrix reduce to zero. Let B' be the reduced 8×9 matrix, whose rank is 8. Thus, we can directly estimate the first two columns of the E matrix and $\mathbf{s}$ by SVD based solution using 8 correspondences.

Let $E = \begin{bmatrix} e_1 & e_4 & x \\ e_2 & e_5 & y \\ e_3 & e_6 & z \end{bmatrix}$, where e_i's are estimated as above and x, y, z are unknown. The last column of E is recovered using Demazure constraints [2, 1] and $det(E) = 0$ constraint.

The constraint $det(E) = 0$ gives

$$xe_2e_6 - xe_3e_5 - ye_1e_6 + ye_3e_4 + ze_1e_5 - ze_2e_4 = 0. \tag{33}$$

Using this, x can be obtained in terms of y and z as

$$x = (e_1e_6y - e_3e_4y - e_1e_5z + e_2e_4z)/(e_2e_6 - e_3e_5) \tag{34}$$

Let

$$K = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + x^2 + y^2 + z^2 \tag{35}$$

The Demazure constraints [2, 1] give following nine equations

$$x\,\left(2\,e_1{}^2 + 2\,e_4{}^2 + 2\,x^2\right) - x\,K + y\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + z\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) = 0 \tag{36}$$
$$y\,\left(2\,e_2{}^2 + 2\,e_5{}^2 + 2\,y^2\right) - y\,K + x\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + z\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{37}$$
$$z\,\left(2\,e_3{}^2 + 2\,e_6{}^2 + 2\,z^2\right) - z\,K + x\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) + y\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{38}$$
$$e_1\,\left(2\,e_1{}^2 + 2\,e_4{}^2 + 2\,x^2\right) - e_1K + e_2\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + e_3\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) = 0 \tag{39}$$
$$e_4\,\left(2\,e_1{}^2 + 2\,e_4{}^2 + 2\,x^2\right) - e_4\,K + e_5\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + e_6\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) = 0 \tag{40}$$
$$e_2\,\left(2\,e_2{}^2 + 2\,e_5{}^2 + 2\,y^2\right) - e_2\,K + e_1\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + e_3\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{41}$$
$$e_5\,\left(2\,e_2{}^2 + 2\,e_5{}^2 + 2\,y^2\right) - e_5\,K + e_4\,\left(2\,e_1\,e_2 + 2\,e_4\,e_5 + 2\,x\,y\right) + e_6\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{42}$$
$$e_3\,\left(2\,e_3{}^2 + 2\,e_6{}^2 + 2\,z^2\right) - e_3\,K + e_1\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) + e_2\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{43}$$
$$e_6\,\left(2\,e_3{}^2 + 2\,e_6{}^2 + 2\,z^2\right) - e_6\,K + e_4\,\left(2\,e_1\,e_3 + 2\,e_4\,e_6 + 2\,x\,z\right) + e_5\,\left(2\,e_2\,e_3 + 2\,e_5\,e_6 + 2\,y\,z\right) = 0 \tag{44}$$

Note that the first three equations have cubic terms of x, y, z, while the next six equations have quadratic terms. We can choose any two of these six quadratic equations. Lets choose the first two of the six quadratic equations and denote them as EQ_2 and EQ_3. Substituting x from above we get two equations of the following form

$$EQ_2: \; k_{11}y^2 + k_{12}yz + k_{13}z^2 + k_{14} = 0 \tag{45}$$
$$EQ_3: \; k_{21}y^2 + k_{22}yz + k_{23}z^2 + k_{24} = 0, \tag{46}$$

where k_{ij} depend on e_i's and are known coefficients. We can eliminate y^2 from the above two equations to get y in terms of z

$$y = \frac{k_{21}(k_{13}z^2 + k_{14}) - k_{11}(k_{23}z^2 + k_{24})}{k_{11}k_{22}z - k_{12}k_{21}z} \tag{47}$$

Substituting y back into EQ_3 gives a fourth degree equation in z

$$g_1 z^4 + g_2 z^2 + g_3 = 0, \tag{48}$$

where

$$\begin{aligned}
g_1 &= k_{11}(k_{11}^2 k_{23}^2 - k_{11}k_{12}k_{22}k_{23} - 2k_{11}k_{13}k_{21}k_{23} + k_{11}k_{13}k_{22}^2 + k_{12}^2 k_{21}k_{23} - k_{12}k_{13}k_{21}k_{22} + k_{13}^2 k_{21}^2) \\
g_2 &= k_{11}(k_{11}k_{14}k_{22}^2 + 2k_{13}k_{14}k_{21}^2 + k_{12}^2 k_{21}k_{24} + 2k_{11}^2 k_{23}k_{24} - k_{11}k_{12}k_{22}k_{24} - \qquad (49) \\
&\quad 2k_{11}k_{13}k_{21}k_{24} - 2k_{11}k_{14}k_{21}k_{23} - k_{12}k_{14}k_{21}k_{22}) \qquad (50) \\
g_3 &= k_{11}(k_{11}k_{24} - k_{14}k_{21})^2 \qquad (51)
\end{aligned}$$

Note that since the above equation has only z^4 and z^2 terms, we can substitute $\gamma = z^2$ and get a quadratic equation in γ. In general, there are two real solutions and two imaginary solutions, where the real solutions differ in sign.

3. Derivation of Forward Projection Equation for Case 1, Case 2 and Case 3

In this section, we present the details of the derivation of Analytical Forward Projection (AFP) Equation for Case 1, Case 2 and Case 3. Matlab code to derive the equations for all three cases is included in the supplementary materials.

As explained in the paper, given a *calibrated* central or non-central camera, the AFP describes an analytical method to compute the projection (or the corresponding camera ray) of a known 3D point. AFP can be used to minimize the image reprojection error. For Case 1, AFP is a 4^{th} degree equation [3]. We derive the AFP equation for two refractions and show that it is a 4^{th} degree equation for Case 2 and a 12^{th} degree equation for Case 3. The analysis can be done on the plane of refraction.

3.1. Coordinate Transformations

We use the following coordinate transformation to do the analysis on the plane of refraction itself. To derive the AFP equation, we are given the calibration parameters axis A, layer thicknesses d_i's and refractive indices μ_i's, and the known 3D point P. The goal is to find the 2D projection (or the corresponding camera ray $\mathbf{vp}_0$) of the 3D point P. The plane of refraction (POR) can be defined by the 3D point P and the axis A, with the camera at the origin of the coordinate system.

Let $[\mathbf{z}_2, \mathbf{z}_1]$ denote an orthogonal coordinate system on the POR. We choose $\mathbf{z}_1$ along the axis. Let $\mathbf{z}_2 = \mathbf{z}_1 \times (\mathbf{z}_1 \times P)$ be the orthogonal direction. The projection of $\mathbf{P}$ on POR is given by $\mathbf{u} = [u^x, u^y]$, where $u^x = \mathbf{z}_2^T\mathbf{P}$ and $u^y = \mathbf{z}_1^T\mathbf{P}$. Note that $\mathbf{z}_1$ and $\mathbf{z}_2$ are 3×1 vectors that define the coordinate system on POR.

The unknown camera ray on the plane of refraction $\mathbf{vp}_0$ can be parameterized as $[x, d_0]$, where x is unknown. On the plane of refraction, the normal $\mathbf{n}$ of the refracting layers is given by $\mathbf{n} = [0; -1]$.

3.2. Case 1: Single Refraction

Figure 1 depicts Case 1. Let $\mathbf{q}_1 = [x, d_0]$ be the point on the refractive medium where refraction happens. The forward projection equation is given by

$$\mathbf{vp}_1 \times (\mathbf{u} - \mathbf{q}_1) = 0, \tag{52}$$

where $\mathbf{vp}_1$ is the refracted ray. This is because $\mathbf{vp}_1$ should be parallel to the line joining $\mathbf{u}$ and $\mathbf{q}_1$. $\mathbf{vp}_1$ is given by

$$\mathbf{vp}_1 = a_1\mathbf{vp}_0 + b_1\mathbf{n} \tag{53}$$

where $a_1 = 1/\mu_1$ and

$$\begin{aligned}
b_1 &= (-\mathbf{vp}_0^T\mathbf{n} - \sqrt{(\mathbf{vp}_0^T\mathbf{n})^2 - (1-\mu_1^2)\mathbf{vp}_0^T\mathbf{vp}_0})/(\mu_1) \qquad (54) \\
&= (d_0 - \sqrt{d_0^2 - (1-\mu_1^2)(x^2+d_0^2)})/\mu_1. \qquad (55)
\end{aligned}$$

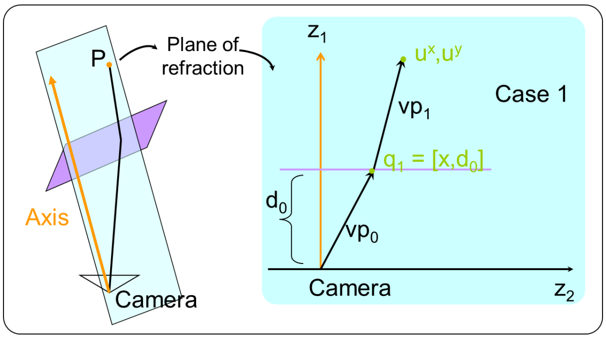

Figure 1. Case 1. The forward projection equation can be derived on the plane of refraction containing the given 3D point P and the axis.

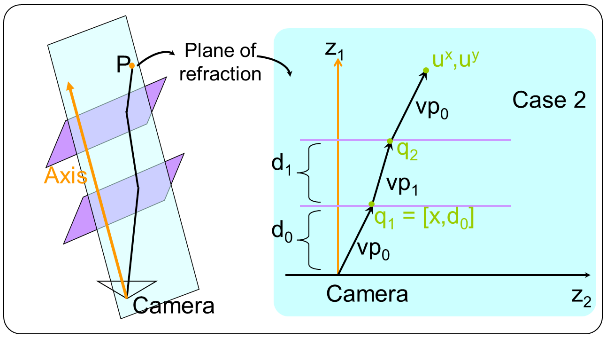

Figure 2. Case 2. Since $\mu_2 = \mu_0$, the final refracted ray $\mathbf{vp}_2$ is parallel to camera ray $\mathbf{vp}_0$.

Using a_1 and b_1 we can obtain $\mathbf{vp}_1$. Substituting in (52), we get

$$u^y x - d_0 x - u^x \sqrt{d_0^2 \mu^2 + {\mu_1}^2 x^2 - x^2} + x\sqrt{d_0^2 {\mu_1}^2 + {\mu_1}^2 x^2 - x^2} = 0 \tag{56}$$

Taking square root terms on one side and squaring, the following AFP equation is obtained.

$$(u^x - x)^2 (d_0^2 \mu_1^2 + \mu_1^2 x^2 - x^2) - (d_0 x - u^y x)^2 = 0 \tag{57}$$

The AFP equation for Case 1 is 4^{th} degree. After solving the AFP equation, we get four solutions. The correct solution is found by removing imaginary solutions and checking Snell's law for each real solution. Once x is found, the camera ray is given by $x\mathbf{z}_2 + d_0\mathbf{z}_1$.

3.3. Case 2: Two Refractions $\mu_2 = \mu_0$

Case 2 can be analyzed in a similar manner to Case 1 as shown in Figure 2. In this case, since $\mu_2 = \mu_0$, $\mathbf{vp}_2$ will be parallel to $\mathbf{vp}_0$. The forward projection equation is given by

$$\mathbf{vp}_0 \times (\mathbf{u} - \mathbf{q}_2) = 0, \tag{58}$$

The refraction point $\mathbf{q}_2$ is given by

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{q}_1 - d_1 \mathbf{vp}_1 / (\mathbf{vp}_1^T \mathbf{n}) && (59) \\ &= [x; d_0] - d_1 \mathbf{vp}_1 / (\mathbf{vp}_1^T \mathbf{n}), && (60) \end{aligned}$$

where $\mathbf{vp}_1$ is given as in (53). After substituting for $\mathbf{vp}_1$ and $\mathbf{q}_2$ using a_1 and b_1 in (58), we get

$$(d_0 u^x + d_1 x - u^y x)\sqrt{d_0^2 \mu_1^2 + \mu_1^2 x^2 - x^2} - d_0 d_1 x = 0 \tag{61}$$

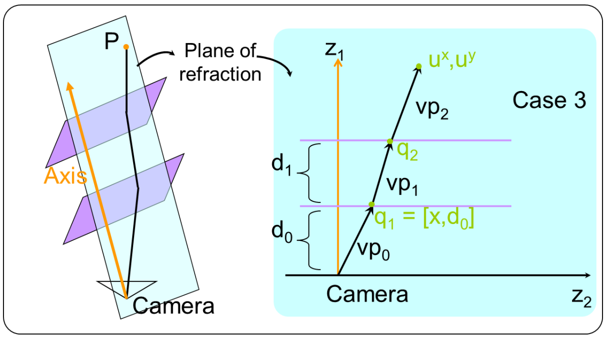

Figure 3. Case 3. Since $\mu_2 \neq \mu_0$, the final refracted ray $\mathbf{vp}_2$ is not parallel to camera ray $\mathbf{vp}_0$.

Squaring, we get the AFP equation

$$(d_0 u^x + d_1 x - u^y x)^2 (d_0^2 \mu_1^2 + \mu_1^2 x^2 - x^2) = (d_0 d_1 x)^2 \tag{62}$$

which is a 4^{th} degree equation in x. After solving the AFP equation, we get four solutions. The correct solution is found by removing imaginary solutions and checking Snell's law for each real solution. Once x is found, the camera ray is given by $x\mathbf{z}_2 + d_0\mathbf{z}_1$.

3.4. Case 3: Two Refractions $\mu_2 \neq \mu_0$

Now we consider Case 3 as shown in Figure 3. In this case, since $\mu_2 \neq \mu_0$, $\mathbf{vp}_2$ will not be parallel to $\mathbf{vp}_0$. The forward projection equation is given by

$$\mathbf{vp}_2 \times (\mathbf{u} - \mathbf{q}_2) = 0, \tag{63}$$

The refraction point $\mathbf{q}_2$ is same as in Case 2. However, the final refracted ray $\mathbf{vp}_2$ is given by

$$\begin{aligned} \mathbf{vp}_2 &= a_2\mathbf{vp}_1 + b_2\mathbf{n} & (64)\\ &= a_2(a_1\mathbf{vp}_0 + b_1\mathbf{n}) + b_2\mathbf{n} & (65)\\ &= a_2 a_1\mathbf{vp}_0 + (a_2 b_1 + b_2)\mathbf{n}. & (66) \end{aligned}$$

We have $a_1 = 1/\mu_1$ and $a_2 = \mu_1/\mu_2$. b_1 is given as in (55)

$$b_1 = (d_0 - \sqrt{d_0^2 - (1-\mu_1^2)(x^2 + d_0^2)})/\mu_1. \tag{67}$$

Similarly, b_2 is given by

$$\begin{aligned} b_2 &= (-\mu_1\mathbf{vp}_1^T\mathbf{n} - \sqrt{\mu_1^2(\mathbf{vp}_1^T\mathbf{n})^2 - (\mu_1^2 - \mu_2^2)\mathbf{vp}_1^T\mathbf{vp}_1})/(\mu_2) & (68)\\ &= (-\mu_1\mathbf{vp}_1^T\mathbf{n} - \sqrt{\mu_1^2(\mathbf{vp}_1^T\mathbf{n})^2 - (\mu_1^2 - \mu_2^2)\mathbf{vp}_0^T\mathbf{vp}_0})/(\mu_2) & (69)\\ &= \frac{\sqrt{(\mu_1{}^2 - 1)\left(d_0{}^2 + x^2\right) + d_0{}^2} - \sqrt{(\mu_1{}^2 - 1)\left(d_0{}^2 + x^2\right) - (\mu_1{}^2 - \mu_2{}^2)\left(d_0{}^2 + x^2\right) + d_0{}^2}}{\mu_2} & (70) \end{aligned}$$

Using a_1, a_2, b_1, b_2, we can obtain $\mathbf{vp}_2$ and $\mathbf{q}_2$. Substituting in AFP equation 63, we get

$$k_1\sqrt{D_1} + k_2\sqrt{D_1 D_2} + k_3\sqrt{D_2} = 0. \tag{71}$$

where

$$
\begin{aligned}
k_1 &= x(d_0 + d_1 - u^y) && (72)\\
k_2 &= (u^x - x) && (73)\\
k_3 &= -d_1 x && (74)\\
D_1 &= d_0^2\mu_1^2 + \mu_1^2 x^2 - x^2 && (75)\\
D_2 &= d_0^2\mu_2^2 + \mu_2^2 x^2 - x^2 && (76)
\end{aligned}
$$

Removing the square root terms, we get

$$(k_1^2 D_1 + k_3^2 D_2 - k_2^2 D_1 D_2)^2 - 4k_1^2 k_3^2 D_1 D_2 = 0. \tag{77}$$

After substituting for k_1, k_2, k_3, D_1 and D_2, we get a 12^{th} degree equation in x. After solving the AFP equation, we get twelve solutions. The correct solution is found by removing imaginary solutions and checking Snell's law for each real solution. Once x is found, the camera ray is given by $x\mathbf{z}_2 + d_0\mathbf{z}_1$.

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