10-701/15-781, Fall 2006, Midterm

- There are 7 questions in this exam (11 pages including this cover sheet).
- Questions are not equally difficult.
- If you need more room to work out your answer to a question, use the back of the page and clearly mark on the front of the page if we are to look at what's on the back.
- This exam is open book and open notes. Computers, PDAs, cell phones are not allowed.
- You have 1 hour and 20 minutes. Good luck!

Name:
Andrew ID:

Q	Topic	Max. Score	Score
1	Conditional Independece, MLE/MAP, Probability	12	
2	Decision Tree	12	
3	Neural Network and Regression	18	
4	Bias-Variance Decomposition	12	
5	Support Vector Machine	12	
6	Generative vs. Discriminative Classifier	20	
7	Learning Theory	14	
	Total	100	

1 Conditional Independence, MLE/MAP, Probability (12 pts)

1. (4 pts) Show that $\Pr(X, Y|Z) = \Pr(X|Z)\Pr(Y|Z)$ if $\Pr(X|Y, Z) = \Pr(X|Z)$.

$$\Pr(X,Y|Z) = \Pr(X|Y,Z)\Pr(Y|Z) \quad \text{(chain rule)}$$
$$= \Pr(X|Z)\Pr(Y|Z)$$

common mistake: $\Pr(X|Y,Z) = \Pr(X|Z) \Rightarrow X \perp Y$ given Z
$\Rightarrow \Pr(X,Y|Z) = \Pr(X|Z)\Pr(Y|Z)$

the first $\Rightarrow$ does not hold if the equation is not for all possible values of X, Y, Z.

2. (4 pts) If a data point y follows the Poisson distribution with rate parameter θ, then the probability of a single observation y is

$$p(y|\theta) = \frac{\theta^y e^{-\theta}}{y!}, \quad \text{for } y = 0, 1, 2, \cdots.$$

You are given data points $y_1, \cdots, y_n$ independently drawn from a Poisson distribution with parameter θ. Write down the log-likelihood of the data as a function of θ.

$$\sum_{i=1}^{n} \left(y_i \log\theta - \theta - \log y_i! \right)$$
$$= \left(\sum_{i=1}^{n} y_i \right) \log\theta - n\theta - \log\left(\prod_{i=1}^{n} y_i! \right)$$

3. (4 pts) Suppose that in answering a question in a multiple choice test, an examinee either knows the answer, with probability p, or he guesses with probability $1 - p$. Assume that the probability of answering a question correctly is 1 for an examinee who knows the answer and $1/m$ for the examinee who guesses, where m is the number of multiple choice alternatives. What is the probability that an examinee knew the answer to a question, given that he has correctly answered it?

$$P(\text{know answer} \mid \text{correct}) = \frac{P(\text{know answer, correct})}{P(\text{correct})}$$
$$= \frac{p}{p + (1-p)\frac{1}{m}}$$

2 Decision Tree (12 pts)

The following data set will be used to learn a decision tree for predicting whether students are lazy (L) or diligent (D) based on their weight (Normal or Underweight), their eye color (Amber or Violet) and the number of eyes they have (2 or 3 or 4).

Weight	Eye Color	Num. Eyes	Output
N	A	2	L
N	V	2	L
N	V	2	L
U	V	3	L
U	V	3	L
U	A	4	D
N	A	4	D
N	V	4	D
U	A	3	D
U	A	3	D

The following numbers may be helpful as you answer this problem without using a calculator: $\log_2 0.1 = -3.32, \log_2 0.2 = -2.32, \log_2 0.3 = -1.73, \log_2 0.4 = -1.32, \log_2 0.5 = -1$.

*You don't need to show the derivation for your answers in this problem.

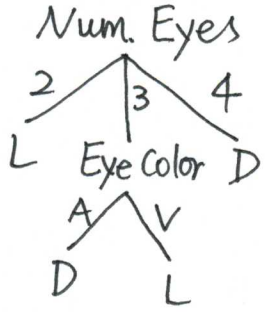
1. (3 pts) What is the conditional entropy $H(EyeColor|Weight = N)$?

$$-(0.4\log_2 0.4 + 0.6\log_2 0.6)$$
$$= 0.4 \times 1.32 + 0.6 \times (1.73 - 1) = 0.966$$

2. (3 pts) What attribute would the ID3 algorithm choose to use for the root of the tree (no pruning)?

Num. Eyes

3. (4 pts) Draw the full decision tree learned for this data (no pruning).



4. (2 pts) What is the training set error of this unpruned tree?

0

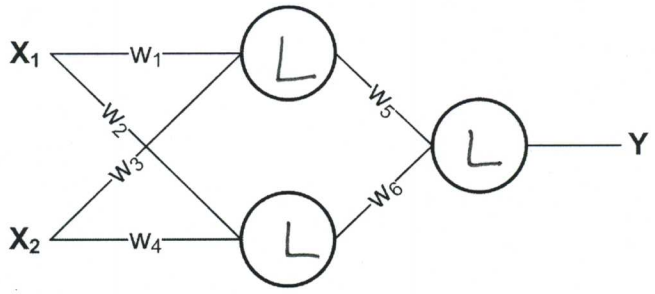
3 Neural Network and Regression (18 pts)

Consider a two-layer neural network to learn a function $f : X \rightarrow Y$ where $X = \langle X_1, X_2 \rangle$ consists of two attributes. The weights, $w_1, \cdots, w_6$, can be arbitrary. There are two possible choices for the function implemented by each unit in this network:

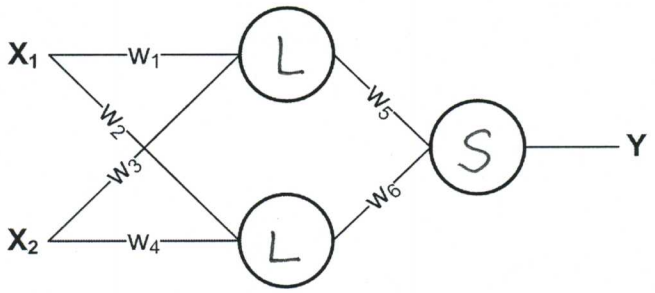
- **S**: signed sigmoid function $S(a) = sign[\sigma(a) - 0.5] = sign[\frac{1}{1+\exp(-a)} - 0.5]$
- **L**: linear function $L(a) = c\,a$

where in both cases $a = \sum_i w_i X_i$

1. (4 pts) Assign proper activation functions (**S** or **L**) to each unit in the following graph so this neural network simulates a linear regression: $Y = \beta_1 X_1 + \beta_2 X_2$.



2. (4 pts) Assign proper activation functions (**S** or **L**) for each unit in the following graph so this neural network simulates a binary logistic regression classifier: $Y = \arg\max_y P(Y = y|X)$, where $P(Y = 1|X) = \frac{\exp(\beta_1 X_1 + \beta_2 X_2)}{1+\exp(\beta_1 X_1 + \beta_2 X_2)}$, $P(Y = -1|X) = \frac{1}{1+\exp(\beta_1 X_1 + \beta_2 X_2)}$.

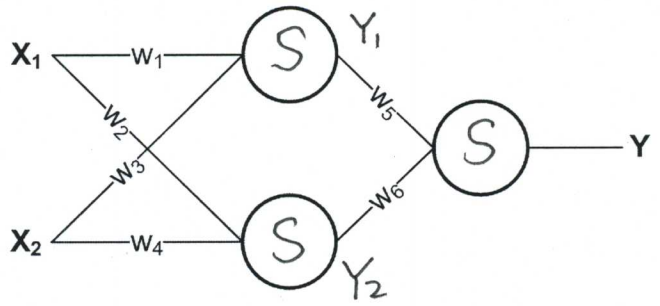


3. (3 pts) Following problem 3.2, derive β_1 and β_2 in terms of $w_1, \cdots, w_6$.

$\beta_1 = c(w_1 w_5 + w_2 w_6)$

$\beta_2 = c(w_3 w_5 + w_4 w_6)$

4. (4 pts) Assign proper activation functions (**S** or **L**) for each unit in the following graph so this neural network simulates a boosting classifier which combines two logistic regression classifiers, $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$, to produce its final prediction: $Y = sign[\alpha_1 Y_1 + \alpha_2 Y_2]$. Use the same definition in problem 3.2 for f_1 and f_2.



5. (3 pts) Following problem 3.4, derive α_1 and α_2 in terms of $w_1, \cdots, w_6$.

$\alpha_1 = w_5$ #

$\alpha_2 = w_6$

4 Bias-Variance Decomposition (12 pts)

1. (6 pts) Suppose you have regression data generated by a polynomial of degree 3. Characterize the bias-variance of the estimates of the following models on the data with respect to the true model by circling the appropriate entry.

	Bias	Variance
Linear regression	low/high	low/high
Polynomial regression with degree 3	low/high	low/high
Polynomial regression with degree 10	low/high	low/high

2. Let $Y = f(X) + \epsilon$, where ϵ has mean zero and variance σ_ϵ^2. In k-nearest neighbor (kNN) regression, the prediction of Y at point x_0 is given by the average of the values Y at the k neighbors closest to x_0.

   (a) (2 pts) Denote the ℓ-nearest neighbor to x_0 by $x_{(\ell)}$ and its corresponding Y value by $y_{(\ell)}$. Write the prediction $\hat{f}(x_0)$ of the kNN regression for x_0 in terms of $y_{(\ell)}, 1 \leq \ell \leq k$.

$$\hat{f}(x_0) = \frac{1}{k} \sum_{\ell=1}^{k} y_{(\ell)}$$

   (b) (2 pts) What is the behavior of the bias as k increases?

increase

   (c) (2 pts) What is the behavior of the variance as k increases?

decrease

5 Support Vector Machine (12 pts)

Consider a supervised learning problem in which the training examples are points in 2-dimensional space. The positive examples are $(1,1)$ and $(-1,-1)$. The negative examples are $(1,-1)$ and $(-1,1)$.

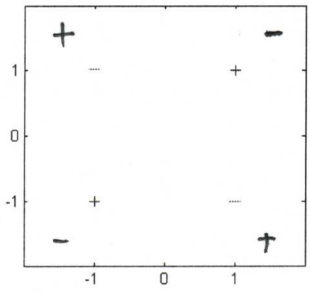
1. (1 pts) Are the positive examples linearly separable from the negative examples in the original space?

   No

2. (4 pts) Consider the feature transformation $\phi(x) = [1, x_1, x_2, x_1 x_2]$, where x_1 and x_2 are, respectively, the first and second coordinates of a generic example x. The prediction function is $y(x) = w^T * \phi(x)$ in this feature space. Give the coefficients, w, of a maximum-margin decision surface separating the positive examples from the negative examples. (You should be able to do this by inspection, without any significant computation.)

   $w = (0,0,0,1)^T$

3. (3 pts) Add one training example to the graph so the total five examples can no longer be linearly separated in the feature space $\phi(x)$ defined in problem 5.2.



4. (4 pts) What kernel $K(x, x')$ does this feature transformation ϕ correspond to?

   $1 + x_1 x_1' + x_2 x_2' + x_1 x_2 x_1' x_2'$

6 Generative vs. Discriminative Classifier (20 pts)

Consider the binary classification problem where class label $Y \in \{0, 1\}$ and each training example X has 2 binary attributes $X_1, X_2 \in \{0, 1\}$.

In this problem, we will always assume X_1 and X_2 are conditional independent given Y, that the class priors are $P(Y = 0) = P(Y = 1) = 0.5$, and that the conditional probabilities are as follows:

$P(X_1\|Y)$	$X_1 = 0$	$X_1 = 1$
$Y = 0$	0.7	0.3
$Y = 1$	0.2	0.8

$P(X_2\|Y)$	$X_2 = 0$	$X_2 = 1$
$Y = 0$	0.9	0.1
$Y = 1$	0.5	0.5

The expected error rate is the probability that a classifier provides an incorrect prediction for an observation: if Y is the true label, let $\hat{Y}(X_1, X_2)$ be the predicted class label, then the expected error rate is

$$P_{\mathcal{D}}\left(Y = 1 - \hat{Y}(X_1, X_2)\right) = \sum_{X_1=0}^{1} \sum_{X_2=0}^{1} P_{\mathcal{D}}\left(X_1, X_2, Y = 1 - \hat{Y}(X_1, X_2)\right).$$

Note that we use the subscript $\mathcal{D}$ to emphasize that the probabilities are computed under the true distribution of the data.

*You don't need to show all the derivation for your answers in this problem.

1. (4 pts) Write down the naïve Bayes prediction for all the 4 possible configurations of X_1, X_2. The following table would help you to complete this problem.

X_1	X_2	$P(X_1, X_2, Y = 0)$	$P(X_1, X_2, Y = 1)$	$\hat{Y}(X_1, X_2)$
0	0	$0.7 \times 0.9 \times 0.5$	$0.2 \times 0.5 \times 0.5$	0
0	1	$0.7 \times 0.1 \times 0.5$	$0.2 \times 0.5 \times 0.5$	1
1	0	$0.3 \times 0.9 \times 0.5$	$0.8 \times 0.5 \times 0.5$	1
1	1	$0.3 \times 0.1 \times 0.5$	$0.8 \times 0.5 \times 0.5$	1

2. (4 pts) Compute the expected error rate of this naïve Bayes classifier which predicts Y given both of the attributes $\{X_1, X_2\}$. Assume that the classifier is learned with infinite training data.

$$\begin{aligned} & 0.2 \times 0.5 \times 0.5 \\ + & 0.7 \times 0.1 \times 0.5 \\ + & 0.3 \times 0.9 \times 0.5 \\ + & 0.3 \times 0.1 \times 0.5 \\ = & \ 0.235 \end{aligned}$$

3. (4 pts) Which of the following two has a smaller expected error rate?

- (•) the naïve Bayes classifier which predicts Y given X_1 only
- ( ) the naïve Bayes classifier which predicts Y given X_2 only

Prediction $X_2=1 \mapsto \hat{Y}=1$, $X_2=0 \to \hat{Y}=0$

$P_D(X_2=1, Y=0)+P_D(X_2=0, Y=1)$
$=0.1\times0.5+0.5\times0.5=0.3$

Prediction
$X_1=1 \to \hat{Y}=1$
$X_1=0 \to \hat{Y}=0$

$P_D(X_1=0, Y=0)+P_D(X_1=0, Y=1)$
$=0.3\times0.5+0.2\times0.5=0.25$

4. (4 pts) Now, suppose that we create a new attribute X_3, which is a deterministic copy of X_2. What is the expected error rate of the naïve Bayes which predicts Y given all the attributes (X_1, X_2, X_3) now? Assume that the classifier is learned with infinite training data.

X_1	X_2	$X_3=X_2$	$P_{NB}(X_1,X_2,X_3,Y=0)$	$P_{NB}(X_1,X_2,X_3,Y=1)$	$\hat{Y}(X_1,X_2,X_3)$
0	0	0	$0.7\times0.9\times0.9\times0.5$	$0.2\times0.5\times0.5\times0.5$	0
0	1	1	$0.7\times0.1\times0.1\times0.5$	$0.2\times0.5\times0.5\times0.5$	1
1	0	0	$0.3\times0.9\times0.9\times0.5$	$0.8\times0.5\times0.5\times0.5$	0
1	1	1	$0.3\times0.1\times0.1\times0.5$	$0.8\times0.5\times0.5\times0.5$	1

Error rate $= 0.2\times0.5\times0.5$ $P_D(X_1=0, X_2=0, Y=1)$
$+ 0.7\times0.1\times0.5$ $P_D(X_1=0, X_2=1, Y=0)$
$+ 0.8\times0.5\times0.5$ $P_D(X_1=1, X_2=0, Y=1)$
$+ 0.3\times0.1\times0.5$ $P_D(X_1=1, X_2=1, Y=0)$
$= 0.3$

5. (4 pts) Explain what is happening with naïve Bayes in problem 6.4? Does logistic regression suffer from the same problem? Why?

The conditional independence assumption of naïve Bayes classifier does not hold. X_2 is overcounted leading to an error prediction when $X_1=0, X_2=0$.

LR does not suffer because it does not make such conditional independence assumption.

7 Learning Theory (14 pts)

You read in the paper that the famous bird migration website, Netflocks, is offering a $1M prize for accurately recommending movies about penguins. Furthermore, it is providing a training data set containing 100,000,000 labeled training examples. Each training example consists of a set of 100 real-valued features describing a movie, along with a boolean label indicating whether to recommend this movie to a person.

You determine that the $1M can be yours if you can train a *linear* Support Vector Machine with a true accuracy of 98%. Of course you understand that PAC learning theory provides only probabilistic bounds, so you decide to enter only if you can prove you have at least a 0.9 probability of achieving an accuracy of 98%.

1. (8 pts) Can you use PAC learning theory to decide whether you can meet your performance objective? If yes, give an expression for the number of training examples sufficient to meet your performance objective. If not, explain why not, then provide the minimum set of additional assumptions needed so that PAC learning theory can be applied, and give an expression of the number of training examples sufficient under your assumptions. (you may leave your expression as an unsolved arithmetic expression, but it should contain only constants - no variables).

No. Need to assume the true concept $C \in H$ to apply PAC learning theory.

$$m = \frac{1}{\varepsilon}\left(4\log_2\frac{1}{\delta} + 8\,VC(H)\log_2\frac{13}{\varepsilon}\right)$$

with

$$\varepsilon = 0.02,\quad \delta = 0.1,\quad VC(H) = 100+1 = 101$$

then

$$m = 50\left(4\log_2 10 + 808\log_2 650\right) << 100{,}000{,}000.$$

2. (3 pts) Consider the PAC-style statement "we can achieve true accuracy of at least 98% with probability 0.9." What is the meaning of "with probability 0.9"? Answer this by describing a randomized experiment which you could perform repeatedly to test whether the statement is true.

We could do M repititions of the following experiments (M is a large number, say 10^6).

generate 100,000,000 training data from the true data distribution D, train a linear SVM, then compute its accuracy on the true distribution D. If the accuracy is greater than 98%, report a positive result; otherwise, report a negative one.

If we get a number greater than 0.9M of positive results, we validate the PAC statement

3. (3 pts) Your friend already has a private dataset of 100,000,000 labeled movies, so she will end up with twice as much training data as you. You train using the Netflocks data to produce a classifier h_1. She uses the same learning algorithm, but trains with twice as much data to produce her output hypothesis, h_2. You are interested in how well the training errors of h_1 and h_2 predict their true errors. Consider the ratio

$$\frac{error_{train}(h_1) - error_{true}(h_1)}{error_{train}(h_2) - error_{true}(h_2)}$$

Which of these is the most likely value for this ratio? Circle the answer and give a *one-sentence* explanation.

$$4, 2, \textcircled{\sqrt{2}}, 1, -1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{4}$$

$$error_{true}(h) < error_{train}(h) + \sqrt{\frac{VC(H)\left(\ln\frac{2m}{VC(H)}+1\right)+\ln\frac{4}{\delta}}{m}}$$

the difference is approx. $\propto m^{-\frac{1}{2}}$

* Any resemblance to real persons, animals, or organizations, living or dead, is purely coincidental.