15-354: Midterm

Tuesday, March 16, 2021
Posted: 10:00

Instructions

- You have 24 hours to work on this test.
- Feel free to email the course staff with questions.
- You may also post to piazza, but make your posts private.
- When you are done, submit to Gradescope.
- Good luck.

Problem 1: Wurzelbrunft Functions (30 pts.)

Someone has given Wurzelbrunft a recursion theory book for his birthday. He finds large function values produced by monsters such as the Ackermann function confusing, so he decides to study **k-Wurzelbrunft** functions: any total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) < k$ for all x. Here $k \in \mathbb{N}$.

Wurzelbrunft thinks that, for sufficiently small k, all k-Wurzelbrunft functions are primitive recursive. Help him come up with an actual theorem and a proof. The stronger the theorem, the better.

A. Clearly state your k-Wurzelbrunft theorem.

B. Prove your theorem.

Problem 2: Growth and Decidability (30 pts.)

Suppose $A \subseteq \mathbb{N}$. It is often important to understand the behavior of the corresponding **growth function** $f_A : \mathbb{N} \to \mathbb{N}$ defined by

$$f_A(n) = \text{ cardinality of } A \cap \{0, 1, \ldots, n-1\}$$

For example, when A is the set of primes, the corresponding prime-counting function has been studied in great detail in number theory. It behaves roughly like $n/\ln n$, but the details are very messy.

A. Show that f_A is computable whenever A is decidable.

B. Show that A must be decidable when f_A is computable.

C. Now assume that A is semidecidable and f_A is computable. Give another decision algorithm for membership in A (different from the one in part (B)).

   Comment: This is not frivolous; it is important in Kolmogorov complexity.

Problem 3: Magic (40 pts.)

Background

Fix some set $A \subseteq \mathbb{N}$. Suppose we modify a register machine by adding a an instruction `magic k l` which executes as follows: the machine reads the number r in register R_0. Then it magically figures out whether r is in A; if so, execution continues in line k, and in line l otherwise.

Call these gizmos **A-RMs**.

Task

A. Show that all A-RM can be simulated by an ordinary RM iff A is decidable.

B. For any A, construct a set H that cannot be decided by an A-RM.

C. Suppose you have access to a K-RM where K is the Halting set (for ordinary RMs). Explain how to exploit this to prove or disprove the Goldbach conjecture: every even number larger than 2 is the sum of two primes.

Problem 4: Tally Languages (30 pts.)

Languages $L \subseteq \{a\}^\star$ are referred to as **tally languages**. Let $c(n) = |L \cap \{\varepsilon, a, \ldots, a^{n-1}\}|$ and define the **density** of L to be $\Delta(L) = \lim_{n\to\infty} c(n)/n$. For example, $\Delta((aa)^\star) = 1/2$.

Regular tally languages are fairly easy to describe.

A. Characterize the minimal DFAs associated with regular tally languages.

B. Explain how to compute the density $\Delta(L)$ for any regular tally language L.

Problem 5: Word Shuffle (30 pts.)

The **word shuffle** operation, in symbols $\|$, is a map from $\Sigma^\star \times \Sigma^\star$ to $\mathfrak{P}(\Sigma^\star)$ defined by

$$\begin{aligned} \varepsilon \,\|\, y = y \,\|\, \varepsilon &= \{y\} \\ xa \,\|\, yb = (x \,\|\, yb)\, a &\cup (xa \,\|\, y)\, b. \end{aligned}$$

As usual, we can extend the operation to languages by

$$K \,\|\, L = \bigcup \{\, x \,\|\, y \mid x \in K, y \in L \,\}$$

For example,

$$aa \,\|\, bbb = \{aabbb, ababb, abbab, abbba, baabb, babab, babba, bbaab, bbaba, bbbaa\}$$

Also, $(aa)^\star \,\|\, (bb)^\star$ is the set of all even/even words.

A. Given two words x and y, construct a finite state machine M that accepts $x \,\|\, y$.

B. Let $M_i = \langle Q_i, \Sigma, \delta_i; q_{0i}, F_i \rangle$, $i = 1, 2$, be two DFAs accepting regular languages L_i. Show how to construct a finite state machine M that accepts the shuffle language $L_1 \,\|\, L_2$. Make sure to specify the state set, the transitions, initial and final states.

Problem 6: Regularity (40 pts.)

Background

Let $\Sigma = \{a, b\}$. Define the **mod counter languages** (with parameters m and n) to be

$$L_{m,n} = \{\, x \in \Sigma^\star \mid \#_a x = 0 \pmod{m}, \#_b x = 0 \pmod{n} \,\}$$

and define the **copy language** to be

$$L_{\mathrm{cp}} = \{\, zz \in \Sigma^\star \mid z \in \Sigma^\star \,\}$$

Task

A. Construct the "natural" minimal DFA $\mathcal{A}_{m,n}$ that recognizes $L_{m,n}$.

B. Find a "one-line" proof that all these automata are indeed minimal.

C. Use quotients to show that the copy language fails to be regular.