Support Vector Machines - Dual formulation

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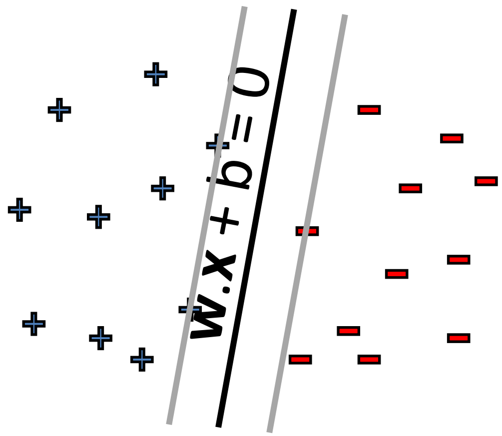
SVM – linearly separable case

n training points $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$

d features $\mathbf{x}_j$ is a d-dimensional vector



- Primal problem: $\text{minimize}_{\mathbf{w},b} \quad \frac{1}{2}\mathbf{w}.\mathbf{w}$

$$\left(\mathbf{w}.\mathbf{x}_j + b\right) y_j \geq 1, \ \forall j$$

w – weights on features (d-dim problem)

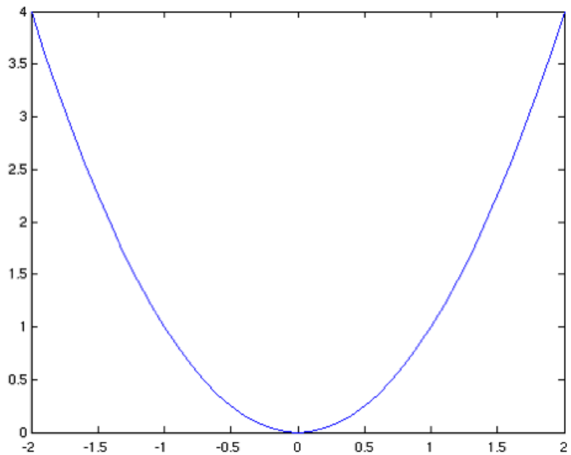
- Convex quadratic program – quadratic objective, linear constraints
- But expensive to solve if d is very large
- Often solved in dual form (n-dim problem)

Constrained Optimization

$$\min_x \ x^2 \quad \text{s.t.} \quad x \geq b$$

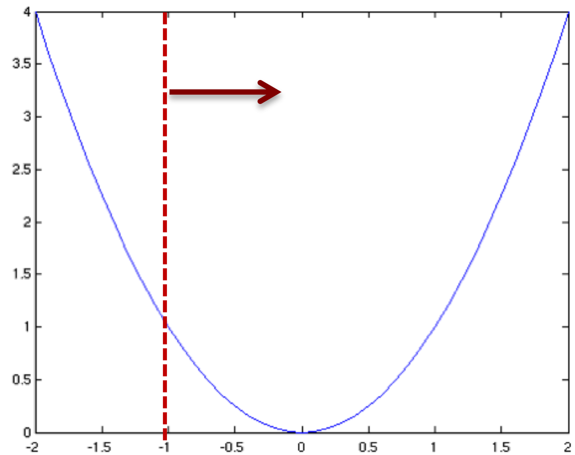
$$x^* = \max(b, 0)$$

$\min_x \ x^2$



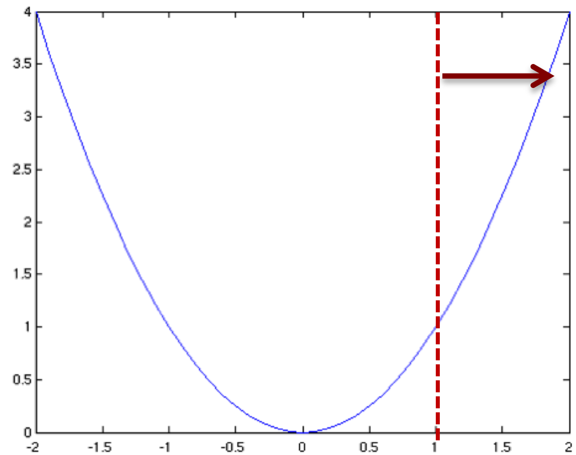
$x^* = 0$

$\min_x \ x^2$
s.t. $x \geq -1$



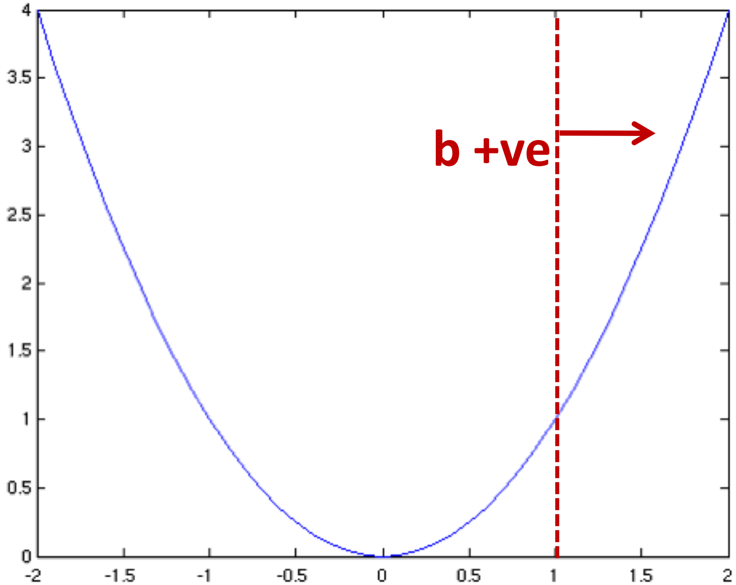
$x^* = 0$
Constraint inactive

$\min_x \ x^2$
s.t. $x \geq 1$



$x^* = 1$
Constraint active and tight

Constrained Optimization – Dual Problem



$$x^* = b$$

α **= 0 constraint is inactive**

α **> 0 constraint is active**

Primal problem:

$$\min_x \; x^2$$
$$\text{s.t.} \quad x \geq b$$

Moving the constraint to objective function

Lagrangian:

$$L(x, \alpha) = x^2 - \alpha(x - b)$$
$$\text{s.t.} \quad \alpha \geq 0$$

Dual problem:

$$\max_\alpha \; d(\alpha) \rightarrow \min_x L(x, \alpha)$$
$$\text{s.t.} \quad \alpha \geq 0$$

Connection between Primal and Dual

Dual problem: d* = $\max_\alpha \; d(\alpha)$ s.t. $\alpha \geq 0$ = $\max_\alpha \min_x L(x,\alpha)$ s.t. $\alpha \geq 0$

Notice that

Primal problem: p* = $\min_x \; x^2$ s.t. $x \geq b$ = $\min_x \max_{\alpha \geq 0} L(x,\alpha)$

Why? $L(x,\alpha) = x^2 - \alpha(x-b)$

$$\max_{\alpha \geq 0} L(x,\alpha) = x^2 - \min_{\alpha \geq 0} \alpha(x-b) = \begin{cases} x^2 & \text{if } x \geq b \\ \infty & \text{if } x < b \end{cases}$$

Connection between Primal and Dual

Primal problem: p* = $\min_x \; x^2$
s.t. $x \geq b$

= $\min_x \max_{\alpha \geq 0} L(x, \alpha)$

Dual problem: d* = $\max_\alpha \; d(\alpha)$
s.t. $\alpha \geq 0$

= $\max_\alpha \min_x L(x, \alpha)$
s.t. $\alpha \geq 0$

- **Dual problem (maximization) is always concave even if primal is not convex**

Why? Pointwise infimum of concave functions is concave.
[Pointwise supremum of convex functions is convex.]

$$L(x, \alpha) = x^2 - \alpha(x - b)$$

Connection between Primal and Dual

Primal problem: p* = $\min_x \; x^2$
s.t. $x \geq b$

Dual problem: d* = $\max_\alpha \; d(\alpha)$
s.t. $\alpha \geq 0$

- **Weak duality:** The dual solution d* lower bounds the primal solution p* i.e. d* ≤ p*

To see this, recall $L(x, \alpha) = x^2 - \alpha(x - b)$

For every feasible x (i.e. x ≥ b) and feasible α (i.e. α ≥ 0) , notice that

$$d(\alpha) = \min_x L(x, \alpha) \leq x^2 - \alpha(x-b) \leq x^2$$

Since this holds for all feasible x, in particular it holds for x* achieving the min of p*, hence d(a) ≤ p* for all feasible α ≥ 0.

Connection between Primal and Dual

Primal problem: p* = $\min_x \; x^2$ s.t. $x \geq b$

Dual problem: d* = $\max_\alpha \; d(\alpha)$ s.t. $\alpha \geq 0$

- **Weak duality:** The dual solution d* lower bounds the primal solution p* i.e. d* ≤ p*

- **Strong duality:** d* = p* holds often for many problems of interest e.g. if the primal is a feasible convex objective with linear constraints

Connection between Primal and Dual

What does strong duality say about α^* (the α that achieved optimal value of dual) and x^* (the x that achieves optimal value of primal problem)?

Whenever strong duality holds, the following conditions (known as KKT conditions) are true for α^* and x^*:

- 1. $\bigtriangledown L(x^*, \alpha^*) = 0$ i.e. Gradient of Lagrangian at x^* and α^* is zero.
- 2. $x^* \geq b$ i.e. x^* is primal feasible
- 3. $\alpha^* \geq 0$ i.e. α^* is dual feasible
- 4. $\alpha^*(x^* - b) = 0$ (called as complementary slackness)

We use the first one to relate x^* and α^*. We use the last one (complimentary slackness) to argue that $\alpha^* = 0$ if constraint is inactive and $\alpha^* > 0$ if constraint is active and tight.

Solving the dual

Solving:

$$\overbrace{\max_\alpha \min_x \; x^2 - \alpha(x-b)}^{L(x,\alpha)}$$

s.t. $\alpha \geq 0$

Find the dual: Optimization over x is unconstrained.

$$\frac{\partial L}{\partial x} = 2x - \alpha = 0 \Rightarrow x^* = \frac{\alpha}{2}$$

$$L(x^*, \alpha) = \frac{\alpha^2}{4} - \alpha\left(\frac{\alpha}{2} - b\right) = -\frac{\alpha^2}{4} + b\alpha$$

Solve: Now need to maximize $L(x^*,\alpha)$ over $\alpha \geq 0$

Solve unconstrained problem to get α' and then take $\max(\alpha',0)$

$$\frac{\partial}{\partial \alpha} L(x^*, \alpha) = -\frac{\alpha}{2} + b \;\; \Rightarrow \alpha' = 2b$$

$$\Rightarrow \alpha^* = \max(2b, 0) \qquad \Rightarrow x^* = \frac{\alpha^*}{2} = \max(b, 0)$$

α = 0 constraint is inactive, α > 0 constraint is active and tight

Dual SVM – linearly separable case

n training points, d features ($\mathbf{x}_1$, …, $\mathbf{x}_n$) where x_i is a d-dimensional vector

- Primal problem: $\text{minimize}_{\mathbf{w},b} \quad \frac{1}{2}\mathbf{w}.\mathbf{w}$
  $\left(\mathbf{w}.\mathbf{x}_j + b\right) y_j \geq 1, \ \forall j$

w – weights on features (d-dim problem)

- Dual problem (derivation):

$$L(\mathbf{w}, b, \alpha) = \frac{1}{2}\mathbf{w}.\mathbf{w} - \sum_j \alpha_j \left[\left(\mathbf{w}.\mathbf{x}_j + b\right) y_j - 1\right]$$
$$\alpha_j \geq 0, \ \forall j$$

α – weights on training pts (n-dim problem)

Dual SVM – linearly separable case

- Dual problem:

$$\max_{\alpha} \min_{\mathbf{w},b} L(\mathbf{w}, b, \alpha) = \frac{1}{2}\mathbf{w}.\mathbf{w} - \sum_j \alpha_j \left[\left(\mathbf{w}.\mathbf{x}_j + b\right) y_j - 1\right]$$

$$\alpha_j \geq 0, \ \forall j$$

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \qquad \Rightarrow \mathbf{w} = \sum_j \alpha_j y_j \mathbf{x}_j$$

$$\frac{\partial L}{\partial b} = 0 \qquad \Rightarrow \sum_j \alpha_j y_j = 0$$

If we can solve for αs (dual problem), then we have a solution for **w**,b (primal problem)

Dual SVM – linearly separable case

$$\text{maximize}_\alpha \quad \sum_i \alpha_i - \frac{1}{2}\sum_{i,j} \alpha_i\alpha_j y_i y_j \mathbf{x}_i.\mathbf{x}_j$$

$$\sum_i \alpha_i y_i = 0$$

$$\alpha_i \geq 0$$

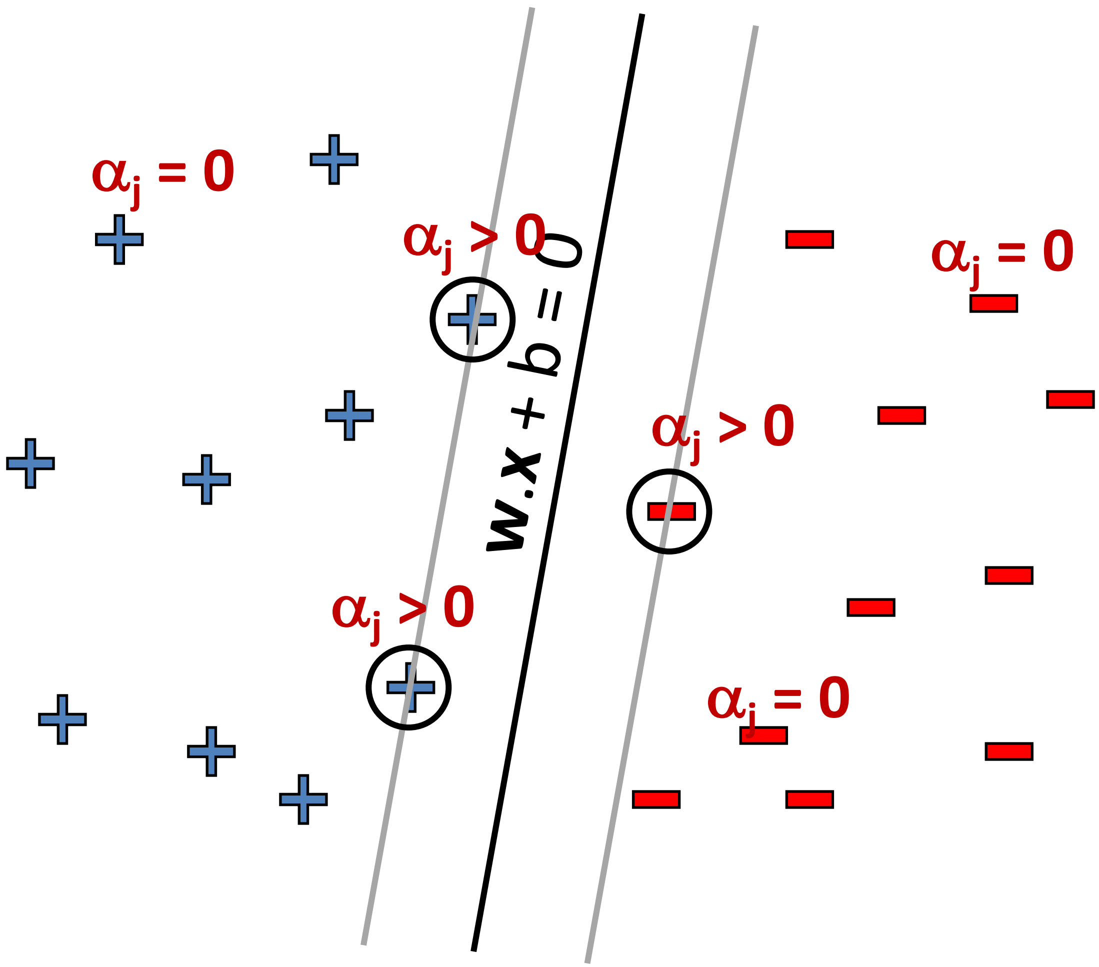
Dual problem is also QP

Solution gives α_js ⟶

$$\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i$$

What about b?

Dual SVM: Sparsity of dual solution



$$\mathbf{w} = \sum_j \alpha_j y_j \mathbf{x}_j$$

Only few α_js can be non-zero : where constraint is active and tight

$$(\mathbf{w}.\mathbf{x}_j + b)y_j = 1$$

Support vectors – training points j whose α_js are non-zero

Dual SVM – linearly separable case

maximize$_\alpha$ $\sum_i \alpha_i - \frac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j \mathbf{x}_i . \mathbf{x}_j$

$\sum_i \alpha_i y_i = 0$

$\alpha_i \geq 0$

Dual problem is also QP

Solution gives α_js →

$$\mathbf{w} = \sum_i \alpha_i y_i \mathbf{x}_i$$

$$b = y_k - \mathbf{w}.\mathbf{x}_k$$

for any k where $\alpha_k > 0$

Use support vectors with α_k>0 to compute b since constraint is tight $(w.x_k + b)y_k = 1$