

Inference Rules for Constructive Logic

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Natural Deduction

Conjunction

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I$$

$$\frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E1$$

$$\frac{A \wedge B \text{ true}}{B \text{ true}} \wedge E2$$

Disjunction

$$\frac{A \text{ true}}{A \vee B \text{ true}} \vee I1$$

$$\frac{B \text{ true}}{A \vee B \text{ true}} \vee I2$$

$$\frac{A \vee B \text{ true} \quad \begin{array}{c} [A \text{ true}]_u \\ \vdots \\ C \text{ true} \end{array} \quad \begin{array}{c} [B \text{ true}]_v \\ \vdots \\ C \text{ true} \end{array}}{C \text{ true}} \vee E^{u,v}$$

Implication

$$\frac{\begin{array}{c} [A \text{ true}]_u \\ \vdots \\ B \text{ true} \end{array}}{A \supset B \text{ true}} \supset I^u$$

$$\frac{A \supset B \text{ true} \quad A \text{ true}}{B \text{ true}} \supset E$$

Truth and Falsehood

$$\frac{}{T \text{ true}} TI$$

$$\frac{F \text{ true}}{C \text{ true}} FE$$

Natural Deduction with Contexts

Hypothesis

$$\frac{A \text{ true} \in \Gamma}{\Gamma \vdash A \text{ true}} \text{ hyp}$$

Conjunction

$$\frac{\Gamma \vdash A \text{ true} \quad \Gamma \vdash B \text{ true}}{\Gamma \vdash A \wedge B \text{ true}} \wedge I \qquad \frac{\Gamma \vdash A \wedge B \text{ true}}{\Gamma \vdash A \text{ true}} \wedge E1 \qquad \frac{\Gamma \vdash A \wedge B \text{ true}}{\Gamma \vdash B \text{ true}} \wedge E2$$

Disjunction

$$\frac{\Gamma \vdash A \text{ true}}{\Gamma \vdash A \vee B \text{ true}} \vee I1 \qquad \frac{\Gamma \vdash B \text{ true}}{\Gamma \vdash A \vee B \text{ true}} \vee I2$$
$$\frac{\Gamma \vdash A \vee B \text{ true} \quad \Gamma, A \text{ true} \vdash C \text{ true} \quad \Gamma, B \text{ true} \vdash C \text{ true}}{\Gamma \vdash C \text{ true}} \vee E$$

Implication

$$\frac{\Gamma, A \text{ true} \vdash B \text{ true}}{\Gamma \vdash A \supset B \text{ true}} \supset I \qquad \frac{\Gamma \vdash A \supset B \text{ true} \quad \Gamma \vdash A \text{ true}}{\Gamma \vdash B \text{ true}} \supset E$$

Truth and Falsehood

$$\frac{}{\Gamma \vdash T \text{ true}} TI \qquad \frac{\Gamma \vdash F \text{ true}}{\Gamma \vdash C \text{ true}} FE$$

Proof Terms

Conjunction

$$\frac{M : A \quad N : B}{\langle M, N \rangle : A \wedge B} \wedge I$$

$$\frac{M : A \wedge B}{\text{fst } M : A} \wedge E1$$

$$\frac{M : A \wedge B}{\text{snd } M : B} \wedge E2$$

Disjunction

$$\frac{M : A}{\text{inl } M : A \vee B} \vee I1$$

$$\frac{M : B}{\text{inr } M : A \vee B} \vee I2$$

$$\frac{\begin{array}{c} [x : A] \quad [y : B] \\ \vdots \quad \vdots \\ M : A \vee B \quad M_1 : C \quad M_2 : C \end{array}}{\text{case}(M, x.M_1, y.M_2) : C} \vee E$$

Implication

$$\frac{\begin{array}{c} [x : A] \\ \vdots \\ M : B \end{array}}{\lambda x. M : A \supset B} \supset I$$

$$\frac{M : A \supset B \quad N : A}{MN : B} \supset E$$

Truth and Falsehood

$$\frac{}{\langle \rangle : T} TI$$

$$\frac{M : F}{\text{abort } M : C} FE$$

Verifications and Uses

Conjunction

$$\frac{A \uparrow \quad B \uparrow}{A \wedge B \uparrow} \wedge \uparrow$$

$$\frac{A \wedge B \downarrow}{A \downarrow} \wedge \downarrow 1$$

$$\frac{A \wedge B \downarrow}{B \downarrow} \wedge \downarrow 2$$

Disjunction

$$\frac{A \uparrow}{A \vee B \uparrow} \vee \uparrow 1$$

$$\frac{B \uparrow}{A \vee B \uparrow} \vee \uparrow 2$$

$$\frac{A \vee B \downarrow \quad \begin{array}{c} [A \downarrow]_u \\ \vdots \\ C \uparrow \end{array} \quad \begin{array}{c} [B \downarrow]_v \\ \vdots \\ C \uparrow \end{array}}{C \uparrow} \vee \downarrow^{u,v}$$

Implication

$$\frac{\begin{array}{c} [A \downarrow]_u \\ \vdots \\ B \uparrow \end{array}}{A \supset B \uparrow} \supset \uparrow^u$$

$$\frac{A \supset B \downarrow \quad A \uparrow}{B \downarrow} \supset \downarrow$$

Truth and Falsehood

$$\overline{T} \uparrow \quad T \uparrow$$

$$\frac{F \downarrow}{C \uparrow} \quad F \downarrow$$

Shift

$$\frac{A \downarrow \quad A \text{ atomic}}{A \uparrow} \downarrow \uparrow$$

Quantification

Universal Quantification

$$\frac{[a : \tau] \quad \vdots \quad A(a) \text{ true}}{\forall x:\tau. A(x) \text{ true}} \forall I^a \qquad \frac{\forall x:\tau. A(x) \text{ true} \quad m : \tau}{A(m) \text{ true}} \forall E$$

Existential Quantification

$$\frac{m : \tau \quad A(m) \text{ true}}{\exists x:\tau. A(x) \text{ true}} \exists I \qquad \frac{\exists x:\tau. A(x) \text{ true} \quad \begin{array}{c} [a : \tau] \quad [A(a) \text{ true}]_u \\ \vdots \\ C \text{ true} \end{array}}{C \text{ true}} \exists E^{a,u}$$

Heyting Arithmetic

Natural Numbers

$$\overline{0 : \text{nat}} \text{ nat}I_0 \quad \frac{n : \text{nat}}{s n : \text{nat}} \text{ nat}I_s \quad \frac{n : \text{nat} \quad C(0) \text{ true} \quad \begin{array}{c} [x : \text{nat}] \quad [C(x) \text{ true}]_u \\ \vdots \\ C(s x) \text{ true} \end{array}}{C(n) \text{ true}} \text{ nat}E^{x,u}$$

Equality

$$\overline{0 = 0 \text{ true}} = I_{00} \quad \frac{m = n \text{ true}}{s m = s n \text{ true}} = I_{ss} \\ \frac{s m = s n \text{ true}}{m = n \text{ true}} = E_{ss} \quad \frac{0 = s n \text{ true}}{C \text{ true}} = E_{0s} \quad \frac{s n = 0 \text{ true}}{C \text{ true}} = E_{s0}$$

Primitive Recursion

$$\frac{n : \text{nat} \quad t_0 : \tau \quad \begin{array}{c} [x : \text{nat}] \quad [r : \tau] \\ \vdots \\ t_s : \tau \end{array}}{R(n, t_0, x r.t_s) : \tau} \text{ nat}E^{x,r} \quad \begin{array}{l} R(0, t_0, x r.t_s) = t_0 \\ R(s n, t_0, x r.t_s) = [R(n, t_0, x r.t_s)/r][n/x]t_s \end{array}$$

Sequent Calculus

Conjunction

$$\frac{\Delta \Rightarrow A \quad \Delta \Rightarrow B}{\Delta \Rightarrow A \wedge B} \wedge R \quad \frac{\Delta, A \wedge B, A \Rightarrow C}{\Delta, A \wedge B \Rightarrow C} \wedge L1 \quad \frac{\Delta, A \wedge B, B \Rightarrow C}{\Delta, A \wedge B \Rightarrow C} \wedge L2$$

Disjunction

$$\frac{\Delta \Rightarrow A}{\Delta \Rightarrow A \vee B} \vee R1 \quad \frac{\Delta \Rightarrow B}{\Delta \Rightarrow A \vee B} \vee R2 \quad \frac{\Delta, A \vee B, A \Rightarrow C \quad \Delta, A \vee B, B \Rightarrow C}{\Delta, A \vee B \Rightarrow C} \vee L$$

Implication

$$\frac{\Delta, A \Rightarrow B}{\Delta \Rightarrow A \supset B} \supset R \quad \frac{\Delta, A \supset B \Rightarrow A \quad \Delta, A \supset B, B \Rightarrow C}{\Delta, A \supset B \Rightarrow C} \supset L$$

Truth and Falsehood

$$\overline{\Delta \Rightarrow T} \text{ TR} \quad \overline{\Delta, F \Rightarrow C} \text{ FL}$$

Initial

$$\frac{A \text{ atomic}}{\Delta, A \Rightarrow A} \text{ init}$$

Classical Logic

Conjunction

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge T$$

$$\frac{A \text{ false}}{A \wedge B \text{ false}} \wedge F1$$

$$\frac{B \text{ false}}{A \wedge B \text{ false}} \wedge F2$$

Disjunction

$$\frac{A \text{ true}}{A \vee B \text{ true}} \vee T1$$

$$\frac{B \text{ true}}{A \vee B \text{ true}} \vee T2$$

$$\frac{A \text{ false} \quad B \text{ false}}{A \vee B \text{ false}} \vee F$$

Implication

$$\frac{[A \text{ true}]_u \quad \vdots \quad B \text{ true}}{A \supset B \text{ true}} \supset T^u$$

$$\frac{A \text{ true} \quad B \text{ false}}{A \supset B \text{ false}} \supset F$$

Truth and Falsehood

$$\overline{T \text{ true}} \quad TT$$

$$\overline{F \text{ false}} \quad FF$$

Negation

$$\frac{A \text{ false}}{\neg A \text{ true}} \neg T$$

$$\frac{A \text{ true}}{\neg A \text{ false}} \neg F$$

Contradiction

$$\frac{[A \text{ false}]_u \quad \vdots \quad \#}{A \text{ true}} T\#^u$$

$$\frac{[A \text{ true}]_u \quad \vdots \quad \#}{A \text{ false}} F\#^u$$

$$\frac{A \text{ true} \quad A \text{ false}}{\#} \#$$

Reduced Sequent Calculus

Conjunction

$$\frac{\Delta \longrightarrow A \quad \Delta \longrightarrow B}{\Delta \longrightarrow A \wedge B} \wedge R \qquad \frac{\Delta, A, B \longrightarrow C}{\Delta, A \wedge B \longrightarrow C} \wedge L$$

Disjunction

$$\frac{\Delta \longrightarrow A}{\Delta \longrightarrow A \vee B} \vee R1 \qquad \frac{\Delta \longrightarrow B}{\Delta \longrightarrow A \vee B} \vee R2 \qquad \frac{\Delta, A \longrightarrow C \quad \Delta, B \longrightarrow C}{\Delta, A \vee B \longrightarrow C} \vee L$$

Implication

$$\frac{\Delta, A \longrightarrow B}{\Delta \longrightarrow A \supset B} \supset R \qquad \frac{\Delta, A \supset B \longrightarrow A \quad \Delta, B \longrightarrow C}{\Delta, A \supset B \longrightarrow C} \supset L$$

Truth and Falsehood

$$\overline{\Delta \longrightarrow T} \text{ TR} \qquad \frac{\Delta \longrightarrow C}{\Delta, T \longrightarrow C} \text{ TL} \qquad \overline{\Delta, F \longrightarrow C} \text{ FL}$$

Initial

$$\frac{A \text{ atomic}}{\Delta, A \longrightarrow A} \text{ init}$$

Inversion Calculus

right inversion $\Delta; \Omega \xrightarrow{R} A$
left inversion $\Delta; \Omega \xrightarrow{L} C^+$

Right Inversion

$$\frac{\Delta; \Omega \xrightarrow{R} A \quad \Delta; \Omega \xrightarrow{R} B}{\Delta; \Omega \xrightarrow{R} A \wedge B} \wedge R \quad \frac{}{\Delta; \Omega \xrightarrow{R} T} TR \quad \frac{\Delta; \Omega, A \xrightarrow{R} B}{\Delta; \Omega \xrightarrow{R} A \supset B} \supset R$$

Switch

$$\frac{\Delta; \Omega \xrightarrow{L} P}{\Delta; \Omega \xrightarrow{R} P} LR_P \quad \frac{\Delta; \Omega \xrightarrow{L} A \vee B}{\Delta; \Omega \xrightarrow{R} A \vee B} LR_{\vee} \quad \frac{\Delta; \Omega \xrightarrow{L} F}{\Delta; \Omega \xrightarrow{R} F} LR_F$$

Left Inversion

$$\frac{\Delta; \Omega, A, B \xrightarrow{L} C^+}{\Delta; \Omega, A \wedge B \xrightarrow{L} C^+} \wedge L \quad \frac{\Delta; \Omega \xrightarrow{L} C^+}{\Delta; \Omega, T \xrightarrow{L} C^+} TL \quad \frac{\Delta; \Omega, A \xrightarrow{L} C^+ \quad \Delta; \Omega, B \xrightarrow{L} C^+}{\Delta; \Omega, A \vee B \xrightarrow{L} C^+} \vee L$$

$$\frac{}{\Delta; \Omega, F \xrightarrow{L} C^+} FL$$

Shift

$$\frac{\Delta, P; \Omega \xrightarrow{L} C^+}{\Delta; \Omega, P \xrightarrow{L} C^+} shift_P \quad \frac{\Delta, A \supset B; \Omega \xrightarrow{L} C^+}{\Delta; \Omega, A \supset B \xrightarrow{L} C^+} shift_{\supset}$$

Search

$$\frac{P \in \Delta}{\Delta; \cdot \xrightarrow{L} P} init \quad \frac{\Delta; \cdot \xrightarrow{R} A}{\Delta; \cdot \xrightarrow{L} A \vee B} \vee R1 \quad \frac{\Delta; \cdot \xrightarrow{R} B}{\Delta; \cdot \xrightarrow{L} A \vee B} \vee R2$$

$$\frac{\Delta, A \supset B; \cdot \xrightarrow{R} A \quad \Delta; B \xrightarrow{L} C^+}{\Delta, A \supset B; \cdot \xrightarrow{L} C^+} \supset L$$

Backchaining

$$\begin{aligned}
A^-, B^- &::= P^- \mid A^+ \supset B^- \mid \forall x. A^-(x) \\
A^+, B^+ &::= \downarrow P^- \mid A^+ \wedge B^+ \mid \exists x. A^+(x) \\
\Delta^- &::= \cdot \mid \Delta^-, A^-
\end{aligned}$$

Stable Sequent

$$\frac{A^- \in \Delta^- \quad \Delta^-; [A^-] \longrightarrow P^-}{\Delta^- \longrightarrow P^-} \text{focusL}$$

Left Focus

$$\frac{}{\Delta^-; [P^-] \longrightarrow P^-} \text{init}^- \quad \frac{\Delta^-; [B^-] \longrightarrow P^- \quad \Delta^- \longrightarrow [A^+]}{\Delta^-; [A^+ \supset B^-] \longrightarrow P^-} \supset L \quad \frac{\Delta^-; [A^-(M)] \longrightarrow P^-}{\Delta^-; [\forall x. A^-(x)] \longrightarrow P^-} \forall L$$

Right Focus

$$\frac{\Delta^- \longrightarrow P^-}{\Delta^- \longrightarrow [\downarrow P^-]} \downarrow R \quad \frac{\Delta^- \longrightarrow [A^+] \quad \Delta^- \longrightarrow [B^+]}{\Delta^- \longrightarrow [A^+ \wedge B^+]} \wedge^+ R \quad \frac{\Delta^- \longrightarrow [A^+(M)]}{\Delta^- \longrightarrow [\exists x. A^+(x)]} \exists R$$

Forward Chaining

$$\begin{aligned}
 A^-, B^- &::= \uparrow P^+ \mid A^+ \supset B^- \mid \forall x. A^-(x) \\
 A^+, B^+ &::= P^+ \mid A^+ \wedge B^+ \mid \exists x. A^+(x) \\
 \Delta &::= \cdot \mid \Delta, A^- \mid \Delta, P^+
 \end{aligned}$$

Stable Sequent

$$\frac{A^- \in \Delta \quad \Delta; [A^-] \longrightarrow C^+}{\Delta \longrightarrow C^+} \text{focusL} \qquad \frac{\Delta \longrightarrow [A^+]}{\Delta \longrightarrow A^+} \text{focusR}$$

Left Focus

$$\frac{\Delta, P^+ \longrightarrow C^+}{\Delta; [\uparrow P^+] \longrightarrow C^+} \uparrow L \qquad \frac{\Delta \longrightarrow [A^+] \quad \Delta; [B^-] \longrightarrow C^+}{\Delta; [A^+ \supset B^-] \longrightarrow C^+} \supset L \qquad \frac{\Delta; [A^-(M)] \longrightarrow P^-}{\Delta; [\forall x. A^-(x)] \longrightarrow P^-} \forall L$$

Right Focus

$$\frac{P^+ \in \Delta}{\Delta \longrightarrow [P^+]} \text{init}^+ \qquad \frac{\Delta \longrightarrow [A^+] \quad \Delta \longrightarrow [B^+]}{\Delta \longrightarrow [A^+ \wedge B^+]} \wedge R \qquad \frac{\Delta \longrightarrow [A^+(M)]}{\Delta \longrightarrow [\exists x. A^+(x)]} \exists R$$

Focused Logic

$$\begin{aligned}
A^-, B^- &::= P^- \mid \uparrow A^+ \mid A^+ \supset B^- \mid A^- \wedge^- B^- \mid T^- \\
A^+, B^+ &::= P^+ \mid \downarrow A^- \mid A^+ \wedge^+ B^+ \mid T^+ \mid A^+ \vee B^+ \mid F \\
\rho &::= A^+ \mid P^- \\
\Delta &::= \cdot \mid \Delta, A^- \mid \Delta, P^+ \\
\Omega^+ &::= \cdot \mid \Delta, A^+
\end{aligned}$$

$$\begin{aligned}
\text{right inversion} & \quad \Delta; \Omega^+ \xrightarrow{R} A^- \\
\text{left inversion} & \quad \Delta; \Omega^+ \xrightarrow{L} \rho \\
\text{stable sequent} & \quad \Delta \longrightarrow \rho \\
\text{left focus} & \quad \Delta; [A^-] \longrightarrow \rho \\
\text{right focus} & \quad \Delta \longrightarrow [A^+]
\end{aligned}$$

Right Inversion

$$\begin{aligned}
\frac{\Delta; \Omega^+ \xrightarrow{L} P^-}{\Delta; \Omega^+ \xrightarrow{R} P^-} PR & \quad \frac{\Delta; \Omega^+ \xrightarrow{L} A^+}{\Delta; \Omega^+ \xrightarrow{R} \uparrow A^+} \uparrow R & \quad \frac{\Delta; \Omega^+, A^+ \xrightarrow{R} B^-}{\Delta; \Omega^+ \xrightarrow{R} A^+ \supset B^-} \supset R \\
\frac{\Delta; \Omega^+ \xrightarrow{R} A^- \quad \Delta; \Omega^+ \xrightarrow{R} B^-}{\Delta; \Omega^+ \xrightarrow{R} A^- \wedge^- B^-} \wedge^- R & \quad \frac{}{\Delta; \Omega^+ \xrightarrow{R} T^-} T^- R
\end{aligned}$$

Left Inversion

$$\begin{aligned}
\frac{\Delta, P^+; \Omega^+ \xrightarrow{L} \rho}{\Delta; \Omega^+, P^+ \xrightarrow{L} \rho} PL & \quad \frac{\Delta, A^-; \Omega^+ \xrightarrow{L} \rho}{\Delta; \Omega^+, \downarrow A^- \xrightarrow{L} \rho} \downarrow L & \quad \frac{\Delta; \Omega^+, A^+, B^+ \xrightarrow{L} \rho}{\Delta; \Omega^+, A^+ \wedge^+ B^+ \xrightarrow{L} \rho} \wedge^+ L \\
\frac{\Delta; \Omega^+ \xrightarrow{L} \rho}{\Delta; \Omega^+, T^+ \xrightarrow{L} \rho} T^+ L & \quad \frac{\Delta; \Omega^+, A^+ \xrightarrow{L} \rho \quad \Delta; \Omega^+, B^+ \xrightarrow{L} \rho}{\Delta; \Omega^+, A^+ \vee B^+ \xrightarrow{L} \rho} \vee L & \quad \frac{}{\Delta; \Omega^+, F \xrightarrow{L} \rho} FL \\
& \quad \frac{\Delta \longrightarrow \rho}{\Delta; \cdot \xrightarrow{L} \rho} \text{stable}
\end{aligned}$$

Stable Sequent

$$\frac{A^- \in \Delta \quad \Delta; [A^-] \longrightarrow \rho}{\Delta \longrightarrow \rho} \text{focusL} \qquad \frac{\Delta \longrightarrow [A^+]}{\Delta \longrightarrow A^+} \text{focusR}$$

Left Focus

$$\frac{}{\Delta; [P^-] \longrightarrow P^-} \text{init}^- \qquad \frac{\Delta; A^+ \xrightarrow{L} \rho}{\Delta; [\uparrow A^+] \longrightarrow \rho} \uparrow L \qquad \frac{\Delta \longrightarrow [A^+] \quad \Delta; [B^-] \longrightarrow \rho}{\Delta; [A^+ \supset B^-] \longrightarrow \rho} \supset L$$

$$\frac{\Delta; [A^-] \longrightarrow \rho}{\Delta; [A^- \wedge^- B^-] \longrightarrow \rho} \wedge^- L1 \qquad \frac{\Delta; [B^-] \longrightarrow \rho}{\Delta; [A^- \wedge^- B^-] \longrightarrow \rho} \wedge^- L2 \qquad \text{No left rule for } [T^-].$$

Right Focus

$$\frac{P^+ \in \Delta}{\Delta \longrightarrow [P^+]} \text{init}^+ \qquad \frac{\Delta; \cdot \xrightarrow{R} A^-}{\Delta \longrightarrow [\downarrow A^-]} \downarrow R \qquad \frac{\Delta \longrightarrow [A^+] \quad \Delta \longrightarrow [B^+]}{\Delta \longrightarrow [A^+ \wedge^+ B^+]} \wedge^+ R \qquad \frac{}{\Delta \longrightarrow [T^+]} T^+ R$$

$$\frac{\Delta \longrightarrow [A^+]}{\Delta \longrightarrow [A^+ \vee B^+]} \vee R1 \qquad \frac{\Delta \longrightarrow [B^+]}{\Delta \longrightarrow [A^+ \vee B^+]} \vee R2 \qquad \text{No right rule for } [F].$$

Linear Logic

Hypothesis

$$\frac{}{\Delta; A \text{ true} \Vdash A \text{ true}} \text{hyp} \qquad \frac{}{\Delta, A \text{ valid}; \cdot \Vdash A \text{ true}} \text{hypv}$$

Multiplicative Conjunction

$$\frac{\Delta; \Gamma_1 \Vdash A \text{ true} \quad \Delta; \Gamma_2 \Vdash B \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash A \otimes B \text{ true}} \otimes I \qquad \frac{\Delta; \Gamma_1 \Vdash A \otimes B \text{ true} \quad \Delta, \Gamma_2, A \text{ true}, B \text{ true} \Vdash C \text{ true}}{\Delta, \Gamma_1, \Gamma_2 \Vdash C \text{ true}} \otimes E$$

Additive Conjunction

$$\frac{\Delta; \Gamma \Vdash A \text{ true} \quad \Delta; \Gamma \Vdash B \text{ true}}{\Delta; \Gamma \Vdash A \& B \text{ true}} \& I \qquad \frac{\Delta; \Gamma \Vdash A \& B \text{ true}}{\Delta; \Gamma \Vdash A \text{ true}} \& E1 \qquad \frac{\Delta; \Gamma \Vdash A \& B \text{ true}}{\Delta; \Gamma \Vdash B \text{ true}} \& E2$$

Disjunction

$$\frac{\Delta; \Gamma \Vdash A \text{ true}}{\Delta; \Gamma \Vdash A \oplus B \text{ true}} \oplus I1 \qquad \frac{\Delta; \Gamma \Vdash B \text{ true}}{\Delta; \Gamma \Vdash A \oplus B \text{ true}} \oplus I2$$

$$\frac{\Delta; \Gamma_1 \Vdash A \oplus B \text{ true} \quad \Delta; \Gamma_2, A \text{ true} \Vdash C \text{ true} \quad \Delta; \Gamma_2, B \text{ true} \Vdash C \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash C \text{ true}} \oplus E$$

Implication

$$\frac{\Delta; \Gamma, A \text{ true} \Vdash B \text{ true}}{\Delta; \Gamma \Vdash A \multimap B \text{ true}} \multimap I \qquad \frac{\Delta; \Gamma_1 \Vdash A \multimap B \text{ true} \quad \Delta; \Gamma_2 \Vdash A \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash B \text{ true}} \multimap E$$

Units

$$\frac{}{\Delta; \cdot \Vdash 1 \text{ true}} 1I \qquad \frac{\Delta; \Gamma_1 \Vdash 1 \text{ true} \quad \Delta; \Gamma_2 \Vdash C \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash C \text{ true}} 1E$$

$$\frac{}{\Delta; \Gamma \Vdash \top \text{ true}} \top I \qquad \frac{\Delta; \Gamma_1 \Vdash 0 \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash C \text{ true}} 0E$$

Exponential

$$\frac{\Delta; \cdot \Vdash A \text{ true}}{\Delta; \cdot \Vdash !A \text{ true}} !I \qquad \frac{\Delta; \Gamma_1 \Vdash !A \text{ true} \quad \Delta, A \text{ valid}; \Gamma_2 \Vdash C \text{ true}}{\Delta; \Gamma_1, \Gamma_2 \Vdash C \text{ true}} !E$$

Ordered Logic

Hypothesis

$$\frac{}{A \text{ true} \Vdash A \text{ true}} \text{hyp}$$

Multiplicative Conjunction

$$\frac{\Gamma_1 \Vdash A \text{ true} \quad \Gamma_2 \Vdash B \text{ true}}{\Gamma_1, \Gamma_2 \Vdash A \bullet B \text{ true}} \bullet I \qquad \frac{\Gamma_2 \Vdash A \bullet B \text{ true} \quad \Gamma_1, A \text{ true}, B \text{ true}, \Gamma_3 \Vdash C \text{ true}}{\Gamma_1, \Gamma_2, \Gamma_3 \Vdash C \text{ true}} \bullet E$$

Additive Conjunction

$$\frac{\Gamma \Vdash A \text{ true} \quad \Gamma \Vdash B \text{ true}}{\Gamma \Vdash A \& B \text{ true}} \& I \qquad \frac{\Gamma \Vdash A \& B \text{ true}}{\Gamma \Vdash A \text{ true}} \& E1 \qquad \frac{\Gamma \Vdash A \& B \text{ true}}{\Gamma \Vdash B \text{ true}} \& E2$$

Disjunction

$$\frac{\Gamma \Vdash A \text{ true}}{\Gamma \Vdash A \oplus B \text{ true}} \oplus I1 \qquad \frac{\Gamma \Vdash B \text{ true}}{\Gamma \Vdash A \oplus B \text{ true}} \oplus I2$$

$$\frac{\Gamma_2 \Vdash A \oplus B \text{ true} \quad \Gamma_1, A \text{ true}, \Gamma_3 \Vdash C \text{ true} \quad \Gamma_1, B \text{ true}, \Gamma_3 \Vdash C \text{ true}}{\Gamma_1, \Gamma_2, \Gamma_3 \Vdash C \text{ true}} \oplus E$$

Implication

$$\frac{A \text{ true}, \Gamma \Vdash B \text{ true}}{\Gamma \Vdash A \multimap B \text{ true}} \multimap I \qquad \frac{\Gamma_1 \Vdash A \text{ true} \quad \Gamma_2 \Vdash A \multimap B \text{ true}}{\Gamma_1, \Gamma_2 \Vdash B \text{ true}} \multimap E$$

$$\frac{\Gamma, A \text{ true} \Vdash B \text{ true}}{\Gamma \Vdash A \multimap B \text{ true}} \multimap I \qquad \frac{\Gamma_1 \Vdash A \multimap B \text{ true} \quad \Gamma_2 \Vdash A \text{ true}}{\Gamma_1, \Gamma_2 \Vdash B \text{ true}} \multimap E$$

Units

$$\frac{}{\cdot \Vdash 1 \text{ true}} 1I \qquad \frac{\Gamma_2 \Vdash 1 \text{ true} \quad \Gamma_1, \Gamma_3 \Vdash C \text{ true}}{\Gamma_1, \Gamma_2, \Gamma_3 \Vdash C \text{ true}} 1E$$

$$\frac{}{\Gamma \Vdash \top \text{ true}} \top I \qquad \frac{\Gamma_2 \Vdash 0 \text{ true}}{\Gamma_1, \Gamma_2, \Gamma_3 \Vdash C \text{ true}} 0E$$

Modal Logic

Hypothesis

$$\frac{A \text{ true} \in \Gamma}{\Delta; \Gamma \vdash A \text{ true}} \text{ hyp}$$

$$\frac{A \text{ valid} \in \Delta}{\Delta; \Gamma \vdash A \text{ true}} \text{ hypv}$$

Conjunction

$$\frac{\Delta; \Gamma \vdash A \text{ true} \quad \Delta; \Gamma \vdash B \text{ true}}{\Delta; \Gamma \vdash A \wedge B \text{ true}} \wedge I$$

$$\frac{\Delta; \Gamma \vdash A \wedge B \text{ true}}{\Delta; \Gamma \vdash A \text{ true}} \wedge E1$$

$$\frac{\Delta; \Gamma \vdash A \wedge B \text{ true}}{\Delta; \Gamma \vdash B \text{ true}} \wedge E2$$

Disjunction

$$\frac{\Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash A \vee B \text{ true}} \vee I1$$

$$\frac{\Delta; \Gamma \vdash B \text{ true}}{\Delta; \Gamma \vdash A \vee B \text{ true}} \vee I2$$

$$\frac{\Delta; \Gamma \vdash A \vee B \text{ true} \quad \Delta; \Gamma, A \text{ true} \vdash C \text{ true} \quad \Delta; \Gamma, B \text{ true} \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \vee E$$

Implication

$$\frac{\Delta; \Gamma, A \text{ true} \vdash B \text{ true}}{\Delta; \Gamma \vdash A \supset B \text{ true}} \supset I$$

$$\frac{\Delta; \Gamma \vdash A \supset B \text{ true} \quad \Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash B \text{ true}} \supset E$$

Truth and Falsehood

$$\frac{}{\Delta; \Gamma \vdash T \text{ true}} TI$$

$$\frac{\Delta; \Gamma \vdash F \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} FE$$

Necessity

$$\frac{\Delta; \cdot \vdash A \text{ true}}{\Delta; \Gamma \vdash \Box A \text{ true}} \Box I$$

$$\frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, A \text{ valid}; \Gamma \vdash C \text{ true}}{\Delta; \Gamma \vdash C \text{ true}} \Box E$$

$$\frac{\Delta; \Gamma \vdash \Box A \text{ true} \quad \Delta, A \text{ valid}; \Gamma \vdash C \text{ poss}}{\Delta; \Gamma \vdash C \text{ poss}} \Box E_p$$

Possibility

$$\frac{\Delta; \Gamma \vdash A \text{ poss}}{\Delta; \Gamma \vdash \Diamond A \text{ true}} \Diamond I$$

$$\frac{\Delta; \Gamma \vdash A \text{ true}}{\Delta; \Gamma \vdash A \text{ poss}} \text{ here}$$

$$\frac{\Delta; \Gamma \vdash \Diamond A \text{ true} \quad \Delta; A \text{ true} \vdash C \text{ poss}}{\Delta; \Gamma \vdash C \text{ poss}} \Diamond E$$