# Constructive Logic (15-317), Fall 2024
# Assignment 8: Inversion and Classical Logic

Constructive Logic Staff
(Instructor: Karl Crary)

Due: Wednesday, October 30, 2024, 11:59 pm

This assignment will have a written portion and a coding portion. You will submit both portions through Gradescope, to the assignments labelled "Homework 8 (written)" and "Homework 8 (code)." Please submit a file named "`hw.pdf`" to the former, and a file named "`hw.deriv`" to the latter.

We recommend that you typeset your written solutions. Most students use LaTeX, but other software is acceptable. (Please put each task on its own page to speed up grading.) If you choose not to typeset your solutions, be aware that you are answerable for your handwriting. Any that the grader has difficulty reading (in the sole judgement of the grader), will be marked wrong.

For the coding portion you will use Dcheck. You can find documentation on Dcheck at `cs.cmu.edu/~crary/dcheck/dcheck.pdf` and a sample file at `cs.cmu.edu/~crary/dcheck/example.deriv`. (Be aware that the sample file uses several logics that we have not seen yet in class.)

## 1 Inversion

For each of the following rules, indicate whether it is invertible or not. If it is invertible, prove it. If not, give an instance of the rule (*i.e.,* fill in the metavariables) so that conclusion of the rule is derivable, but at least one of the premises is not derivable. If there is more that one premise, indicate which one is not derivable. You do not need to give a derivation of the conclusion, nor do you need to prove the premise is not derivable.

In your proofs pertaining to sequent calculus, feel free to use cut admissibility, identity, weakening, and/or contraction, as necessary.

**Task 1** (10 points).

$$\frac{\Gamma \vdash A\ \mathsf{true} \quad \Gamma \vdash B\ \mathsf{true}}{\Gamma \vdash A \wedge B\ \mathsf{true}}\ \wedge I$$

**Task 2** (10 points).

$$\frac{\Gamma \vdash A\ \mathsf{true}}{\Gamma \vdash A \vee B\ \mathsf{true}}\ \vee I1$$

**Task 3** (10 points).

$$\frac{}{\Gamma, A\ \mathsf{true} \vdash A\ \mathsf{true}}\ hyp$$

**Task 4** (10 points).

$$\frac{A\ \mathsf{true} \in \Gamma}{\Gamma \vdash A\ \mathsf{true}}\ hyp$$

**Task 5** (10 points).

$$\frac{\Delta, A \longrightarrow B}{\Delta \longrightarrow A \supset B}\ {\supset}R$$

**Task 6** (10 points).

$$\frac{\Delta, A \supset B \longrightarrow A \quad \Delta, B \longrightarrow C}{\Delta, A \supset B \longrightarrow C}\ {\supset}L$$

## 2 DeMorgan's Revenge

Provide derivations of the following Classical Logic judgements using Dcheck. Remember that negation ("`~ P`") is primitive in classical logic, not defined in terms of implication and false.

**Task 7** (6 points). Define a derivation named `task2` that derives:

$$\neg(A \wedge B) \supset (\neg A \vee \neg B)\ \mathsf{true}$$

**Task 8** (6 points). Define a derivation named `task3` that derives:

$$(A \supset B) \supset (\neg A \vee B)\ \mathsf{true}$$

Note that neither of these are constructively true in general.

## 3 Classical Quantifiers

We can extend classical logic with universal and existential quantifiers by adding the following truth and falsity rules:

$$\frac{\begin{array}{c}[a:\tau] \\ \vdots \\ A(a)\ \mathsf{true}\end{array}}{\forall x{:}\tau.\, A(x)\ \mathsf{true}}\ \forall T^{a} \qquad \frac{m:\tau \quad A(m)\ \mathsf{false}}{\forall x{:}\tau.\, A(x)\ \mathsf{false}}\ \forall F$$

$$\frac{m:\tau \quad A(m)\ \mathsf{true}}{\exists x{:}\tau.\, A(x)\ \mathsf{true}}\ \exists T \qquad \frac{\begin{array}{c}[a:\tau] \\ \vdots \\ A(a)\ \mathsf{false}\end{array}}{\exists x{:}\tau.\, A(x)\ \mathsf{false}}\ \exists F^{a}$$

Note the duality between the $\forall$ and $\exists$.

**Task 9** (16 points). Using these rules (and the other rules of classical logic as necessary), show that the usual *elimination* rules for the universal and existential quantifiers are *derivable*. For reference, those rules are:

$$\frac{\forall x{:}\tau.\, A(x)\ \mathsf{true} \quad m:\tau}{A(m)\ \mathsf{true}}\ \forall E \qquad \frac{\exists x{:}\tau.\, A(x)\ \mathsf{true} \quad \begin{array}{c}[a:\tau] \quad [A(a)\ \mathsf{true}]_u \\ \vdots \\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \exists E^{a,u}$$

# 4 Sequent Calculus Mastery

**Task 10** (5 points). Using Dcheck, give a derivation of the following sequent, using the original sequent calculus (*i.e.,* not the reduced sequent calculus).

$$\Longrightarrow \neg P \vee \neg Q \supset \neg(P \wedge Q)$$

Name your derivation `task10`. (Remember that Dcheck takes the propositions `P` and `Q` to be atomic.) **Instant feedback is turned off for this task, so be extra careful.**

Since instant feedback is turned off, **pay special care to the assumption numbers**. Dcheck numbers assumptions from right to left, starting at zero. Thus, the rightmost assumption is numbered 0. No partial credit will be granted for submissions that would be correct except for assumption numbers.