Constructive Logic (15-317), Fall 2024
Assignment 6: Cut Admissibiity

Constructive Logic Staff
(Instructor: Karl Crary)

Due: Wednesday, October 9, 2024, 11:59 pm

This assignment is written only. Please submit a file named "`hw.pdf`" to "Homework 6."

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. (Please put each task on its own page to speed up grading.) If you choose not to typeset your solutions, be aware that you are answerable for your handwriting. Any that the grader has difficulty reading (in the sole judgement of the grader), will be marked wrong.

1 Cut for a New Connective

Recall the $\heartsuit$ connective from HW3:

$$\frac{\begin{array}{cc} [A\ \mathsf{true}]_v & [A\ \mathsf{true}]_w \\ \vdots & \vdots \\ B\ \mathsf{true} & C\ \mathsf{true} \end{array}}{\heartsuit(A,B,C)\ \mathsf{true}}\ \heartsuit I^{v,w}$$

$$\frac{\heartsuit(A,B,C)\ \mathsf{true} \quad A\ \mathsf{true} \quad \begin{array}{c}[B\ \mathsf{true}]_u \\ \vdots \\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\ \heartsuit E1^u \qquad \frac{\heartsuit(A,B,C)\ \mathsf{true} \quad A\ \mathsf{true} \quad \begin{array}{c}[C\ \mathsf{true}]_u \\ \vdots \\ D\ \mathsf{true}\end{array}}{D\ \mathsf{true}}\ \heartsuit E2^u$$

In sequent calculus, $\heartsuit$ would have the following rules:

$$\frac{\Delta, A \Longrightarrow B \quad \Delta, A \Longrightarrow C}{\Delta \Longrightarrow \heartsuit(A,B,C)}\ \heartsuit R \qquad \frac{\Delta, \heartsuit(A,B,C) \Longrightarrow A \quad \Delta, \heartsuit(A,B,C), B \Longrightarrow D}{\Delta, \heartsuit(A,B,C) \Longrightarrow D}\ \heartsuit L1$$

$$\frac{\Delta, \heartsuit(A,B,C) \Longrightarrow A \quad \Delta, \heartsuit(A,B,C), C \Longrightarrow D}{\Delta, \heartsuit(A,B,C) \Longrightarrow D}\ \heartsuit L2$$

Task 1 (5 points). If we wanted to add this connective (and the sequent calculus rules describing it above) to our logic, what would be the additional cases we have to prove for cut to remain admissible? Please list the additional cut cases that would need to be added to the cut proof from class. For each of these cases, specify A, $\mathcal{D}$ and $\mathcal{E}$ where A is the cut formula. Use as few distinct cases as possible. For example, when the final rule of $\mathcal{D}$ is *init*, it does not matter what $\mathcal{E}$'s final rule is, so it would be listed as one case, not as several cases. You do not need to prove any of these additional cases; you need only list them.

2   Cut Admissibility

Extend the proof of the admissibility of cut from class by filling in the following inductive cases. A valid proof should include a detailed English explanation—not only notation. Make sure to cite weakening explicitly if it is used, and to justify any use of the induction hypothesis.

Task 2 (10 points). $\mathcal{D}$ ends in $\vee R2$ and $\mathcal{E}$ ends in $\vee L$, where $\vee L$ operates on the cut formula.

Task 3 (10 points). $\mathcal{E}$ ends in $\wedge R$.

Task 4 (2 points).   What category of cuts do the cases in tasks 2 and 3 fall into?