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# 15-826: Multimedia (Databases) and Data Mining

Lecture #21: DSP tools –

DFT – Discrete Fourier Transform

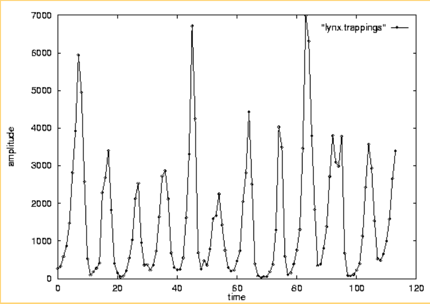
*C. Faloutsos*

# Problem

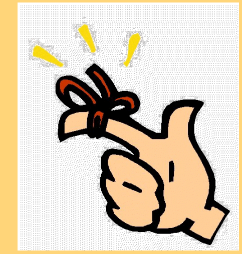
Goal: given a signal (eg., sales over time and/or space)

Q: Find patterns and/or compress

count




year

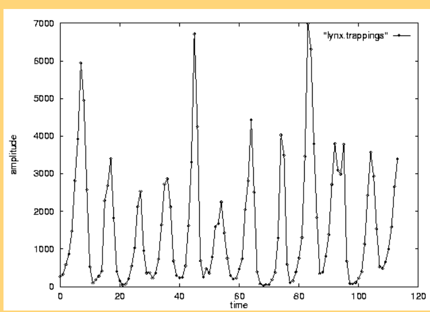


# Solutions:

Goal: given a signal (eg., sales over time and/or space)

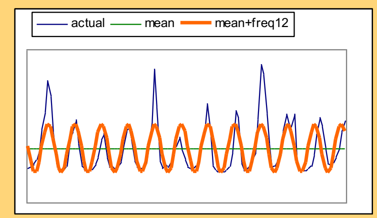
Q: Find patterns and/or compress

count

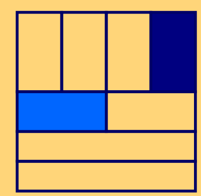


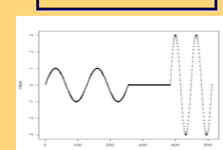

year

A1: Fourier (DFT)




A2: Wavelets (DWT)





# Must-read Material

- DFT/DCT: In PTVF ch. 12.1, 12.3, 12.4; in Textbook Appendix B.
- Wavelets: In PTVF ch. 13.10; in MM Textbook Appendix C

# Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

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# Indexing - Detailed outline

- primary key indexing
- ..
- Multimedia –
  - Digital Signal Processing (DSP) tools
    - Discrete Fourier Transform (DFT)
    - Discrete Wavelet Transform (DWT)

# DSP - Detailed outline

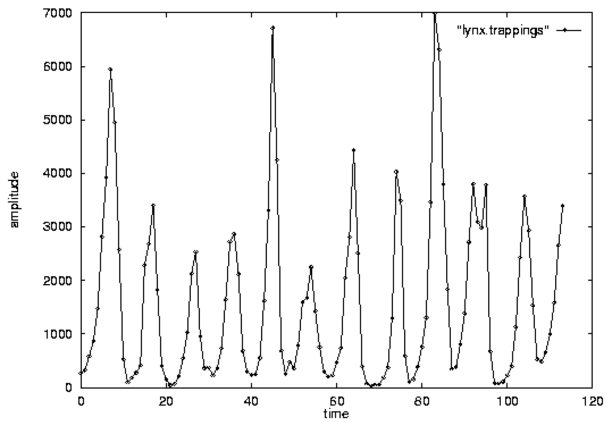
- DFT
  - what
  - why
  - how
  - Arithmetic examples
  - properties / observations
  - DCT
  - 2-d DFT
  - Fast Fourier Transform (FFT)

# Introduction

Goal: given a signal (eg., sales over time and/or space)

Find: patterns and/or compress

count




lynx caught per year

year

# What does DFT do?

A: highlights the periodicities

# Why should we care?

A: several real sequences are periodic

Q: Such as?

# Why should we care?

A: several real sequences are periodic

Q: Such as?

A:

- sales patterns follow seasons;
- economy follows 50-year cycle
- temperature follows daily and yearly cycles

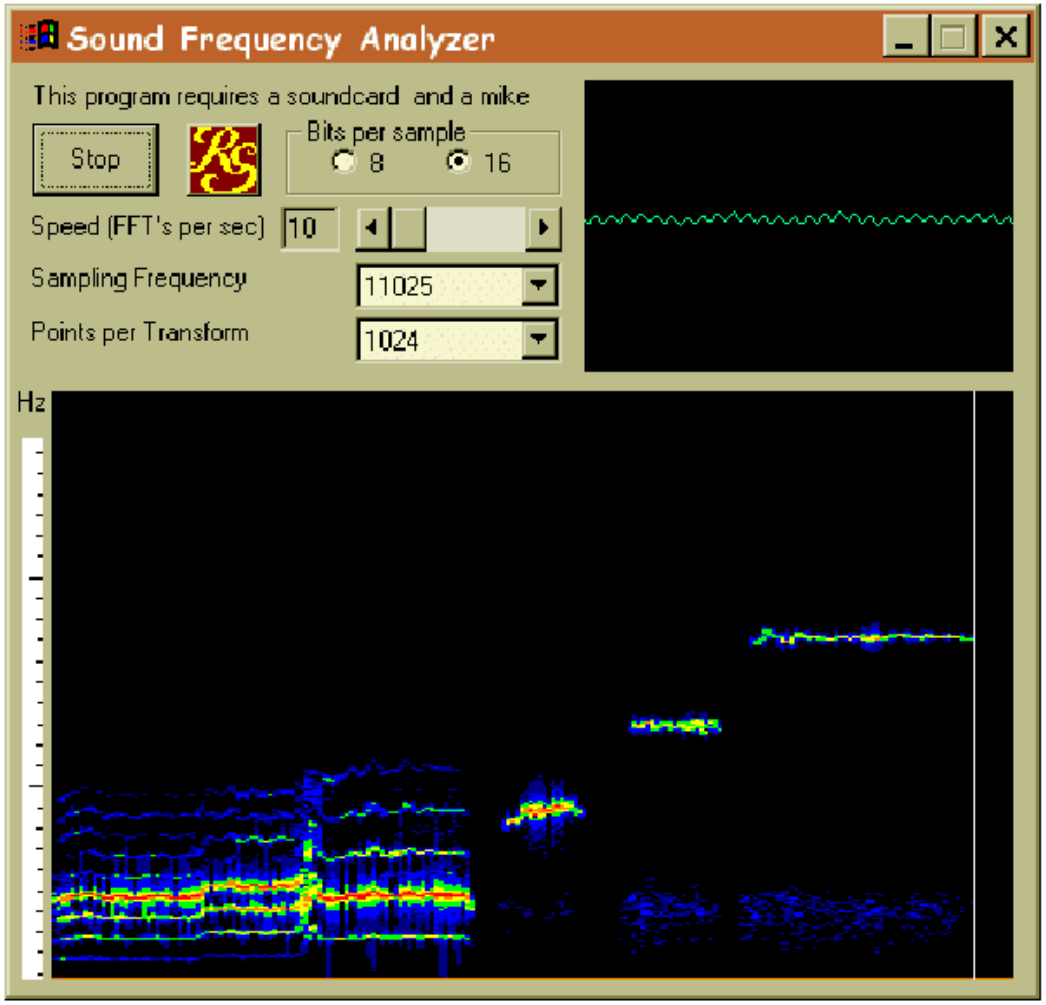
Many real signals follow (multiple) cycles

# Why should we care?

For example: human voice!

- Frequency analyzer
  http://www.relisoft.com/freeware/freq.html
- speaker identification
- impulses/noise -> flat spectrum
- high pitch -> high frequency

# 'Frequency Analyzer'




frequency

time

# DFT: definition

- Discrete Fourier Transform (n-point):

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi f t / n) \qquad f = 0, \ldots, n-1$$

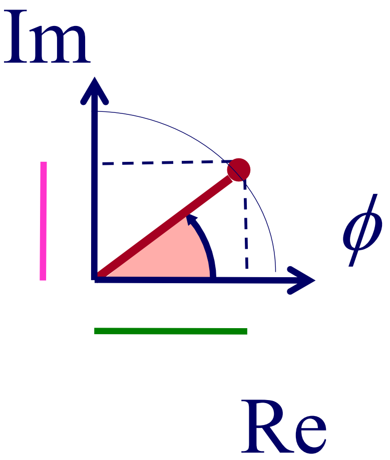
$$(j = \sqrt{-1})$$

inverse DFT

$$x_t = 1/\sqrt{n} \sum_{f=0}^{n-1} X_f * \exp(+j 2\pi f t / n)$$

# (Reminder)

$$\exp(\phi * j) = \cos(\phi) + j * \sin(\phi)$$




(fun fact: the equation with the 5 most important numbers:

$$e^{j\pi} + 1 = 0$$

)

# DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in 'mathematica'

  x = [1,2,1,2];

  X = **Fourier**[x];

  Plot[ Abs[X] ];

# DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in python

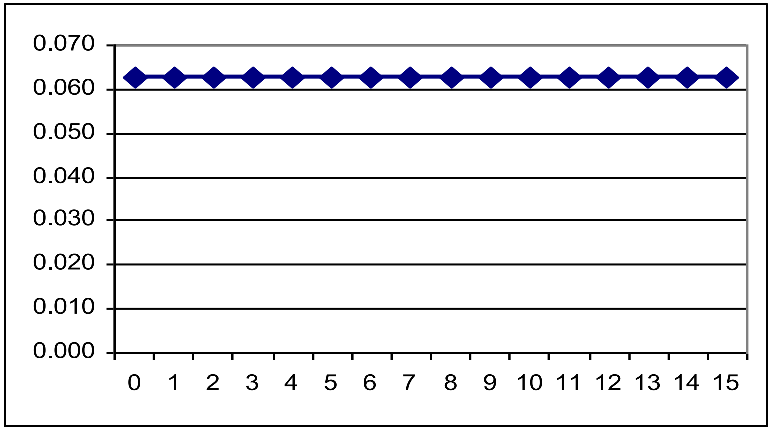
```
x = [1,2,1,2]
f = scipy.fft.fft(x)
matplotlib.pyplot.plot( …,  abs(f))
```

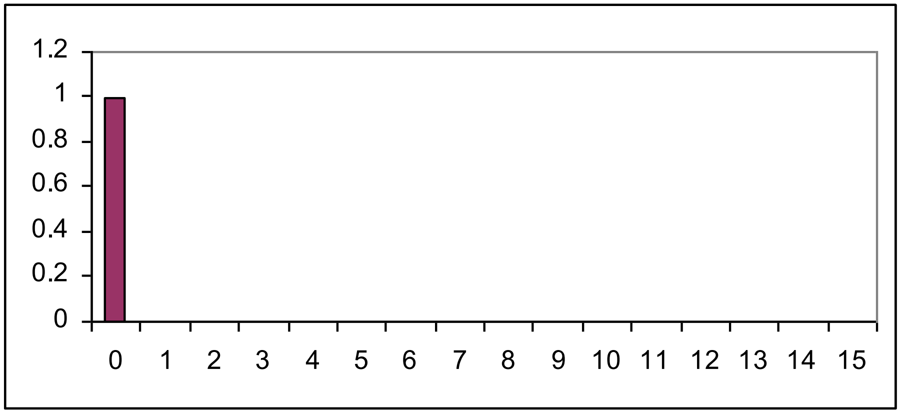
# DFT: examples

*Plot[ Abs[Fourier[x]] ];*

flat

Amplitude




time

freq

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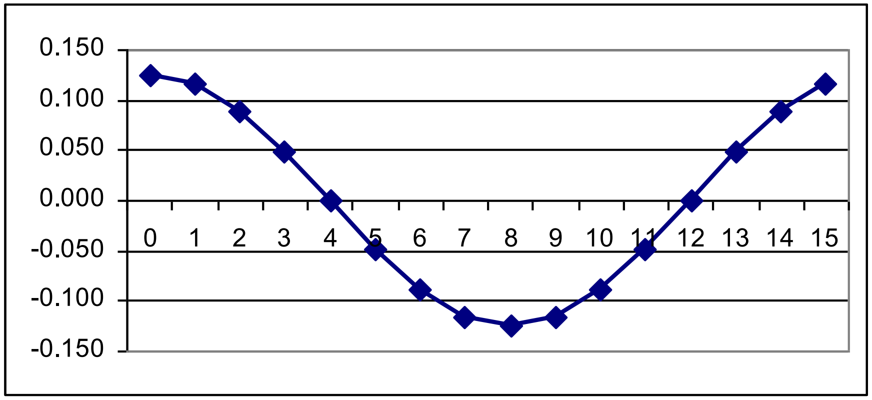
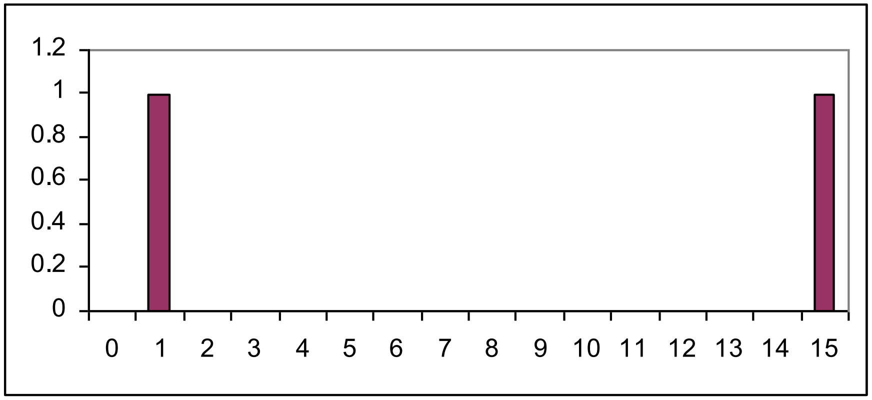
# DFT: examples

Low frequency sinusoid




time

freq

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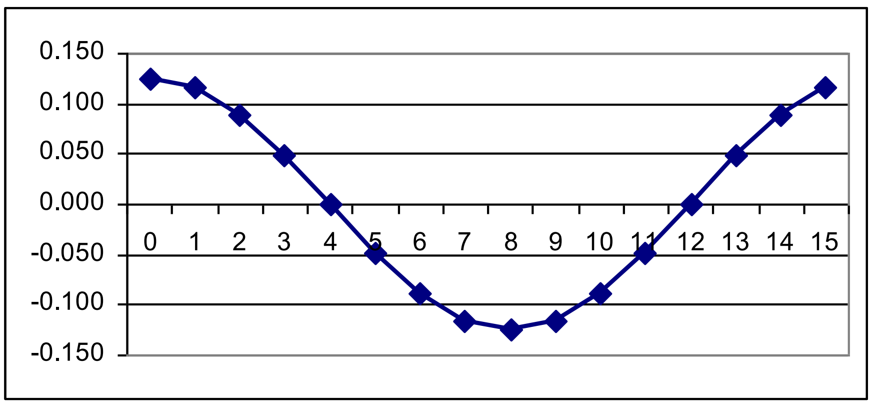
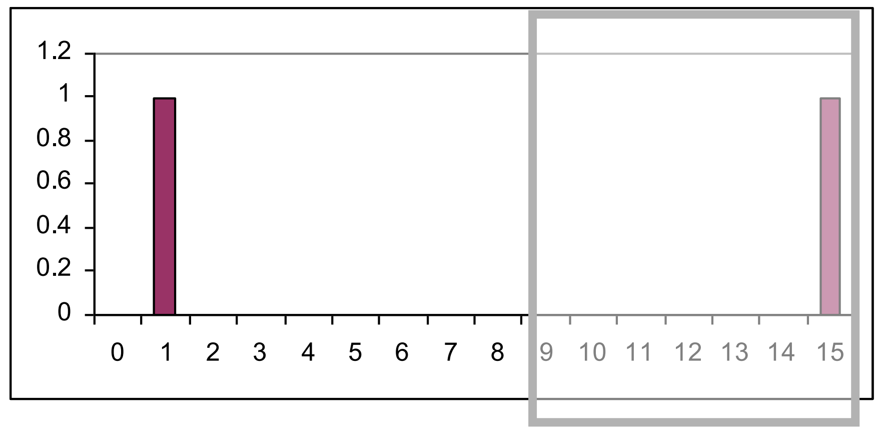
# DFT: examples

- Sinusoid - symmetry property: $X_f = X^*_{n-f}$



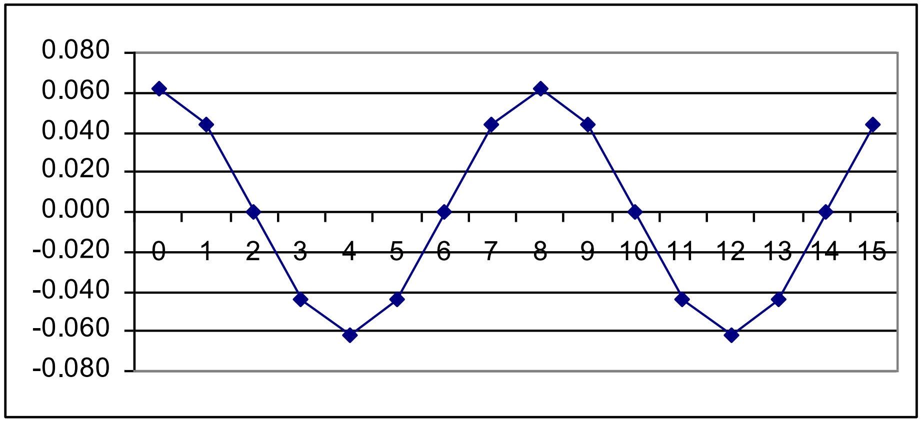




time

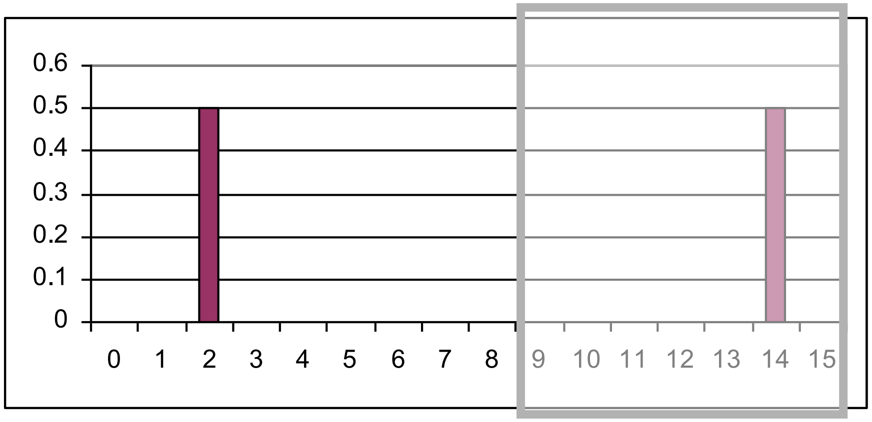
freq

# DFT: examples

- Higher freq. sinusoid



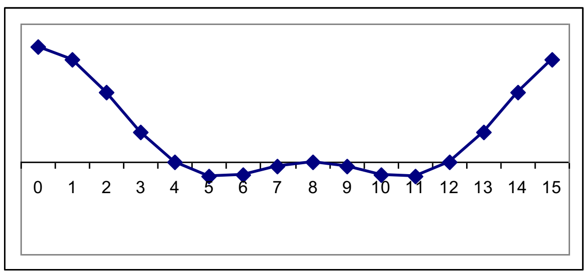

time




freq

# DFT: examples

examples

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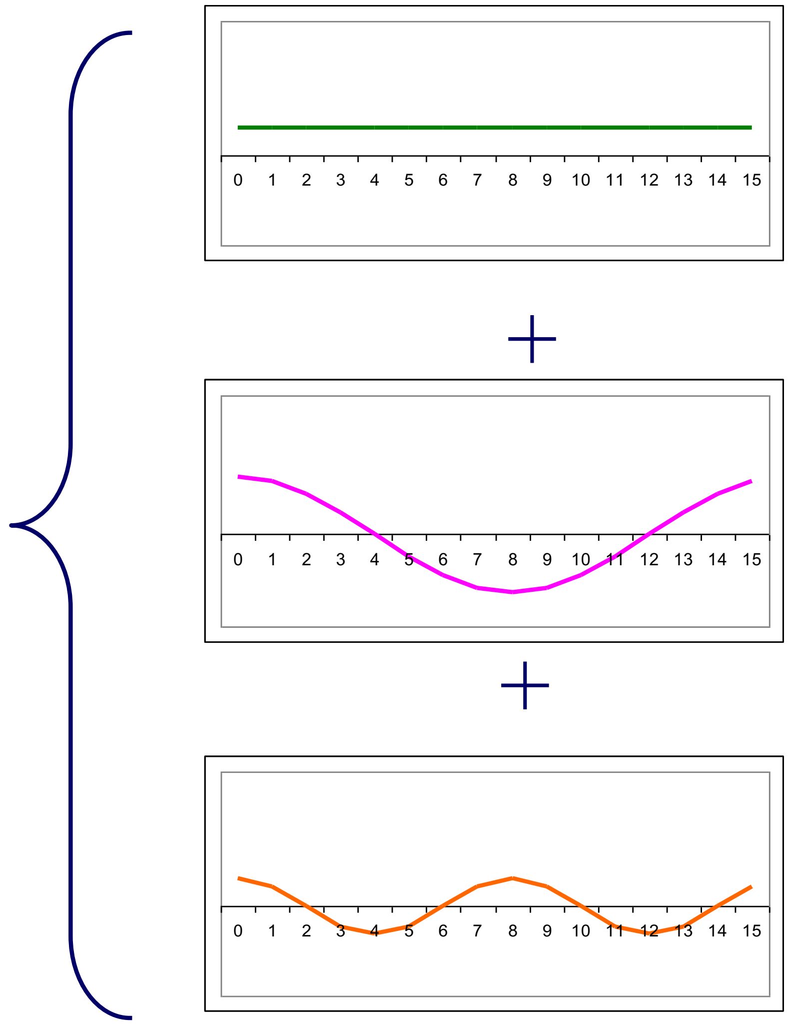
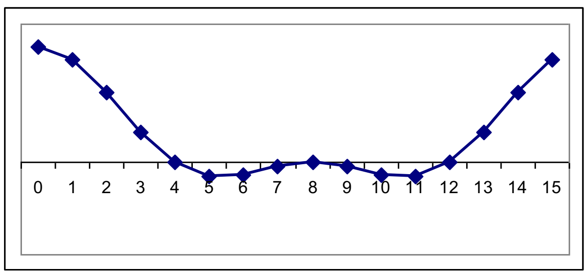
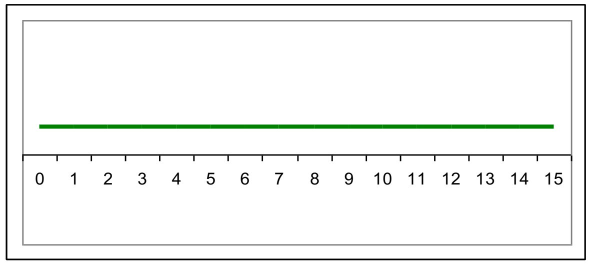
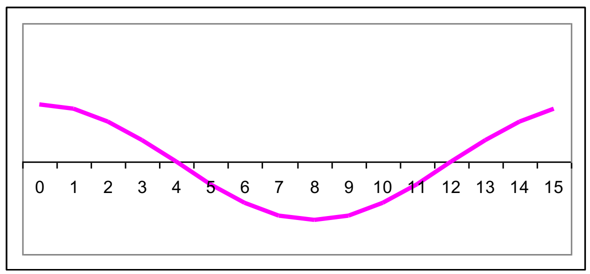


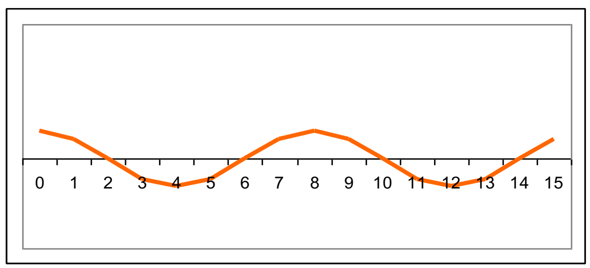
# DFT: examples

examples

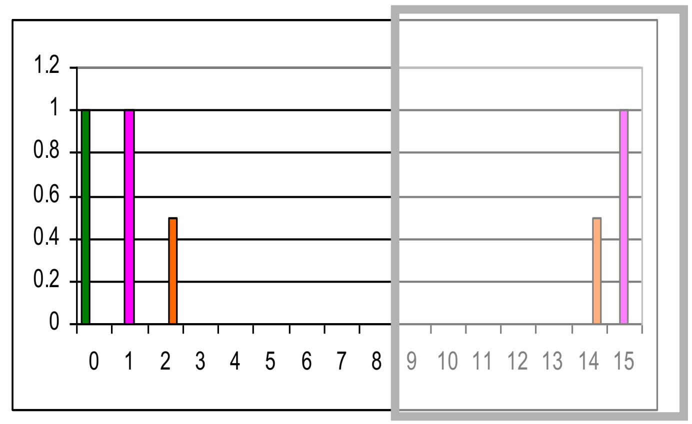



Ampl.




Freq.

# DFT: drills (1 of 2)

- Q1) How does $\sin(2\,\pi\,4\,t\,/128)\;(t = 0 \ldots 127)$ look like?
- Q2) How about its Fourier spectrum?

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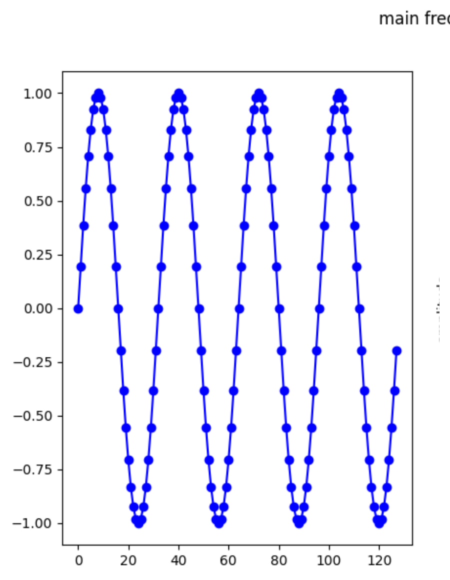
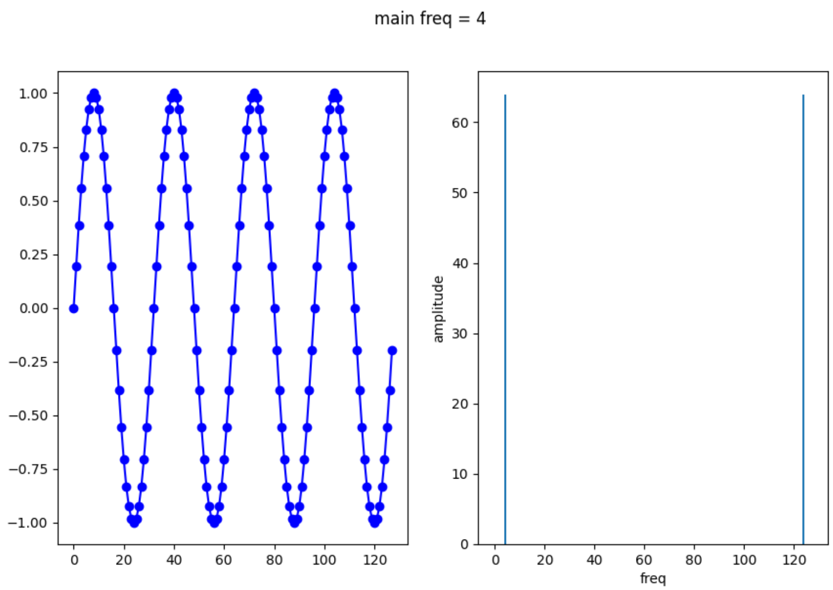


# DFT: drills (1 of 2)

- Q1) How does $\sin(2\,\pi\,4\,t\,/128)$ $(t = 0 \ldots 127)$ look like?
- Q2) How about its Fourier spectrum?



# DFT: drills (2 of 2)

- Q1) How does $\sin(2\,\pi\,3.5\,t\,/128)$ $(t = 0 \ldots 127)$ look like?
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# DFT: drills (2 of 2)

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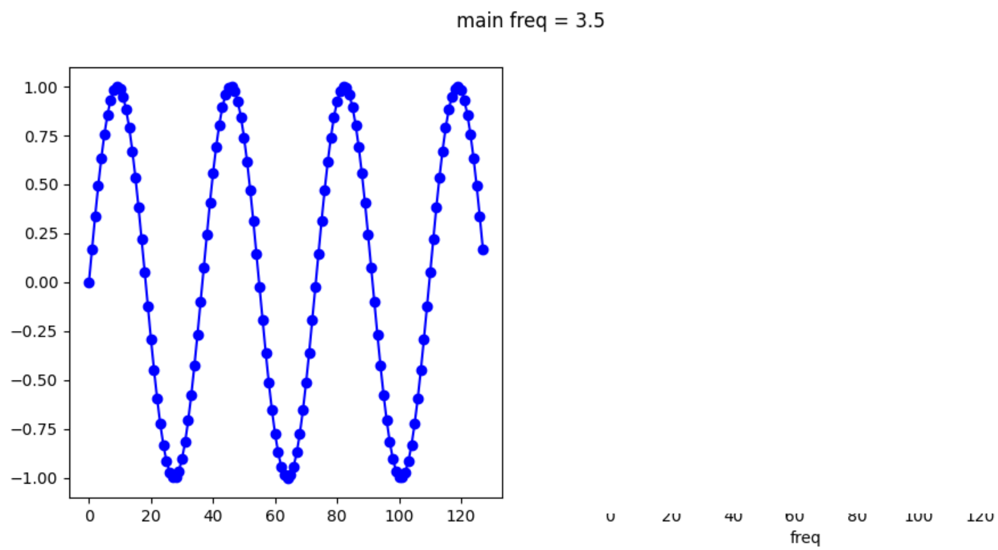

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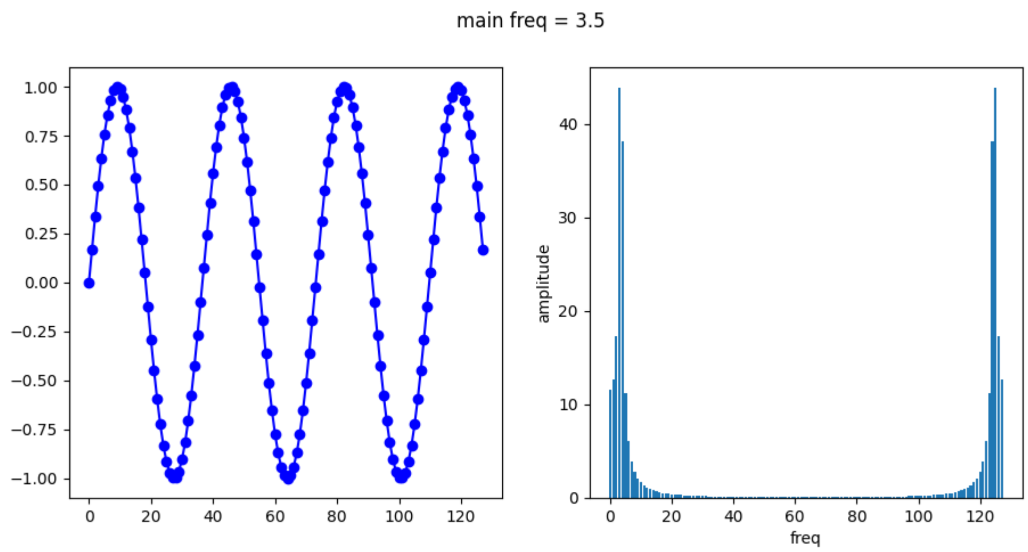

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# DSP - Detailed outline

- DFT
  - what
  - why
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  - Arithmetic examples
  - properties / observations
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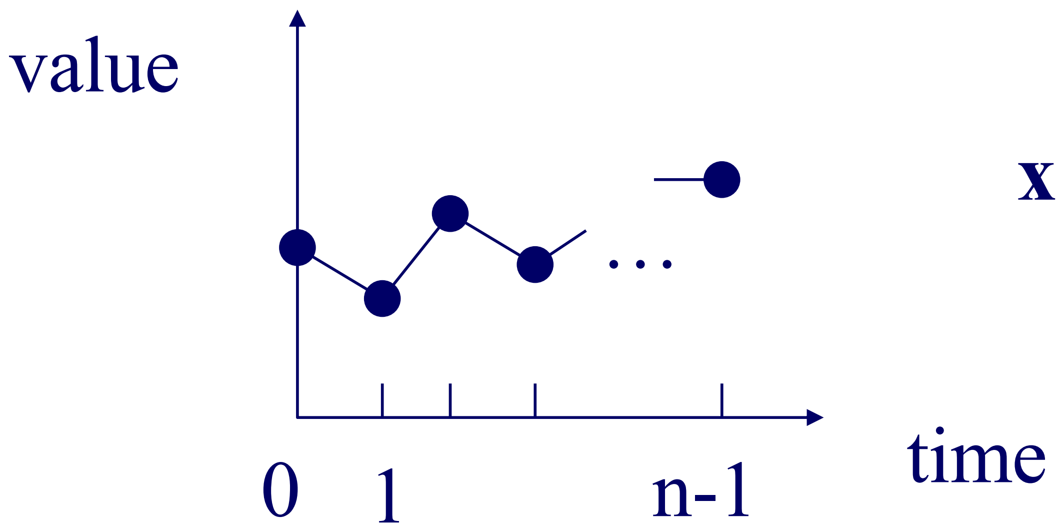
details

# How does it work?

Decomposes signal to a sum of sine (and cosine) waves.

Q:How to assess 'similarity' of **x** with a wave?

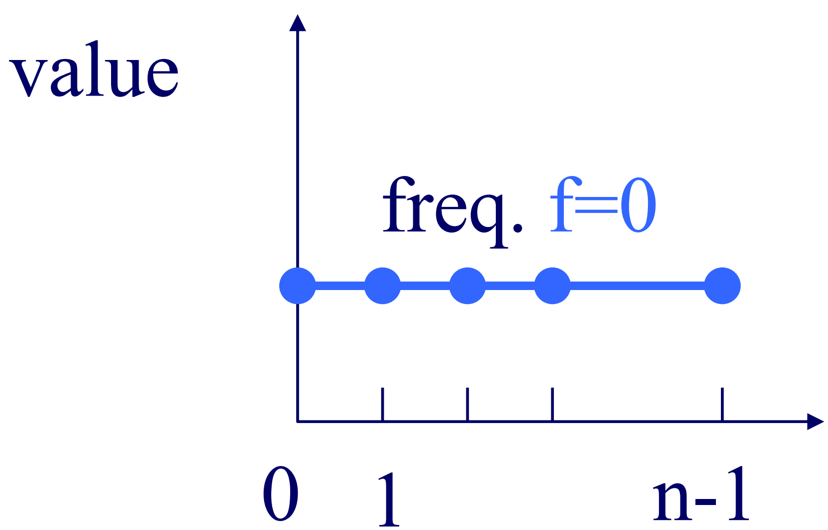


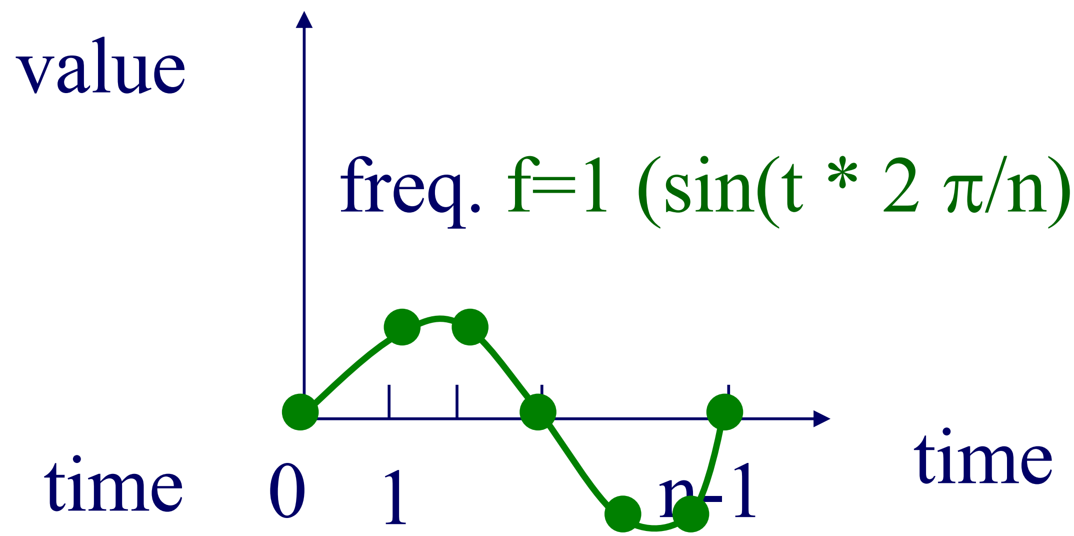

$\mathbf{x} = \{x_0, x_1, \ldots x_{n-1}\}$

details

# How does it work?

A: consider the waves with frequency 0, 1, ...; use the inner-product (~cosine similarity)



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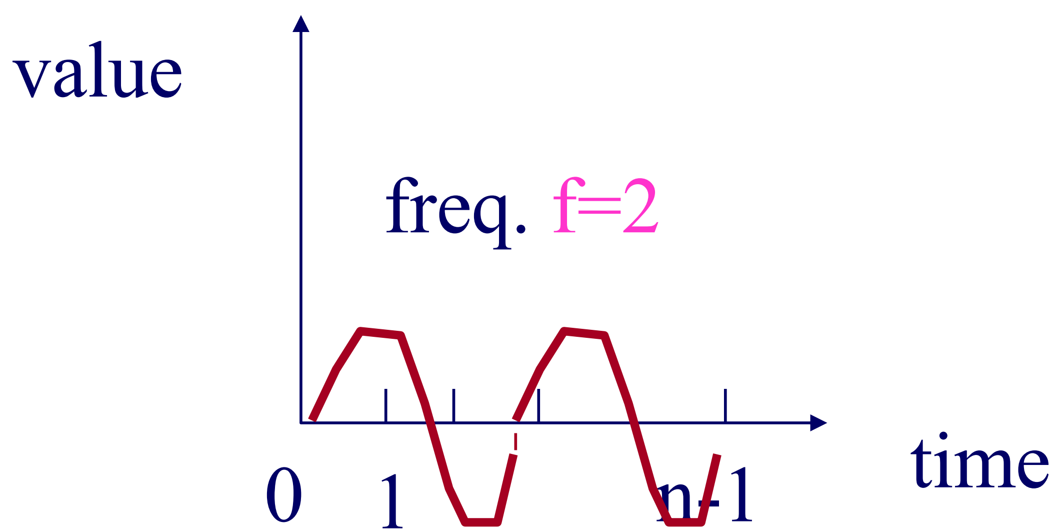
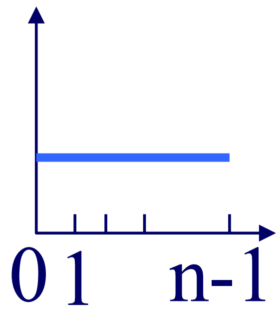


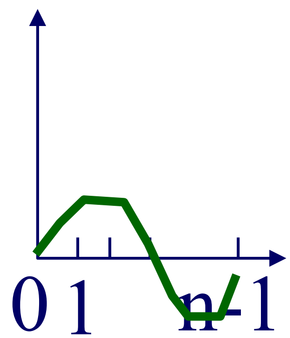
details

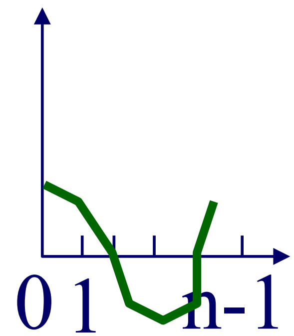
# How does it work?

**‘basis’ functions (vectors)**



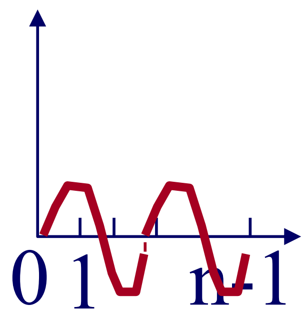

sine, freq =1

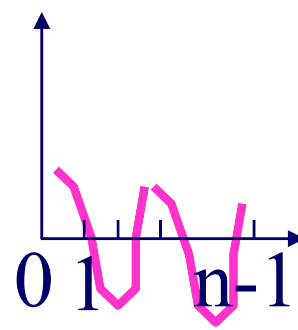



cosine, f=1

sine, freq = 2




cosine, f=2

details

# How does it work?

- Basis functions are actually n-dim vectors, **orthogonal** to each other
- 'similarity' of **x** with each of them: inner product
- DFT: ~ all the similarities of **x** with the basis functions

# DFT: definition

- **Good** news: Available in **all** symbolic math packages, eg., in 'mathematica'

```
x = [1,2,1,2];
X = Fourier[x];
Plot[ Abs[X] ];
```

# DFT: definition

(variations:

- 1/n instead of 1/sqrt(n)
- exp(-...) instead of exp(+...)

# DFT: definition

Observations:

- $X_f$ : are complex numbers except
  - $X_0$ , who is real
- Im ($X_f$ ): ~ amplitude of sine wave of frequency *f*
- Re ($X_f$ ): ~ amplitude of cosine wave of frequency *f*
- **x:** is the sum of the above sine/cosine waves

details

# DFT: definition

Observation - SYMMETRY property:

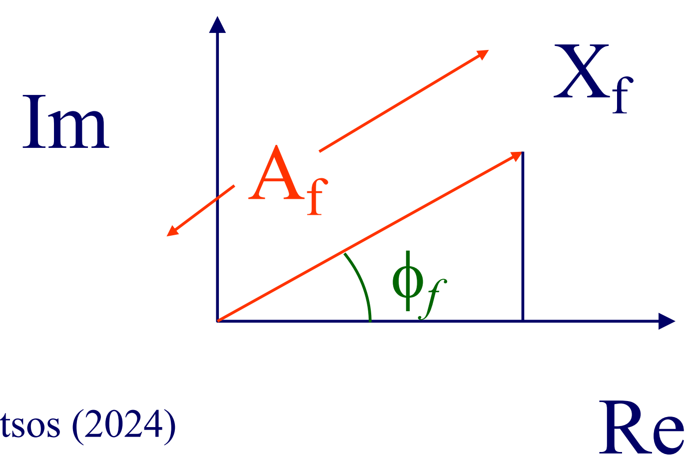
$$X_f = (X_{n-f})^*$$

( "*": complex conjugate: *(a + b j)* = a - b j )*

# DFT: definition

Definitions

- $A_f = |X_f|$ : amplitude of frequency $f$
- $|X_f|^2 = Re(X_f)^2 + Im(X_f)^2$ = energy of frequency $f$
- phase $\phi_f$ at frequency $f$



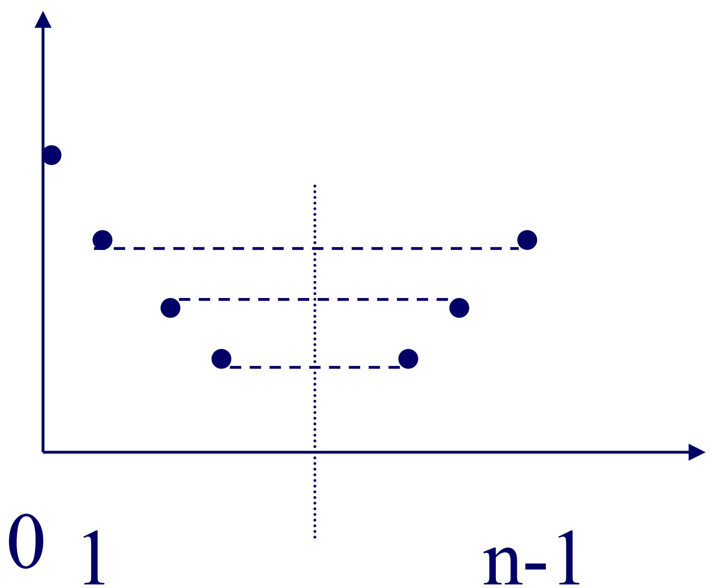

  

details

# DFT: definition

Amplitude spectrum: $| X_f |$ vs $f$ $(f=0, 1, \ldots n-1)$

SYMMETRIC (Thus, we plot the first half only)



details

# DFT: definition

Phase spectrum | $\phi_f$ | vs $f$ ($f$=0, 1, ... $n$-1):

Anti-symmetric

(Rarely used)

# DSP - Detailed outline

- DFT
  - what
  - why
  - how
  - ➡ Arithmetic examples
  - properties / observations
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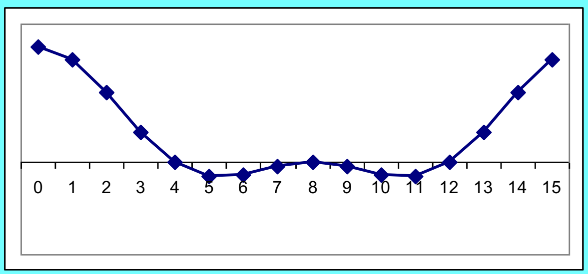
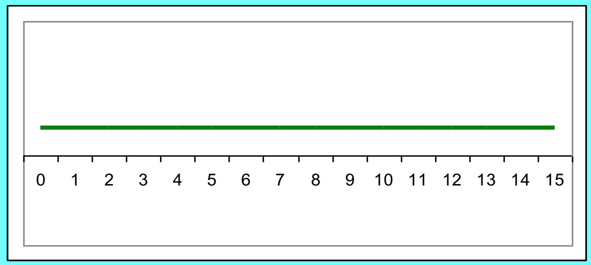
  - Fast Fourier Transform (FFT)

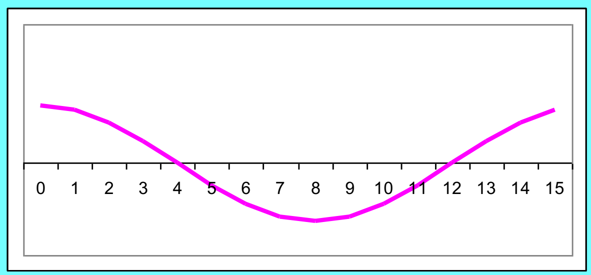
  

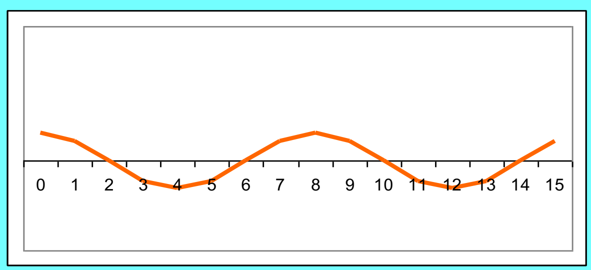
# DFT: examples

examples

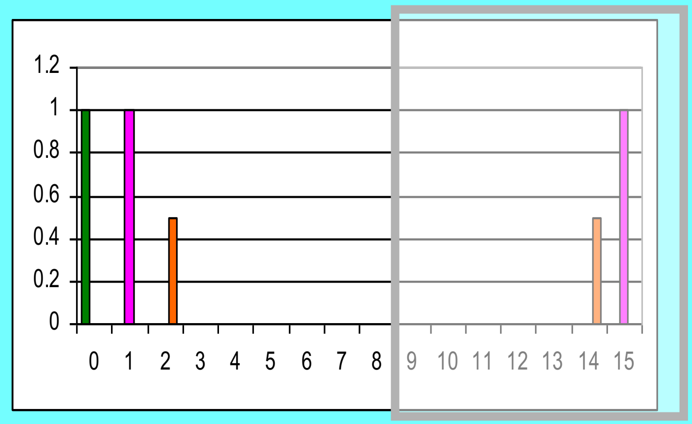



Ampl.

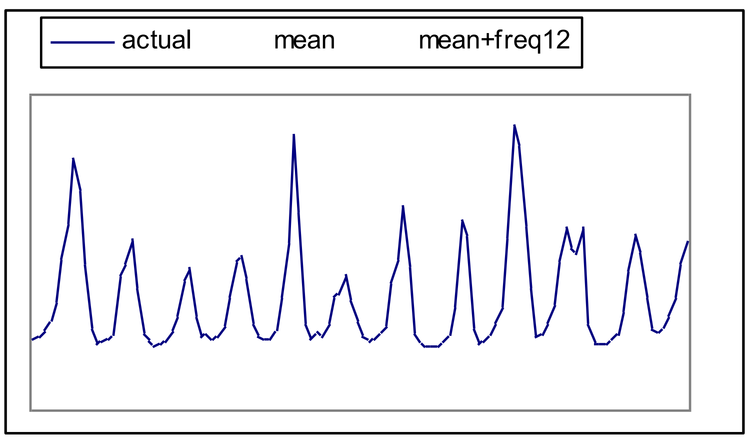



Freq.

# DFT: Amplitude spectrum

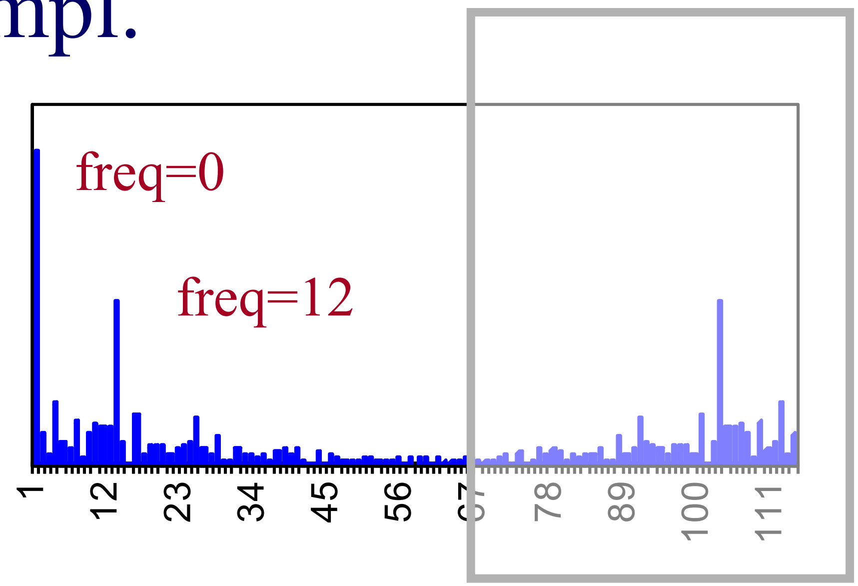
Amplitude: $$A_f^{\,2} = \mathrm{Re}^2(X_f) + \mathrm{Im}^2(X_f)$$

count




year

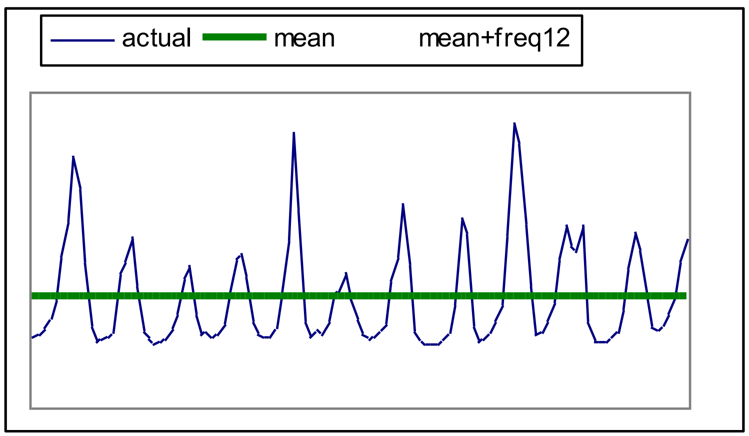
Ampl.



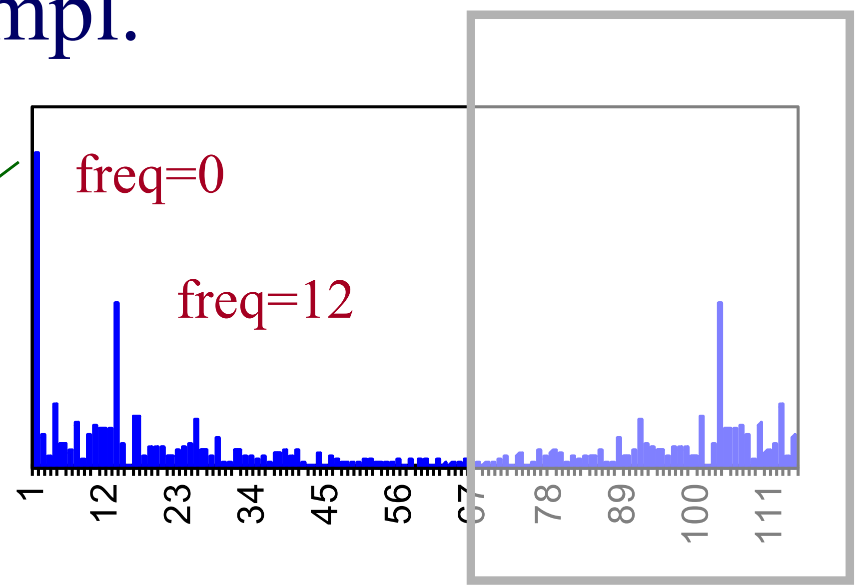

Freq.

# DFT: Amplitude spectrum

count




Ampl.




year

Freq.

# DFT: Amplitude spectrum

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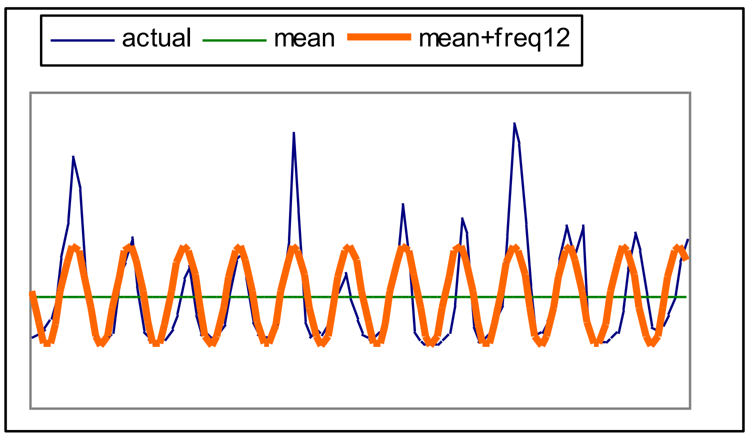
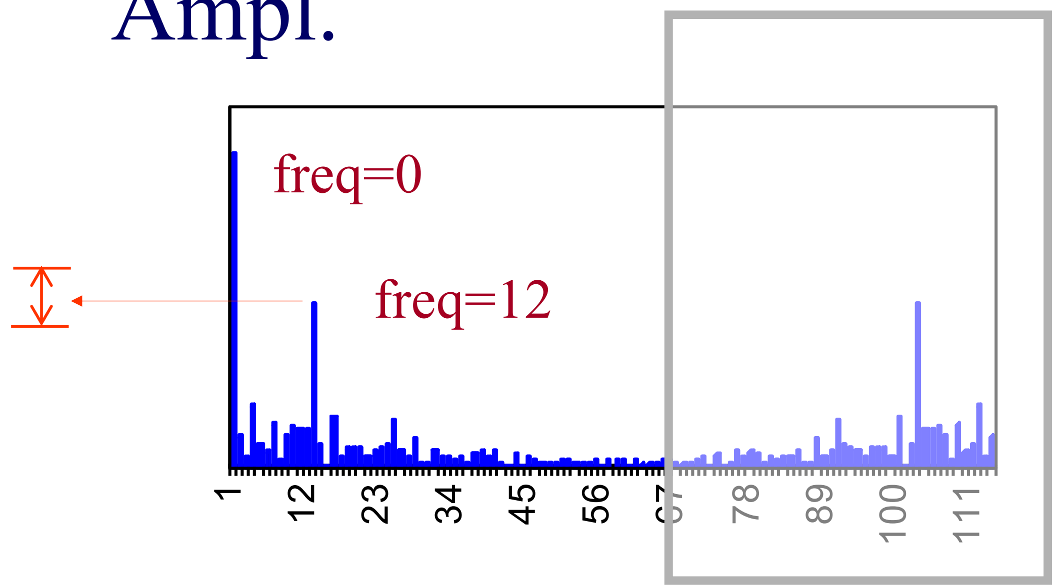



Ampl.

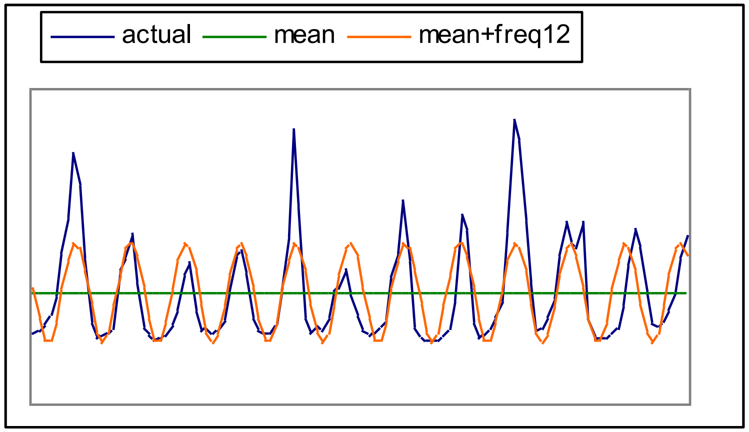


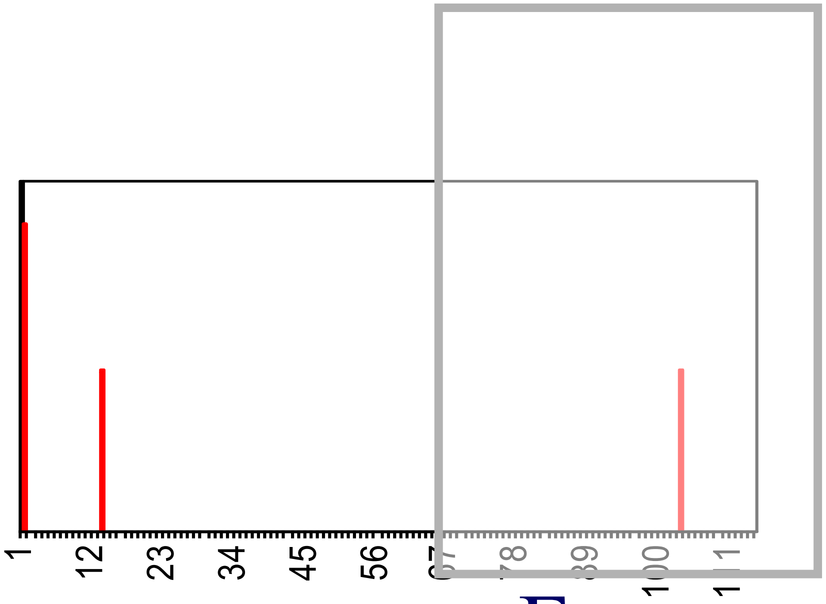

year

Freq.

# DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?



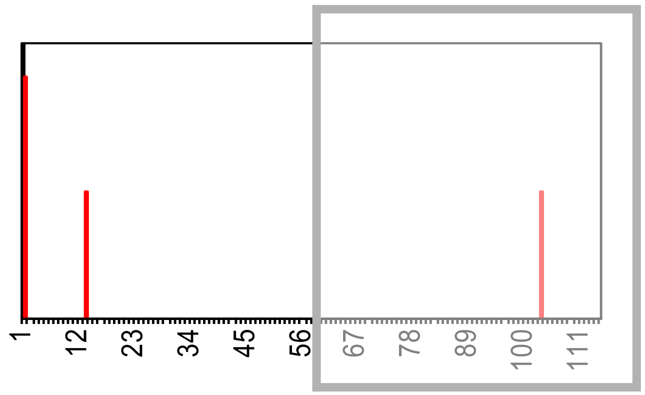



# DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: compression
- A2: pattern discovery
- (A3: forecasting)

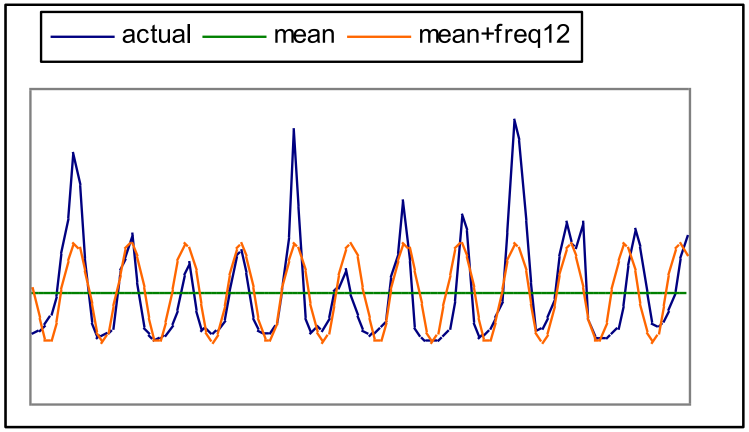
# DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: **(lossy) compression**
- A2: pattern discovery



# DFT: Amplitude spectrum

- excellent approximation, with only 2 frequencies!
- so what?
- A1: (lossy) compression
- A2: **pattern discovery**



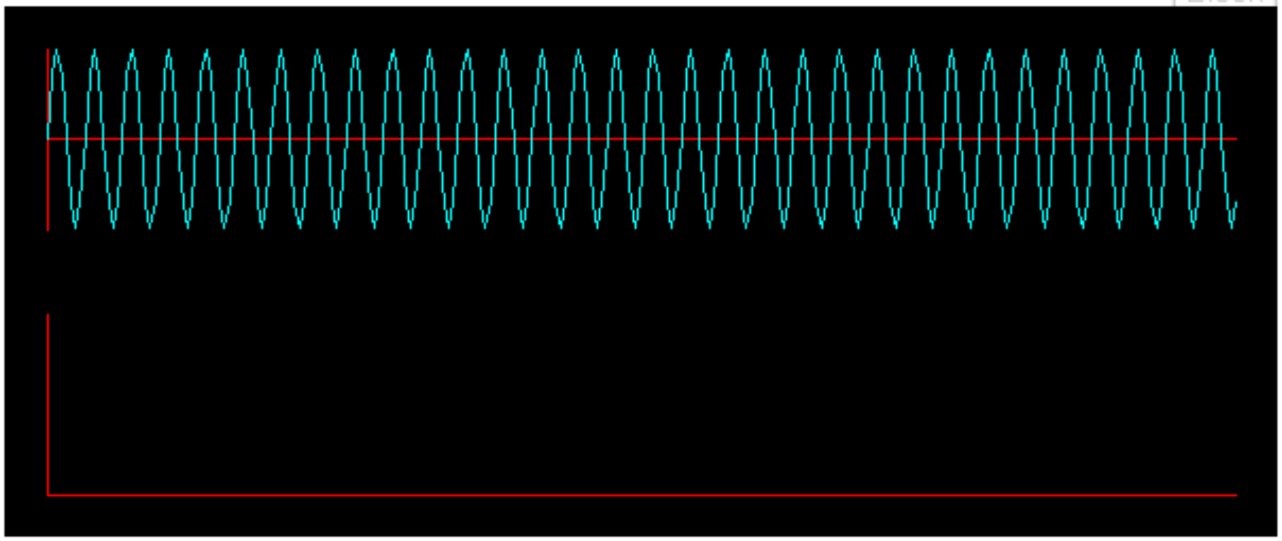
# DSP - Detailed outline

- DFT
  - what
  - why
  - how
  - Arithmetic examples
  - properties / observations
  - DCT
  - 2-d DFT
  - Fast Fourier Transform (FFT)

# DFT: Amplitude spectrum

- Let’s see it in action (defunct now…)
- (http://www.dsptutor.freeuk.com/jsanalyser/FFTSpectrumAnalyser.html)
- plain sine
- phase shift
- two sine waves
- the ‘chirp’ function
- http://ion.researchsystems.com/

# Plain sine



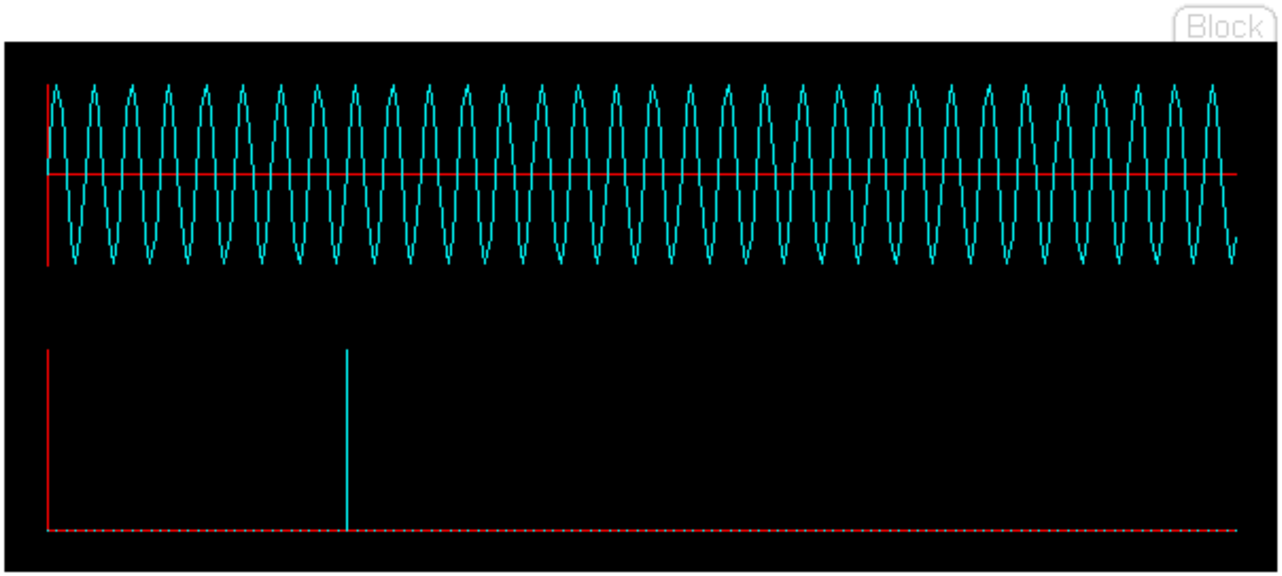
Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

# Plain sine



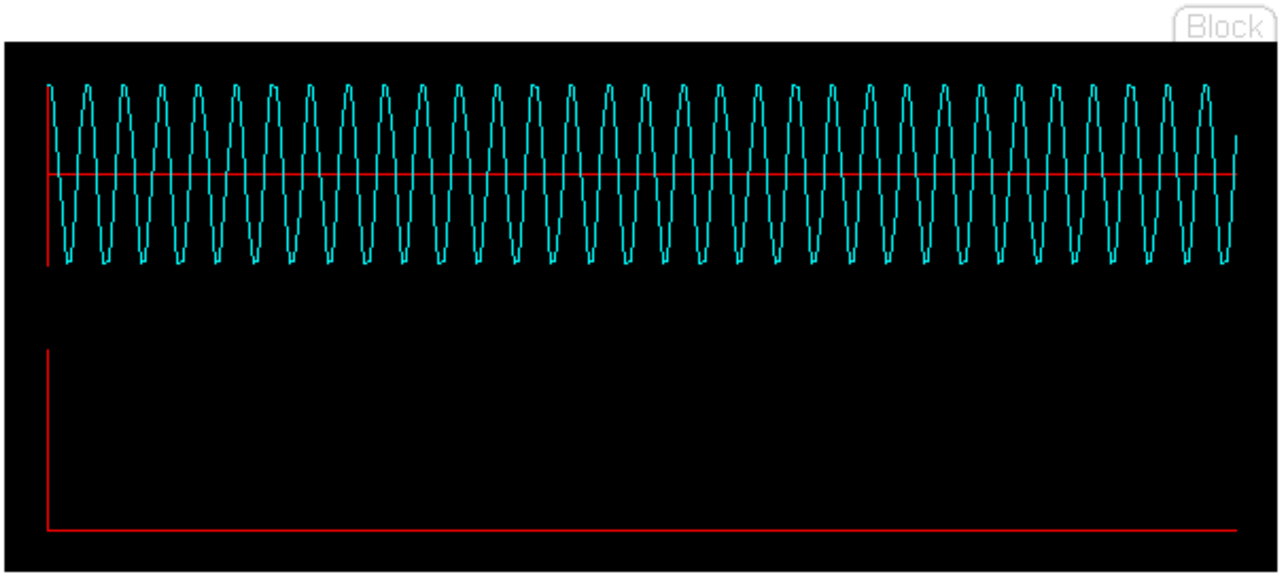

Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

# Plain sine – phase shift



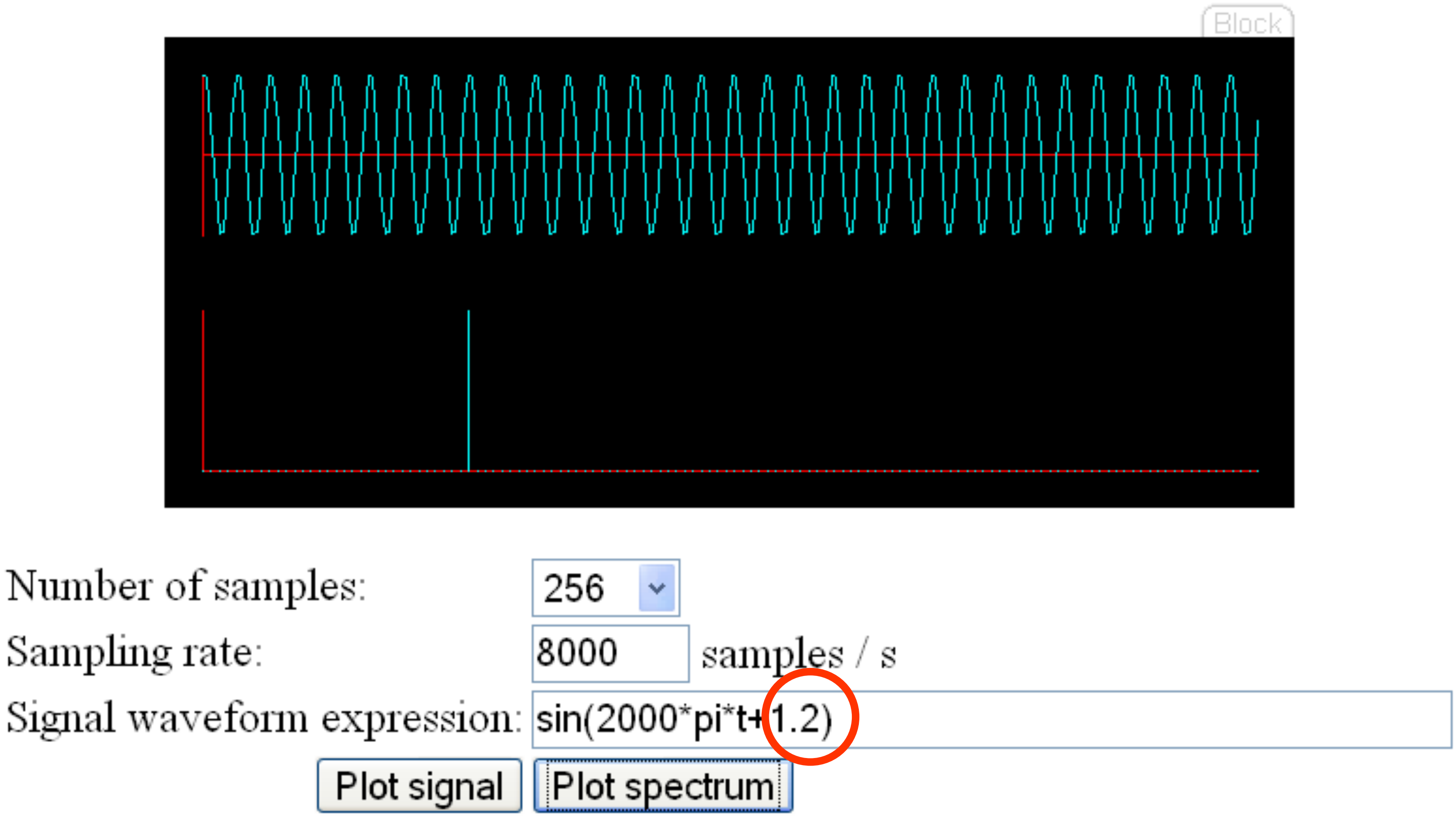

Number of samples: 256

Sampling rate: 8000 samples / s

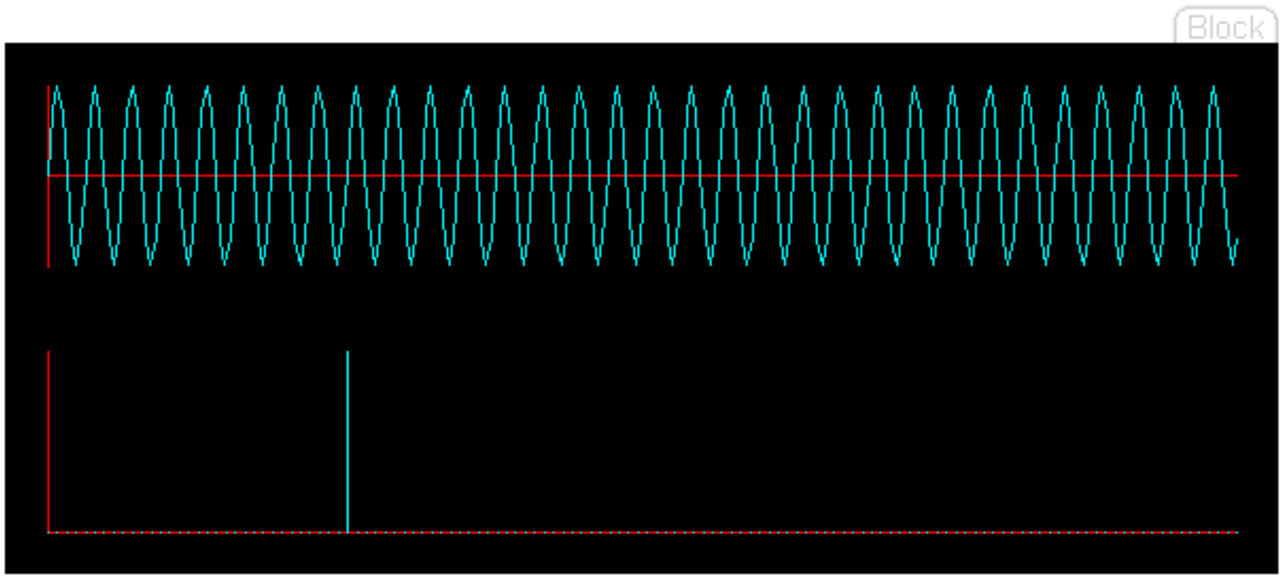
Signal waveform expression: sin(2000*pi*t+1.2)

Plot signal | Plot spectrum

# Plain sine – phase shift



# Plain sine



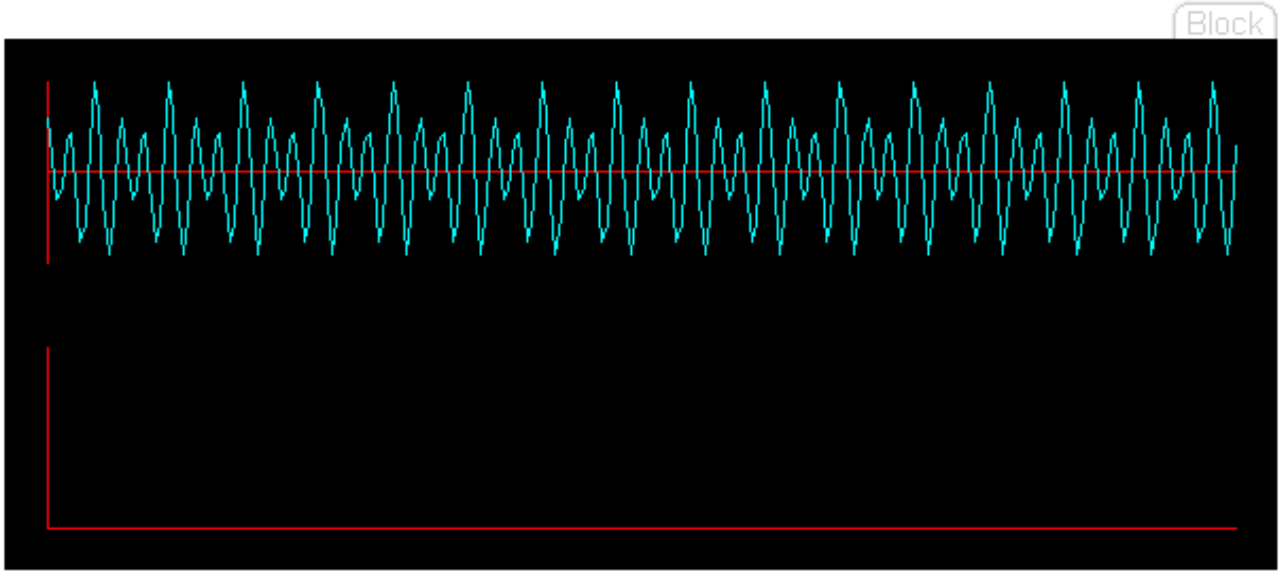

Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)

Plot signal | Plot spectrum

# Two sines



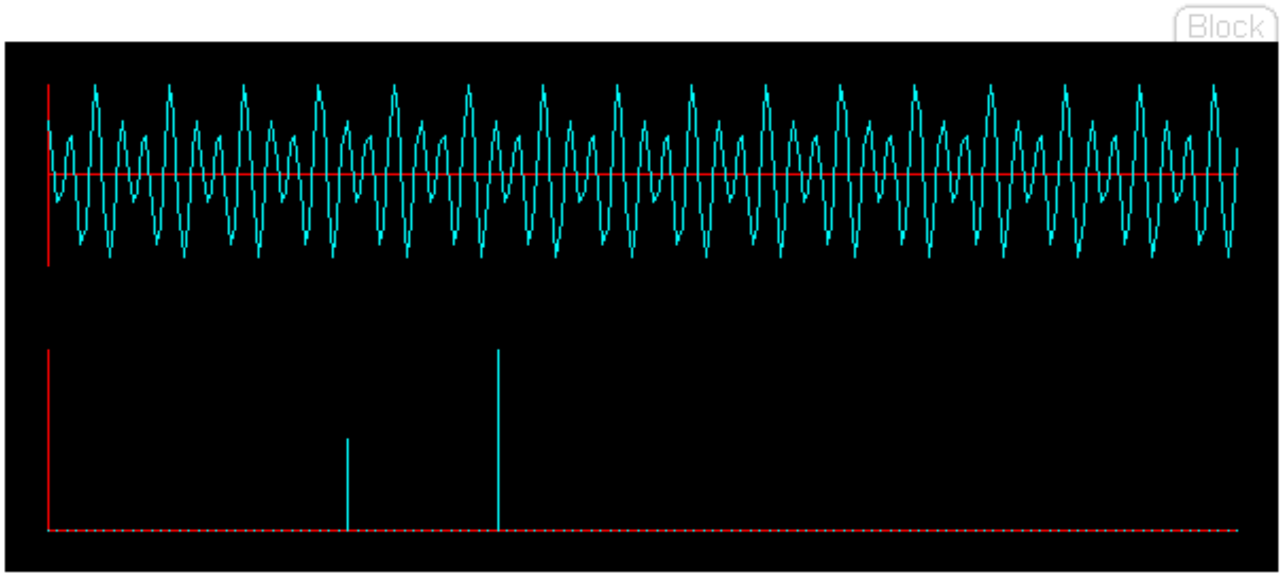

Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)+2*cos(3000*pi*t+0.5)

Plot signal | Plot spectrum

# Two sines




Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(2000*pi*t)+2*cos(3000*pi*t+0.5)

Plot signal | Plot spectrum

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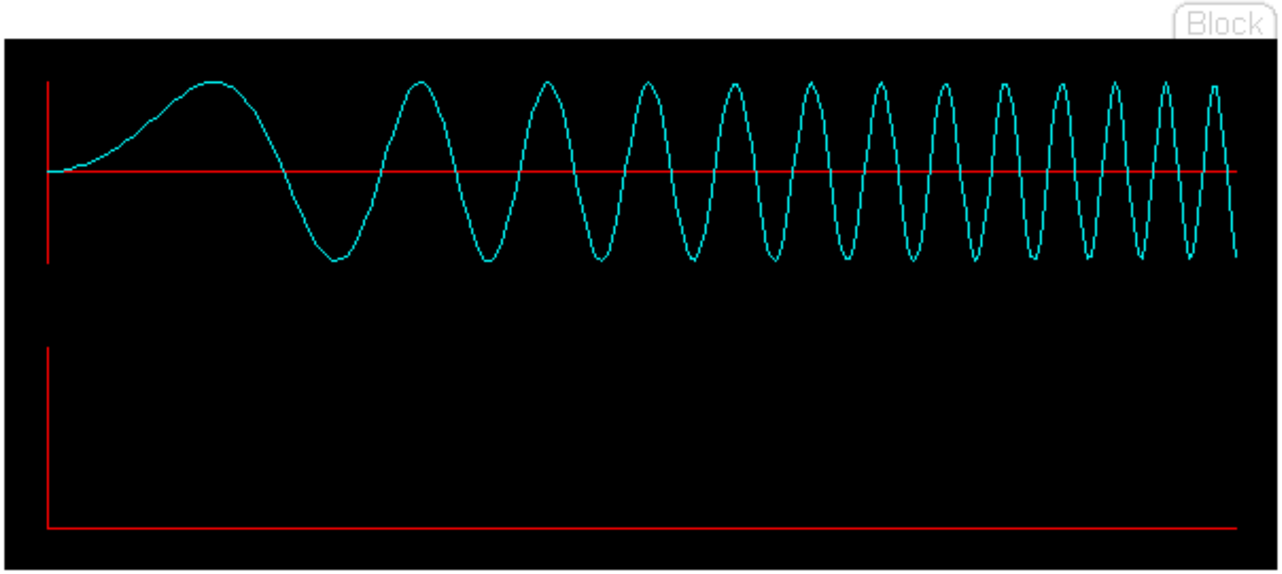
# Chirp



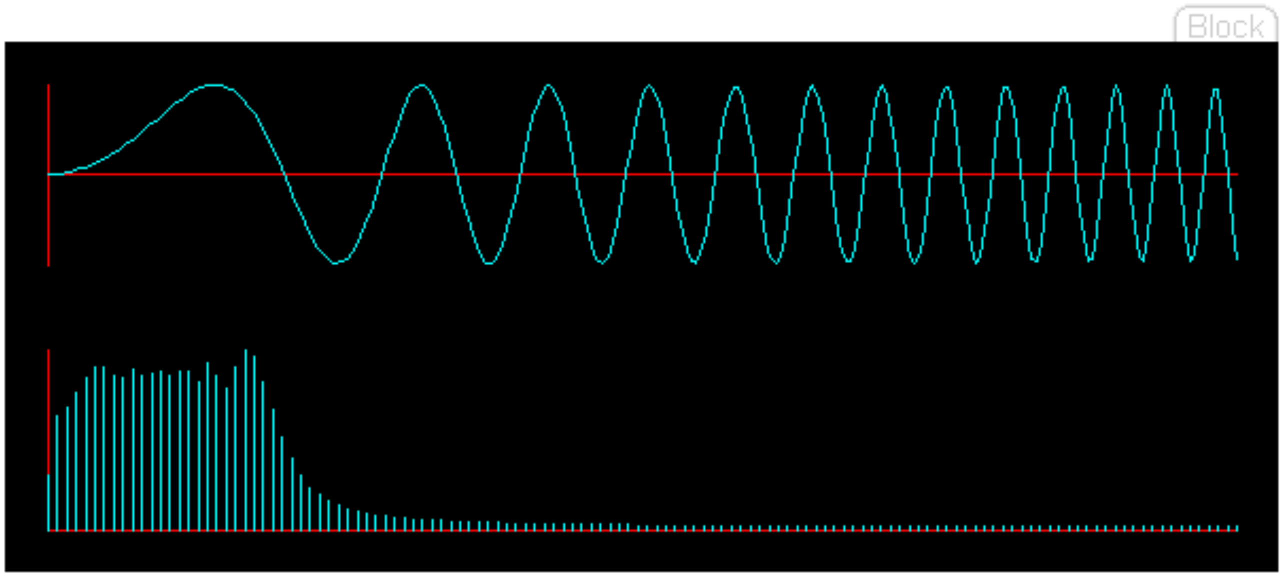

Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(25000*pi*t*t)

Plot signal | Plot spectrum

# Chirp




Number of samples: 256

Sampling rate: 8000 samples / s

Signal waveform expression: sin(25000*pi*t*t)

Plot signal | Plot spectrum

# DSP - Detailed outline

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# Properties

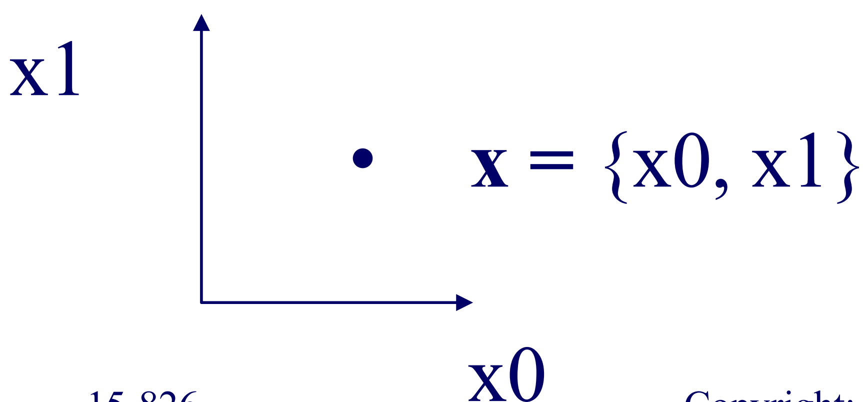
- Time shift sounds the same
  - Changes only phase, not amplitudes
- Sawtooth has almost all frequencies
  - With decreasing amplitude
- Spike has all frequencies

# DFT: Parseval's theorem

$$\text{sum}( x_t^{\,2} ) = \text{sum} ( | X_f |^2 )$$

Ie., DFT preserves the 'energy'

or, alternatively: it does an axis rotation:



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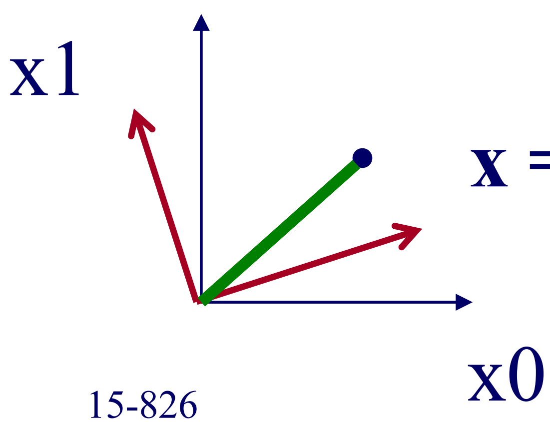


**x** = {x0, x1}

# DFT: Parseval's theorem

$$\text{sum}(x_t^{\,2}) = \text{sum}(|X_f|^2)$$

… equivalently,

matrix **F** (= $\left[\frac{1}{\sqrt{n}} e^{-j2\pi f t}\right]$)

is row-orthonormal

Row: *f*

Column: *t*

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_f \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-j2\pi f t} \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_t \\ \vdots \end{bmatrix}$$

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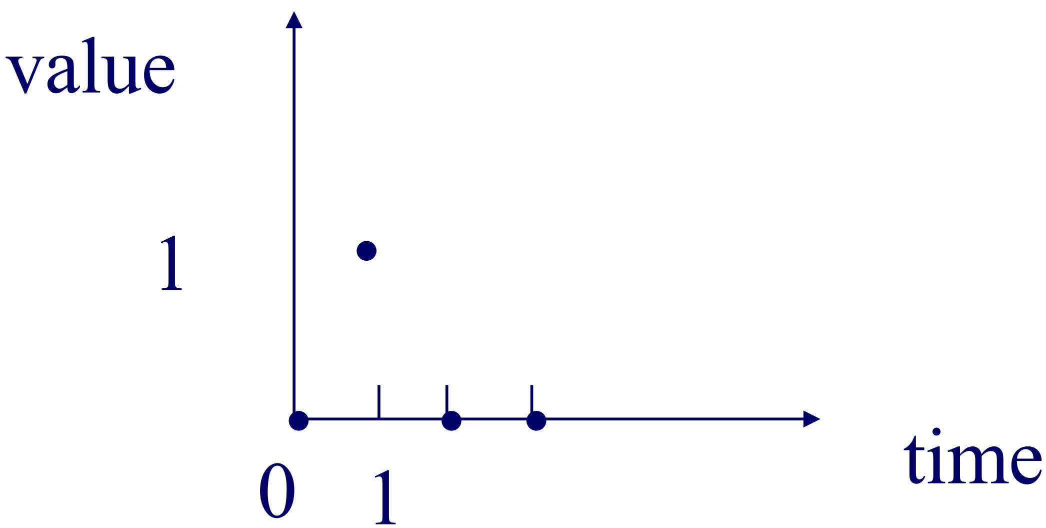
details

# Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- $X_0$=?



details

# Arithmetic examples

- Impulse function: $\mathbf{x} = \{ 0, 1, 0, 0\}$ ($n = 4$)
- $X_0$=?
- A: $X_0$ = 1/sqrt(4) * 1* exp(-j 2 π 0 / n ) = 1/2
- $X_1$=?
- $X_2$=?
- $X_3$=?

details

# Arithmetic examples

- Impulse function: $\mathbf{x}$ = { 0, 1, 0, 0} ($n$ = 4)
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- $X_1$= -1/2 j
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- $X_3$= + 1/2 j
- Q: does the 'symmetry' property hold?

details

# Arithmetic examples

- Impulse function: $\mathbf{x}$ = { 0, 1, 0, 0} ($n$ = 4)
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- $X_3$= + 1/2 j
- Q: does the 'symmetry' property hold?
- A: Yes (of course)

details

# Arithmetic examples

- Impulse function: **x** = { 0, 1, 0, 0} (*n* = 4)
- $X_0$=?
- A: $X_0$ = 1/sqrt(4) * 1* exp(-j 2 π 0 / n ) = 1/2
- $X_1$= -1/2 j
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- $X_3$= + 1/2 j
- Q: check Parseval's theorem

  

details

# Arithmetic examples

- Impulse function: **x** = { 0, 1, 0, 0} ($n$ = 4)
- $X_0$=?
- A: $X_0$ = 1/sqrt(4) * 1* exp(-j 2 π 0 / n ) = 1/2
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- Impulse function: $\mathbf{x}$ = { 0, 1, 0, 0} ($n$ = 4)
- $X_0$=?
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- $X_1$= -1/2 j
- $X_2$= - 1/2
- $X_3$= + 1/2 j
- Q: (Amplitude) spectrum?
- A: FLAT!

  

# Arithmetic examples

- Q: What does this mean?

# Arithmetic examples

- Q: What does this mean?
- A: All frequencies are equally important ->
  - we need *n* numbers in the frequency domain to represent just one non-zero number in the time domain!
  - "*frequency leak*"

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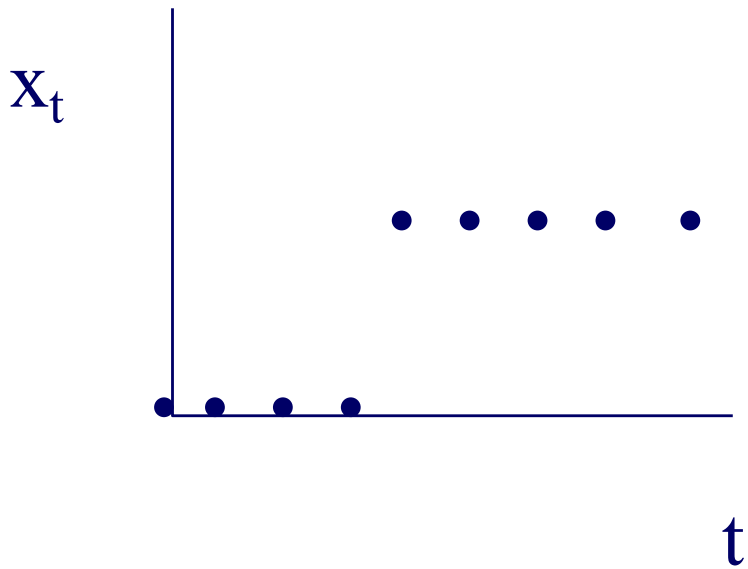
# Observations

- DFT of 'step' function:

  x = { 0, 0, ..., 0, 1, 1, ... 1}



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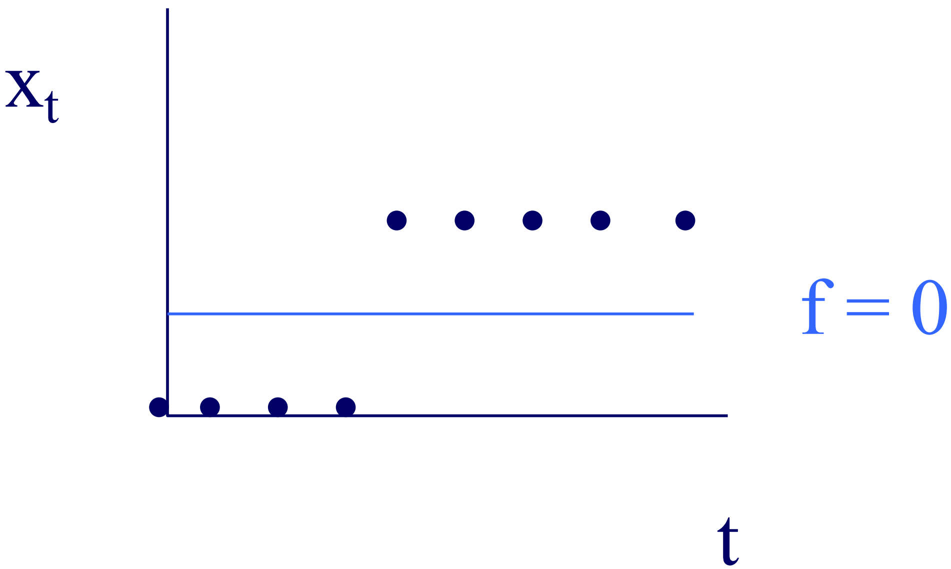

# Observations

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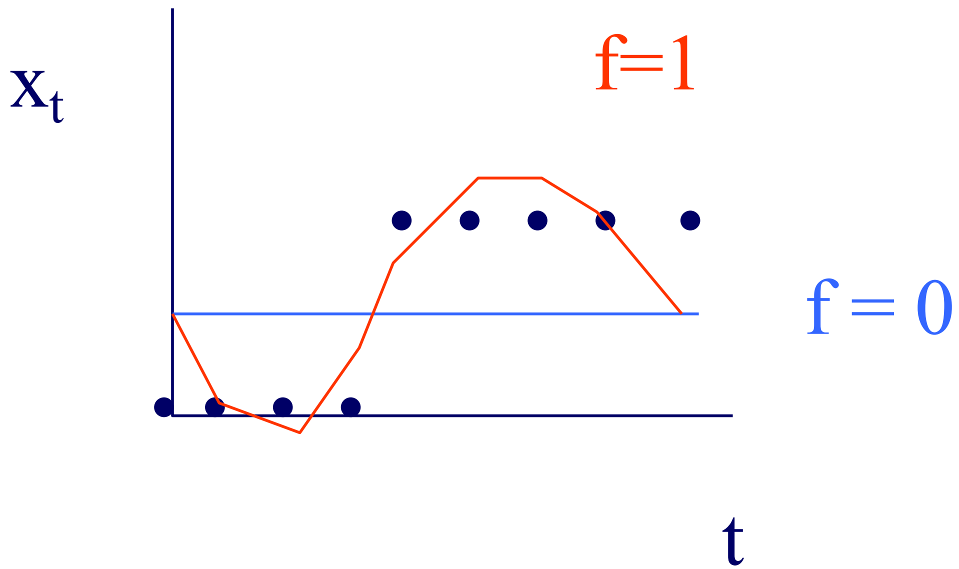
  $x = \{ 0, 0, ..., 0, 1, 1, ... 1\}$



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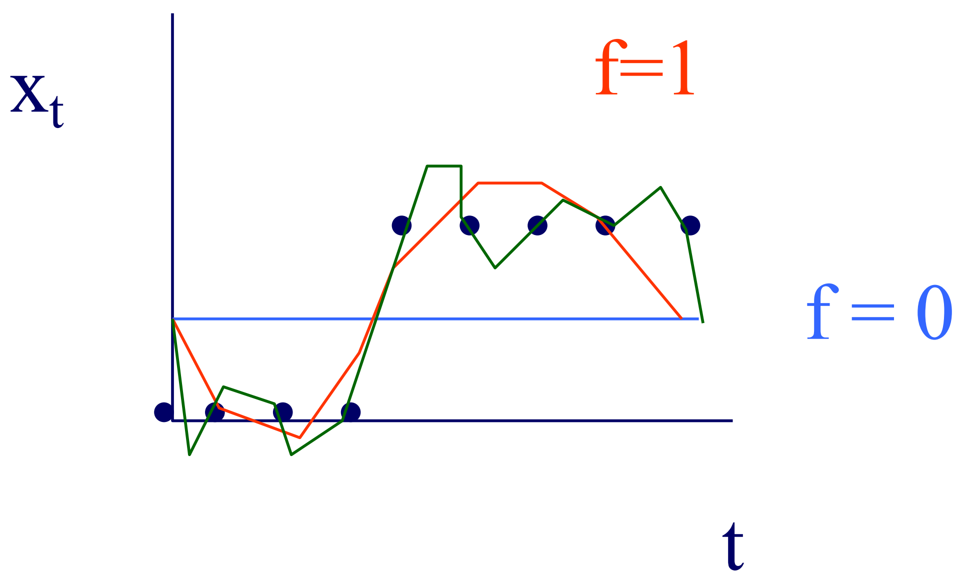
- DFT of 'step' function:

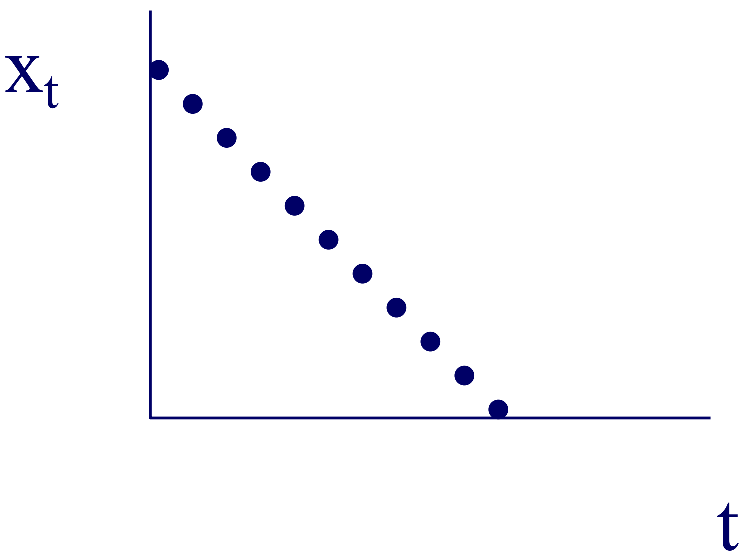
  x = { 0, 0, ..., 0, 1, 1, ... 1}




- the more frequencies, the better the approx.
- 'ringing' becomes worse
- reason: discontinuities; trends

# Observations

- Ringing for trends: because DFT 'sub-consciously' replicates the signal



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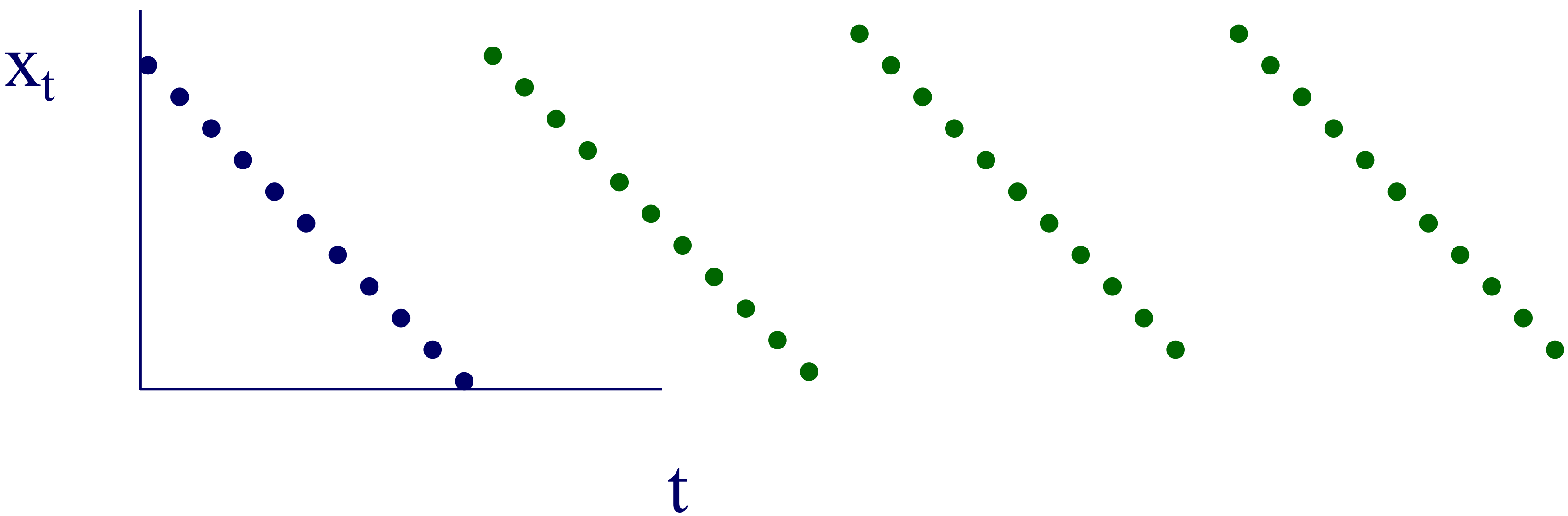

# Observations

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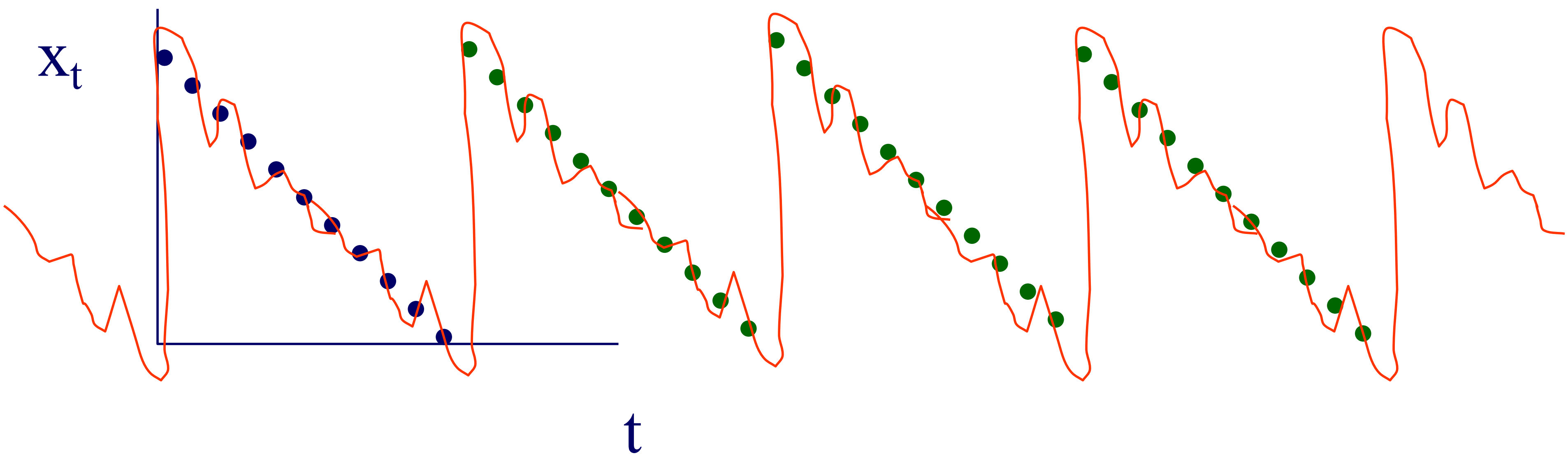
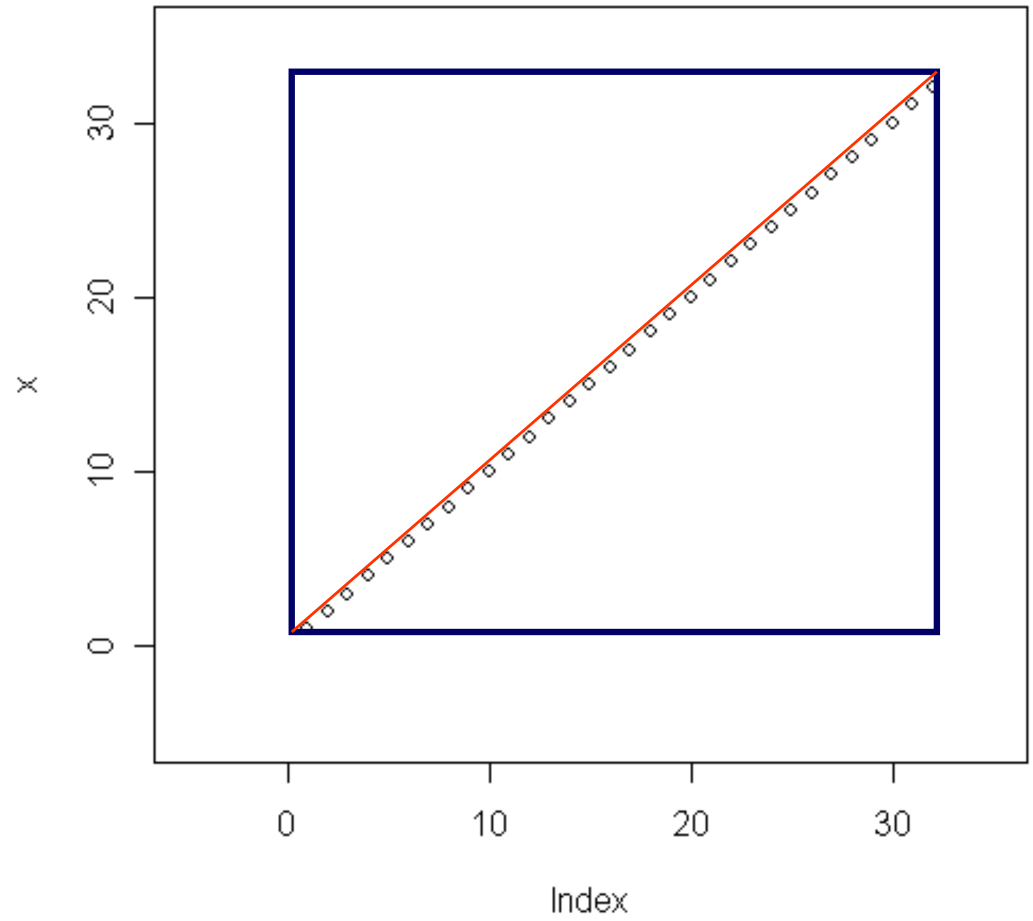


original



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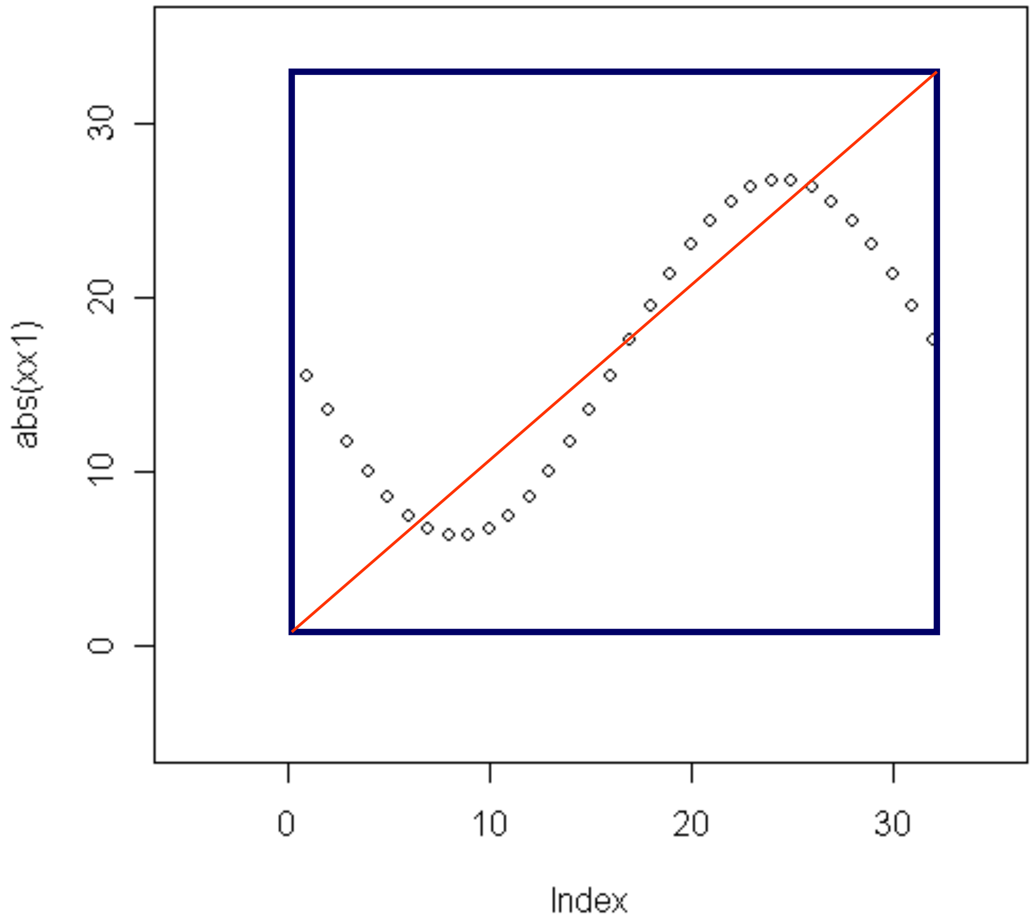
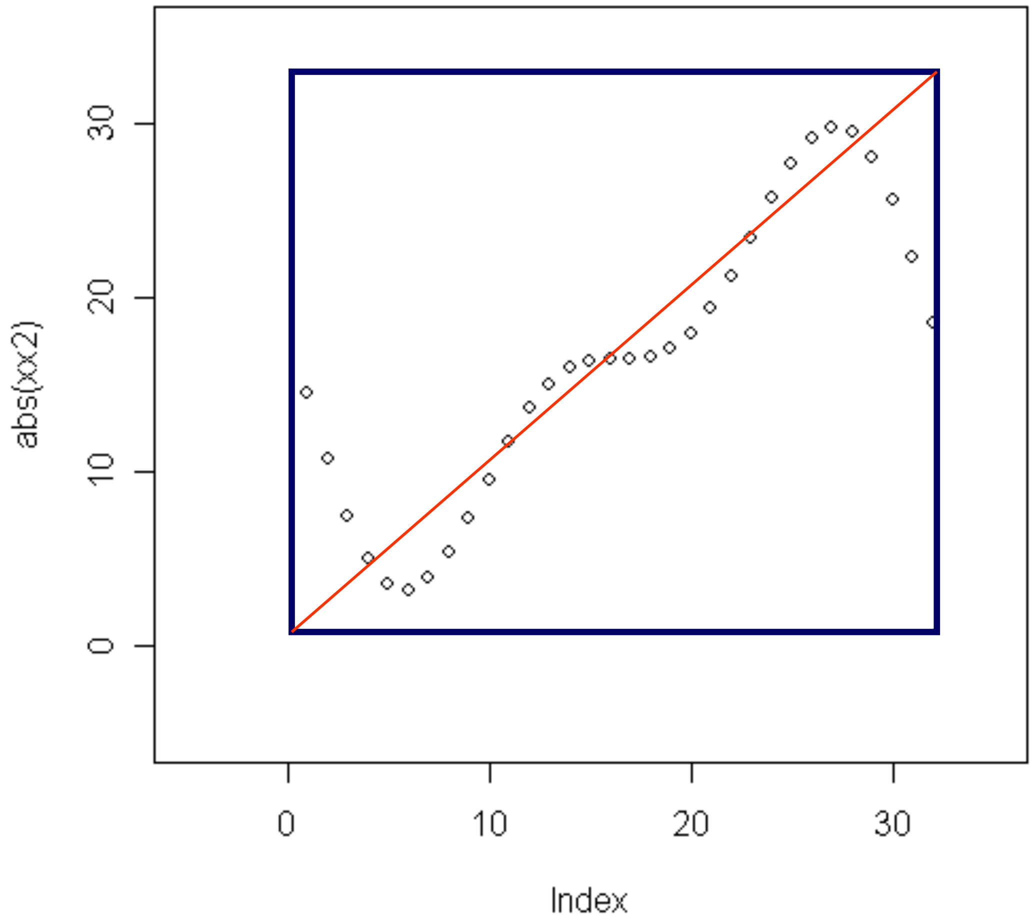
# DC and 1st



DC and 1st

And 2nd



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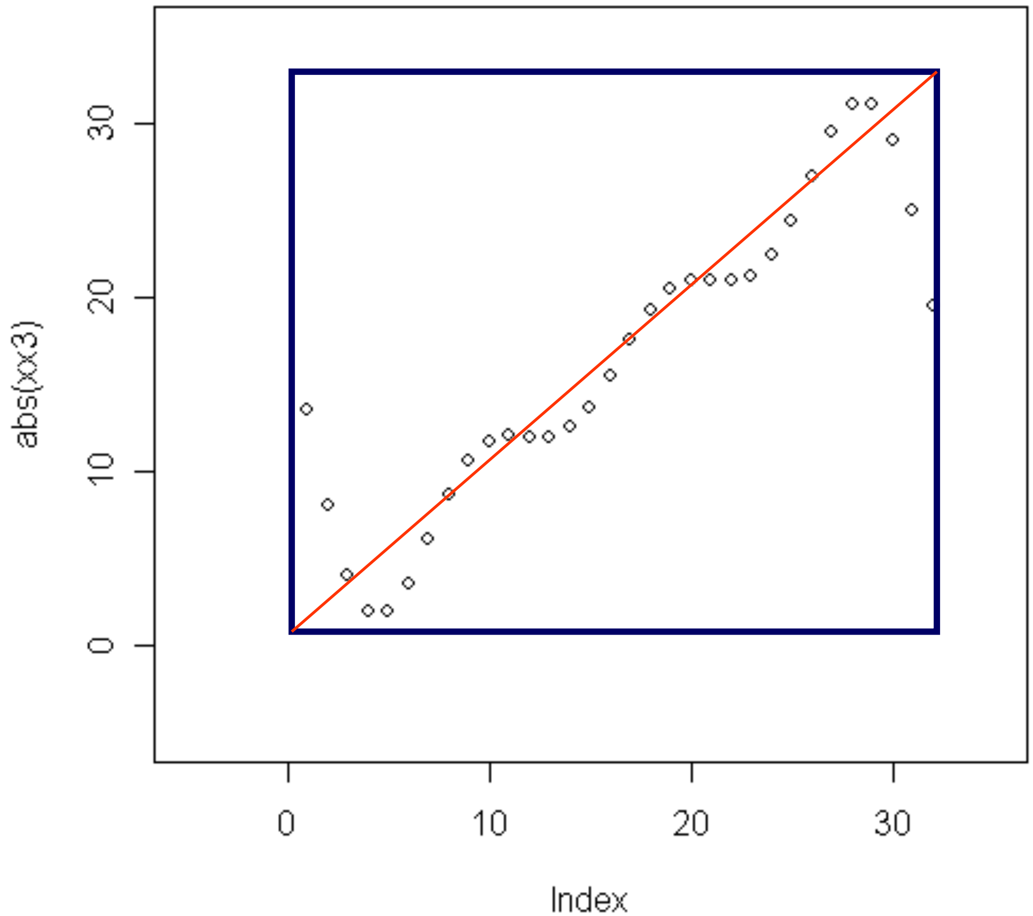
DC and 1st

And 2nd

And 3rd

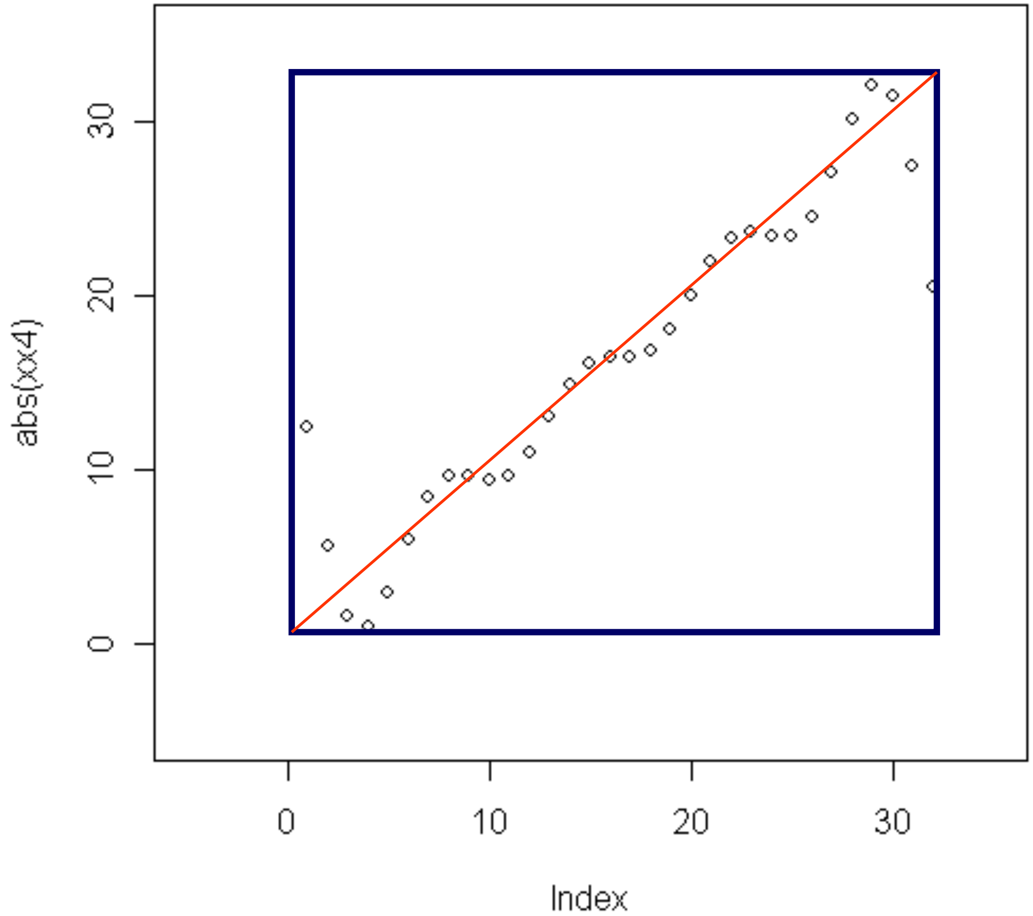


DC and 1st

And 2nd

And 3rd

And 4th



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# Observations

- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin( 2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: $X_0 = ?$
- Q: $X_1 = ?$
- Q: $X_2 = ?$
- Q: $X_3 = ?$

# Observations

- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin( 2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: $X_0 = 0$
- Q: $X_1 = -3 j$
- Q: $X_2 = 0$
- Q: $X_3 = 3j$

•check 'symmetry'
•check Parseval

# Observations

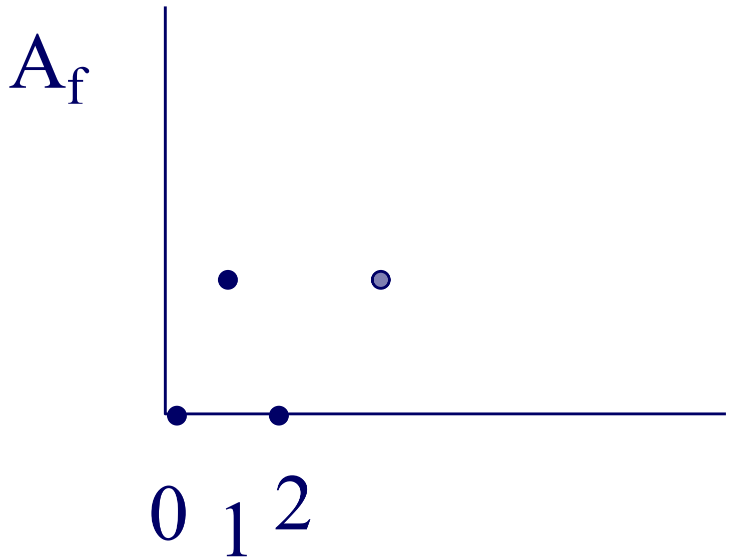
- Q: DFT of a sinusoid, eg.

$$x_t = 3 \sin( 2 \pi / 4 t)$$

(t = 0, ... , 3)

- Q: $X_0 = 0$
- Q: $X_1 = -3 j$
- Q: $X_2 = 0$
- Q: $X_3 = 3j$

•Does this make sense?



# Property

- Shifting **x** in time does NOT change the amplitude spectrum
- eg., **x** = { 0 0 0 1} and **x'** = { 0 1 0 0 }: same (flat) amplitude spectrum
- (only the phase spectrum changes)
- Useful property when we search for patterns that may 'slide'

# Summary of properties

- Spike in time: -> all frequencies
- Step/Trend: -> ringing (~ all frequencies)
- Single/dominant sinusoid: -> spike in spectrum
- Time shift -> same amplitude spectrum

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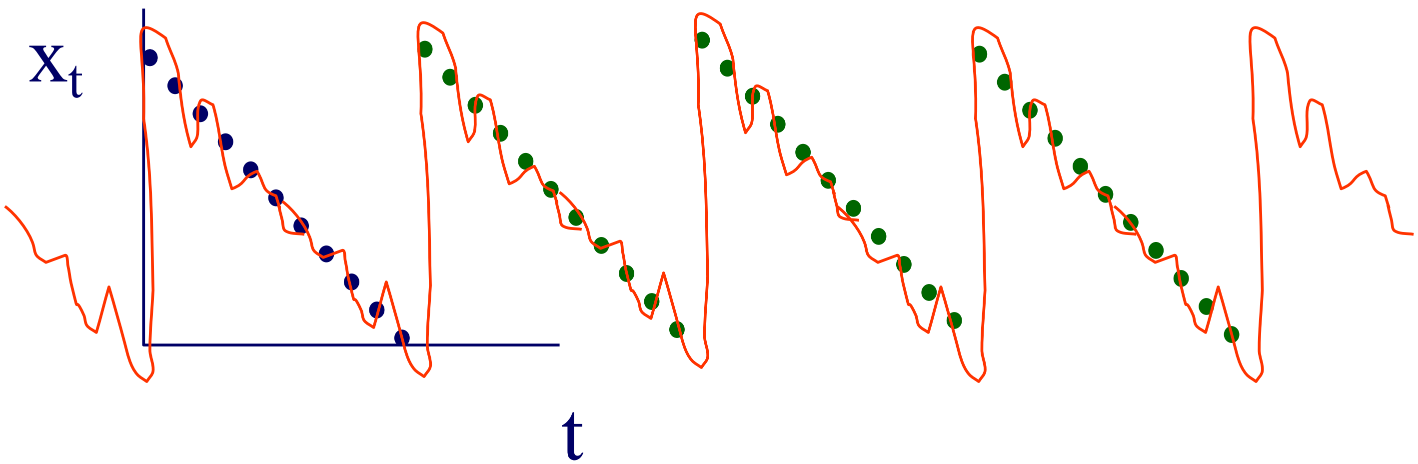
details

# DCT

Discrete Cosine Transform

- motivation#1: DFT gives complex numbers
- motivation#2: how to avoid the 'frequency leak' of DFT on trends?

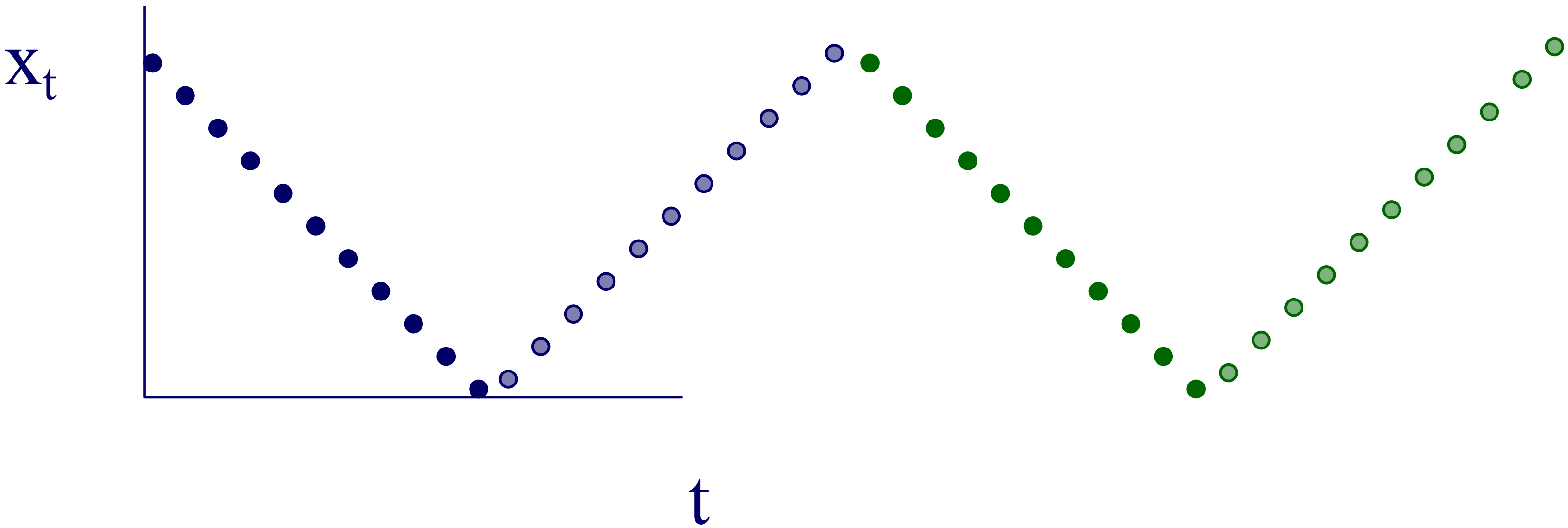


details

# DCT

- brilliant solution to both problems: mirror the sequence, do DFT, and drop the redundant entries!



details

# DCT

- (see Numerical Recipes for exact formulas)

# DCT - properties

- it gives real numbers as the result
- it has no problems with trends
- it is very good when $x_t$ and $x_{(t+1)}$ are correlated

(thus, is used in JPEG, for image compression)

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# 2-d DFT

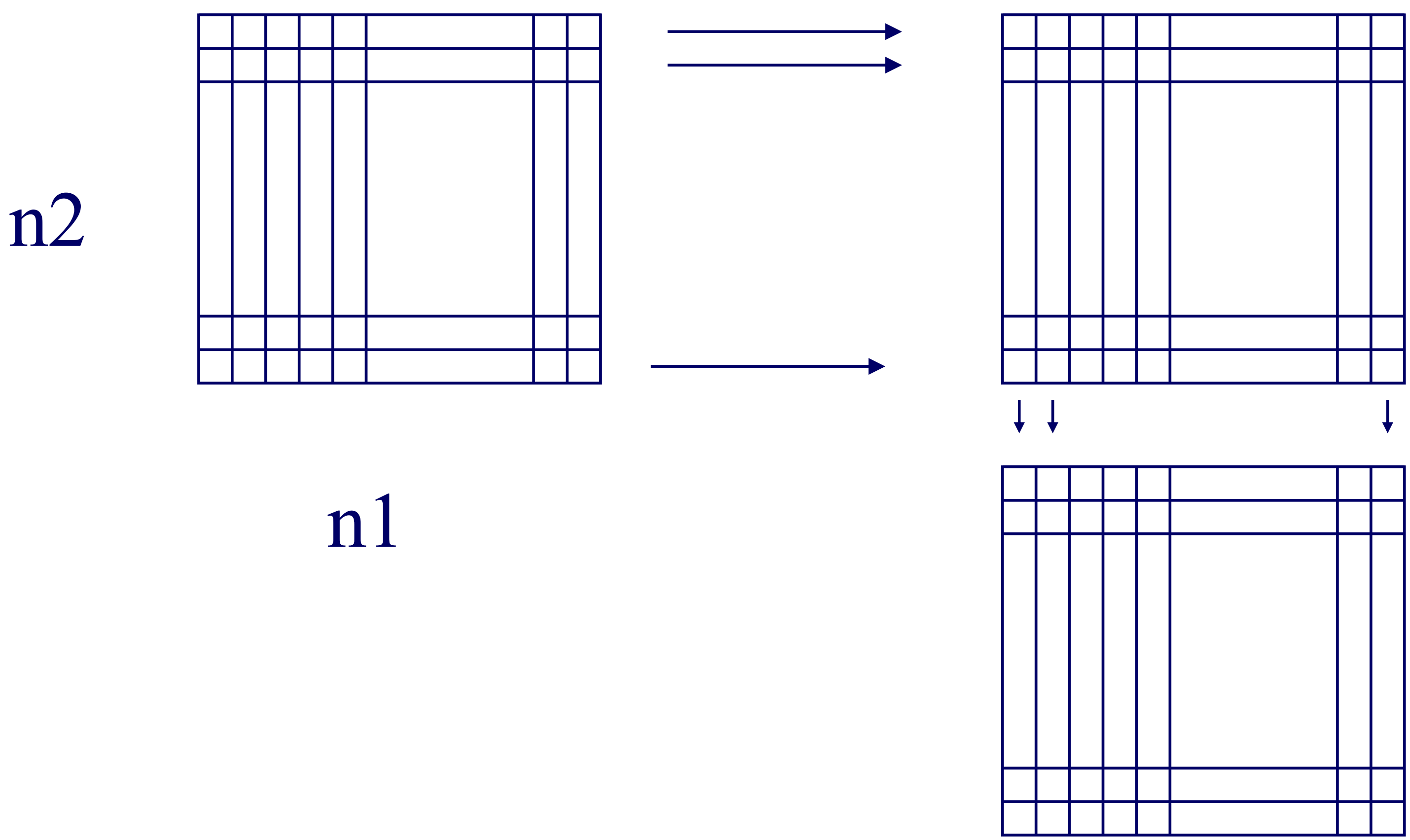
- Definition:

$$X_{f_1,f_2} = \frac{1}{\sqrt{n_1}} \frac{1}{\sqrt{n_2}} \sum_{i_1=0}^{n_1-1} \sum_{i_1=0}^{n_2-1} x_{i_1,i_2} \exp(-2\pi j i_1 f_1 / n_1) \exp(-2\pi j i_2 f_2 / n_2)$$

# 2-d DFT

- Intuition:

do 1-d DFT on each row




and then 1-d DFT on each column

# 2-d DFT

- Quiz: how do the basis functions look like?
- for f1= f2 =0
- for f1=1, f2=0
- for f1=1, f2=1

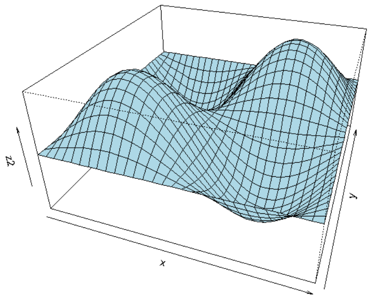
# 2-d DFT

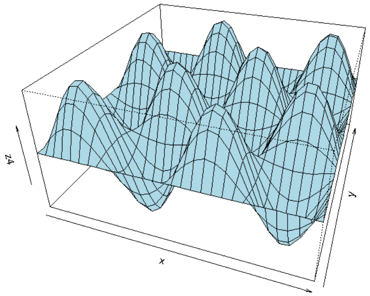
- Quiz: how do the basis functions look like?
- for f1= f2 =0 flat
- for f1=1, f2=0 wave on x; flat on y
- for f1=1, f2=1 ~ egg-carton

# 2-d DFT

- Quiz: how do the basis functions look like?
- for f1= f2 =0 flat
- for f1=1, f2=0 wave on x; flat on y
- for f1=1, f2=1 ~ egg-carton



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# FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j2\pi\, tf / n)$$

details

# FFT

- What is the complexity of DFT?

$$X_f = 1/\sqrt{n} \sum_{t=0}^{n-1} x_t * \exp(-j 2\pi \, tf / n)$$

- A: Naively, O($n^2$)

details

# FFT

- However, if $n$ is a power of 2 (or a number with many divisors), we can make it

$$O(n \log n)$$

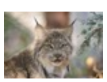
Main idea: if we know the DFT of the odd time-ticks, and of the even time-ticks, we can quickly compute the whole DFT

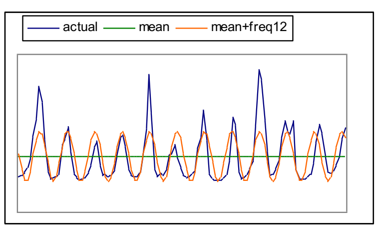
Details: in Num. Recipes

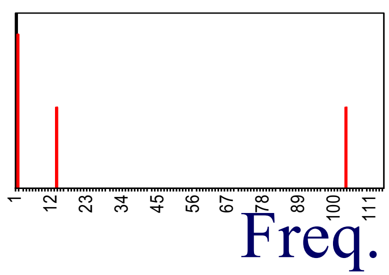
  

# DFT - Conclusions

- It spots periodicities (with the ‘**amplitude spectrum**’)
- can be quickly computed (O( *n* log *n*)), thanks to the FFT algorithm.
- **standard** tool in signal processing (speech, image etc signals)





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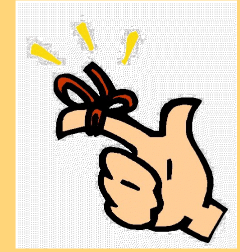
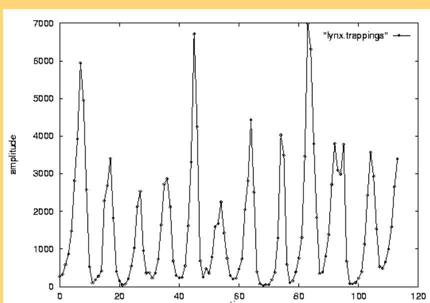


# Solutions:

Goal: given a signal (eg., sales over time and/or space)

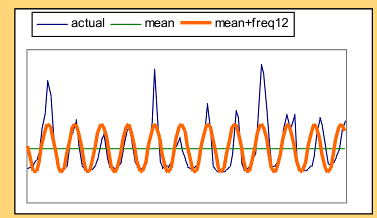
Q: Find patterns and/or compress

count




year

✔A1: Fourier (DFT)




A2: Wavelets (DWT)