School of Computer Science
Carnegie Mellon

# Probabilistic Graphical Models

## Bayesian nonparametrics: The Dirichlet process

Avinava Dubey

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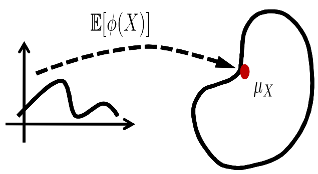







## Parametric vs nonparametric

**Parametric model:**

- Assumes all data can be represented using a fixed, finite number of parameters.
  - Mixture of *K* Gaussians, polynomial regression.

**Nonparametric model:**

- Number of parameters can grow with sample size.
- Number of parameters may be random.
  - Kernel density estimation.

**Bayesian nonparametrics:**

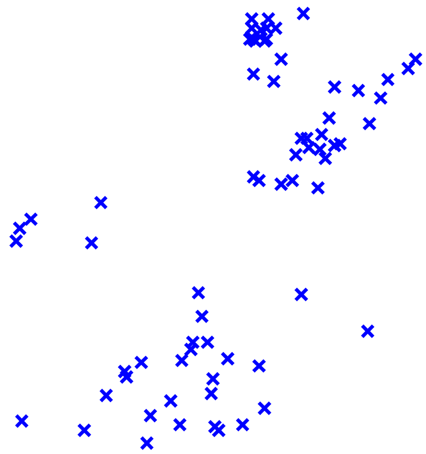
- Allow an *infinite* number of parameters *a priori.*
- A finite data set will only use a finite number of parameters.
- Other parameters are integrated out.

## Clustered data

- How to model this data?
- Mixture of Gaussians:

$$p(x_1, \ldots, x_N | \pi, \{\mu_k\}, \{\Sigma_k\}) \\ = \prod_{n=1}^{\infty} \sum_{k=1}^{K} \pi_k \mathcal{N}(x_k | \mu_k, \Sigma_k)$$

- Parametric model: Fixed finite number of parameters.



## Bayesian finite mixture model

- How to choose the mixing weights and mixture parameters?
- Bayesian choice: Put a prior on them and integrate out:

$$p(x_1, \ldots, x_N) \\ = \int\int\int \left( \prod_{n=1}^{\infty} \sum_{k=1}^{K} \pi_k \mathcal{N}(x_k | \mu_k, \Sigma_k) \right) \\ p(\pi) p(\mu_{1:K}) p(\Sigma_{1:K}) d\pi d\mu_{1:K} d\Sigma_{1:K}$$

- Where possible, use conjugate priors
  - Gaussian/inverse Wishart for mixture parameters
  - What to choose for mixture weights?

## The Dirichlet distribution

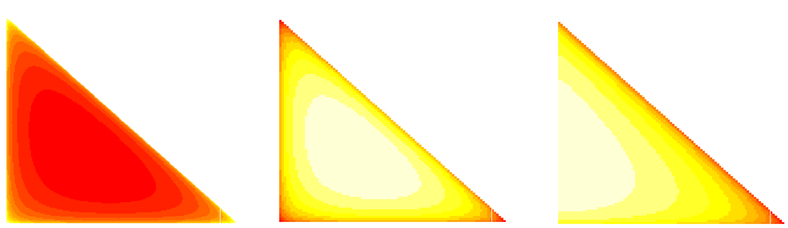
- The Dirichlet distribution is a distribution over the (*K*-*1*)-dimensional simplex.
- It is parametrized by a *K*-dimensional vector $(\alpha_1, \ldots, \alpha_K)$ such that $\alpha_k \geq 0, k = 1, \ldots, K$ and $\sum_k \alpha_k > 0$
- Its distribution is given by

$$\frac{\prod_{k=1}^K \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^K \alpha_k)} \prod_{k=1}^K \pi_k^{\alpha_k - 1}$$




## Samples from the Dirichlet distribution

- If $\pi \sim \mathrm{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ then $\pi_k \geq 0$ for all *k,* and $\sum_{k=1}^K \pi_k = 1.$
- Expectation: $\mathbb{E}\Big[(\pi_1, \ldots, \pi_K)\Big] = \frac{(\alpha_1, \ldots, \alpha_K)}{\sum_k \alpha_k}$



$\alpha = (0.01, 0.01, 0.01)$ $\alpha = (100, 100, 100)$ $\alpha = (5, 50, 100)$

## Conjugacy to the multinomial

- If $\theta \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ and $x_n \overset{iid}{\sim} \theta$

$$
\begin{aligned}
p(\pi|x_1,\ldots,x_n) &\propto p(x_1,\ldots,x_n|\pi)p(\pi)\\
&= \left(\frac{\prod_{k=1}^K \Gamma(\alpha_k)}{\Gamma(\sum_{k=1}^K \alpha_k)} \prod_{k=1}^K \pi_k^{\alpha_k - 1}\right)\left(\frac{n!}{m_1!\ldots m_K!}\pi_1^{m_1}\cdots\pi_K^{m_K}\right)\\
&\propto \frac{\prod_{k=1}^K \Gamma(\alpha_k + m_k)}{\Gamma(\sum_{k=1}^K \alpha_k + m_k)} \prod_{k=1}^K \pi_k^{\alpha_k + m_k - 1}\\
&= \text{Dirichlet}(\pi|\alpha_1 + m_1, \ldots, \alpha_K + m_K)
\end{aligned}
$$

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## Distributions over distributions

- The Dirichlet distribution is a distribution over positive vectors that sum to one.
- We can associate each entry with a set of parameters
  - e.g. finite mixture model: each entry associated with a mean and covariance.
- In a Bayesian setting, we want these parameters to be *random.*
- We can combine the distribution over probability vectors with a distribution over parameters to get a **distribution over distributions over parameters.**

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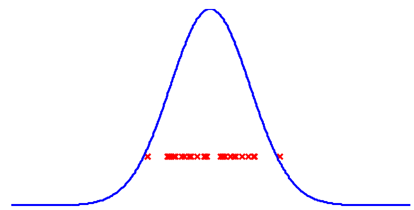
# Example: finite mixture model

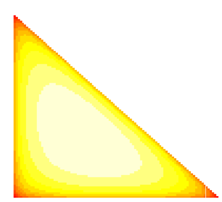
- Gaussian distribution: distribution over means.
  - Sample from a Gaussian is a real-valued number.

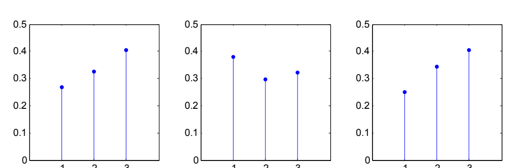


# Example: finite mixture model

- Gaussian distribution: distribution over means.
  - Sample from a Gaussian is a real-valued number.
- Dirichlet distribution:
  - Sample from a Dirichlet distribution is a probability vector.

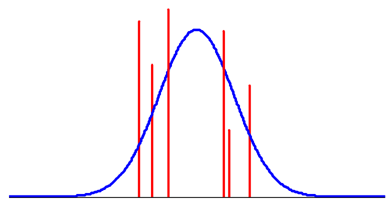




# Example: finite mixture model

- Dirichlet prior
  - Each element of a Dirichlet-distributed vector is associated with a parameter value drawn from some distribution.
  - Sample from a Dirichlet prior is a probability distribution over parameters.







# Properties of the Dirichlet distribution

- Relationship to gamma distribution: If $\eta_k \sim \text{Gamma}(\alpha_k, 1)$ ,

$$\frac{(\eta_1, \ldots, \eta_K)}{\sum_k \eta_k} \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$$

- If $\eta_1 \sim \text{Gamma}(\alpha_1, 1)$ and $\eta_2 \sim \text{Gamma}(\alpha_2, 1)$ then

$$\eta_1 + \eta_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, 1)$$

- Therefore, if $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ then $(\pi_1 + \pi_2, \pi_3, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_K)$

## Properties of the Dirichlet distribution

- The beta distribution is a Dirichlet distribution on the 1-simplex.
- Let $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$ and

  $\theta \sim \text{Beta}(\alpha_1 b, \alpha_1(1-b)), 0 < b < 1.$
- Then

  $(\pi_1\theta, \pi_1(1-\theta), \pi_2, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 b_1, \alpha_1(1-b_1), \alpha_2, \ldots, \alpha_K)$
- More generally, if $\theta \sim \text{Dirichlet}(\alpha_1 b_1, \alpha_1 b_2, \ldots, \alpha_1 b_N), \sum_i b_i = 1.$ then

$$(\pi_1\theta_1, \ldots, \pi_1\theta_N, \pi_2, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1 b_1, \ldots, \alpha_1 b_N, \alpha_2, \ldots, \alpha_K)$$





## Properties of the Dirichlet distribution

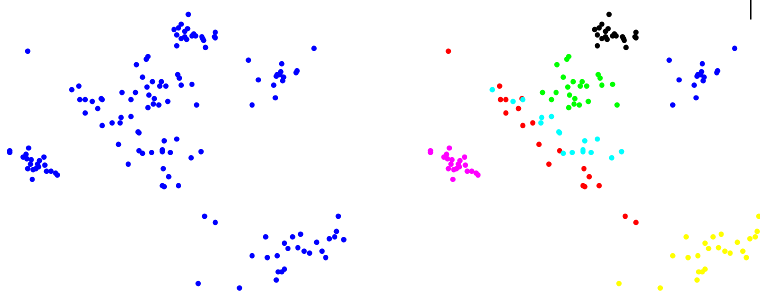
- Renormalization:

  If $(\pi_1 \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_K)$

  then $\dfrac{(\pi_2, \ldots, \pi_K)}{\sum_{k=1}^K \pi_k} \sim ?$

$$\frac{(\pi_2, \ldots, \pi_K)}{\sum_{k=1}^K \pi_k} \sim \text{Dirichlet}(\alpha_2, \ldots, \alpha_K)$$

## Choosing the number of clusters



- Mixture of Gaussians – but how many components?
- What if we see more data – may find new components?

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## Bayesian nonparametric mixture models

- Make sure we always have more clusters than we need.
- Solution – infinite clusters *a priori*!

$$p(x_n|\pi, \{\mu_k\}, \{\Sigma_k\}) = \sum_{k=1}^{\infty} \pi_k \mathcal{N}(x_n|\mu_k, \Sigma_k)$$

- A finite data set will always use a finite – but *random* – number of clusters.
- How to choose the prior?
- We want something *like* a Dirichlet prior – but with an infinite number of components.

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## Constructing an appropriate prior

- Start off with $\pi^{(2)} = (\pi_1^{(2)}, \pi_2^{(2)}) \sim \text{Dirichlet}\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)$
- Split each component according to the splitting rule:

$$\theta_1^{(2)}, \theta_2^{(2)} \overset{iid}{\sim} \text{Beta}\left(\frac{\alpha}{2}\cdot\frac{1}{2}, \frac{\alpha}{2}\cdot\frac{1}{2}\right)$$

$$\begin{aligned}\pi^{(4)} &= (\theta_1^{(2)}\pi_1^{(2)}, (1-\theta_1^{(2)})\pi_1^{(2)}, \theta_2^{(2)}\pi_2^{(2)}, (1-\theta_2^{(2)})\pi_2^{(2)}) \\ &\sim \text{Dirichlet}\left(\frac{\alpha}{4}, \frac{\alpha}{4}, \frac{\alpha}{4}, \frac{\alpha}{4}\right)\end{aligned}$$

- Repeat to get $\pi^{(K)} \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right)$
- As $K \to \infty$, we get a vector with infinitely many components

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## The Dirichlet process

- Let *H* be a distribution on some space Ω – e.g. a Gaussian distribution on the real line.
- Let $\pi \sim \lim_{K\to\infty} \text{Dirichlet}\left(\frac{\alpha}{K} \dots, \frac{\alpha}{K}\right)$
- For $k = 1, \dots, \infty$ let $\theta_k \sim H.$
- Then $G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}$ is an infinite distribution over *H*.
- We write $G \sim \text{DP}(\alpha, H)$

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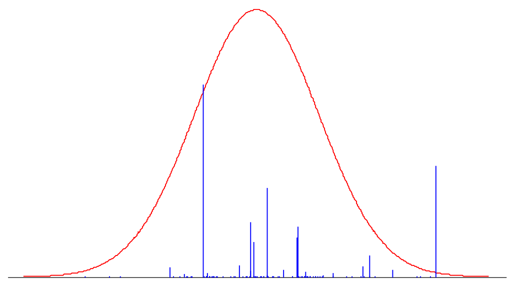
## Samples from the Dirichlet process

- Samples from the Dirichlet process are *discrete*.
- We call the point masses in the resulting distribution, *atoms*.



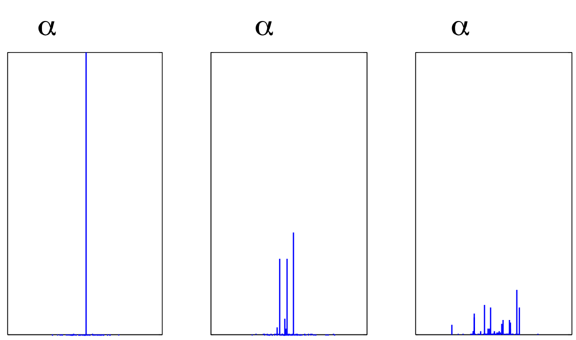
- The *base measure* H determines the *locations* of the atoms.





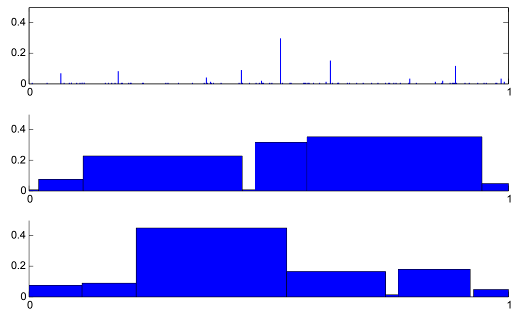
## Samples from the Dirichlet process

- The *concentration parameter* α determines the distribution over atom sizes.
- Small values of α give *sparse* distributions.



## Properties of the Dirichlet process

- For any partition $A_1, \ldots, A_K$ of $\Omega$, the total mass assigned to each partition is distributed according to $Dir(\alpha H(A_1), \ldots, \alpha H(A_K))$



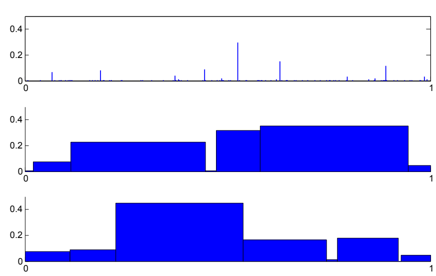




## Definition: Finite marginals

- A Dirichlet process is the unique distribution over probability distributions on some space $\Omega$, such that for any finite partition $A_1, \ldots, A_K$ of $\Omega$,

$$(P(A_1), \ldots, P(A_K)) \sim \text{Dirichlet}(\alpha H(A_1), \ldots, \alpha H(A_K)).$$



[Ferguson, 1973]

# Conjugacy of the Dirichlet process

- Let $A_1,\ldots,A_K$ be a partition of $\Omega$, and let $H$ be a measure on $\Omega$. Let $P(A_k)$ be the mass assigned by $G \sim \mathrm{DP}(\alpha, H)$ to partition $A_k$. Then $(P(A_1),\ldots,P(A_K)) \sim \mathrm{Dirichlet}(\alpha H(A_1),\ldots,\alpha H(A_K))$.
- If we see an observation in the $J^{th}$ segment, then
$$(P(A_1),\ldots,P(A_j),\ldots,P(A_K)|X_1 \in A_j)$$
$$\sim \mathrm{Dirichlet}(\alpha H(A_1),\ldots,\alpha H(A_j)+1,\ldots,\alpha H(A_K)).$$
- This must be true for *all possible partitions of* $\Omega$.
- This is only possible if the posterior of G, given an observation x, is given by
$$G|X_1 = x \sim \mathrm{DP}\left(\alpha+1, \frac{\alpha H + \delta_x}{\alpha+1}\right)$$

# Predictive distribution

- The Dirichlet process clusters observations.
- A new data point can either join an existing cluster, or start a new cluster.
- Question: What is the predictive distribution for a new data point?
- Assume $H$ is a continuous distribution on $\Omega$. This means for every point $\theta$ in $\Omega$, $H(\theta) = 0$.
- First data point:
  - Start a new cluster.
  - Sample a parameter $\theta_1$ for that cluster.

## Predictive distribution

- We have now split our parameter space in two: the singleton $\theta_1$, and everything else.
- Let $\pi_1$ be the atom at $\theta_1$.
- The combined mass of all the other atoms is $\pi_* = 1-\pi_1$.
- *A priori*, $(\pi_1, \pi_*) \sim \text{Dirichlet}(0, \alpha)$
- A posteriori, $(\pi_1, \pi_*)|X_1 = \theta_1 \sim \text{Dirichlet}(1, \alpha)$




## Predictive distribution

- If we integrate out $\pi_1$ we get

$$\begin{aligned} P(X_2 = \theta_k | X_1 = \theta_1) &= \int P(X_2 = \theta_k | (\pi_1, \pi_*)) P((\pi_1, \pi_* | X_1 = \theta_1) d\pi_1 \\ &= \int \pi_k \text{Dirichlet}((\pi_1, 1 - \pi_1)|1, \alpha) d\pi_1 \\ &= \mathbb{E}_{\text{Dirichlet}(1,\alpha)}[\pi_k] \\ &= \begin{cases} \frac{1}{1+\alpha} & \text{if } k = 1 \\ \frac{\alpha}{1+\alpha} & \text{for new } k. \end{cases} \end{aligned}$$

# Predictive distribution

- Lets say we choose to start a new cluster, and sample a new parameter $\theta_2 \sim H$. Let $\pi_2$ be the size of the atom at $\theta_2$.
- A posteriori, $(\pi_1, \pi_2, \pi_*)|X_1 = \theta_1, X_2 = \theta_2 \sim \text{Dirichlet}(1, \alpha)$.
- If we integrate out $\pi = (\pi_1, \pi_2, \pi_*)$ we get

$$\begin{aligned} P(X_3 = \theta_k | X_1 = \theta_1, X_2 = \theta_2) \\ &= \int P(X_3 = \theta_k|\pi) P(\pi | X_1 = \theta_1, X_2 = \theta_2) d\pi \\ &= \mathbb{E}_{\text{Dirichlet}(1,1,\alpha)}[\pi_k] \\ &= \begin{cases} \frac{1}{2+\alpha} & \text{if } k = 1 \\ \frac{1}{2+\alpha} & \text{if } k = 2 \\ \frac{\alpha}{2+\alpha} & \text{for new } k. \end{cases} \end{aligned}$$

# Predictive distribution

- In general, if $m_k$ is the number of times we have seen $X_i$=$k$, and $K$ is the total number of observed values,

$$\begin{aligned} P(X_{n+1} = \theta_k | X_1, \dots, X_n) &= \int P(X_{n+1} = \theta_k|\pi) P(\pi | X_1, \dots, X_n) d\pi \\ &= \mathbb{E}_{\text{Dirichlet}(m_1, \dots, m_K, \alpha)}[\pi_k] \\ &= \begin{cases} \frac{m_k}{n+\alpha} & \text{if } k \leq K \\ \frac{\alpha}{n+\alpha} & \text{for new cluster.} \end{cases} \end{aligned}$$

- We tend to see observations that we have seen before
  – *rich-get-richer property.*
- We can always add new features – *nonparametric.*

## Polya urn scheme

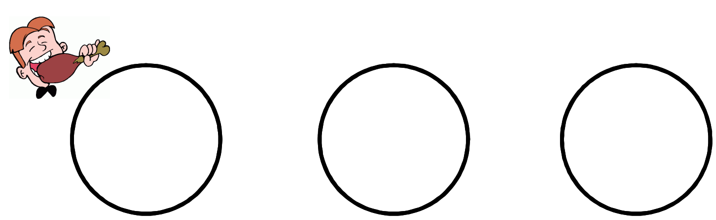
- The resulting distribution over data points can be thought of using the following urn scheme.
- An urn initially contains a black ball of mass α.
- For *n*=1,2,… sample a ball from the urn with probability proportional to its mass.
- If the ball is black, choose a previously unseen color, record that color, and return the black ball plus a unit-mass ball of the new color to the urn.
- If the ball is not black, record it's color and return it, plus another unit-mass ball of the same color, to the urn

[Blackwell and MacQueen,1973]

## Chinese restaurant process

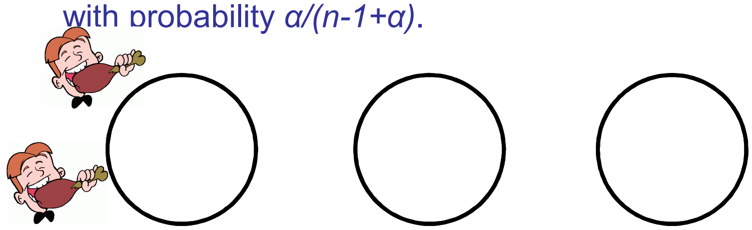
- The distribution over partitions can be described in terms of the following restaurant metaphor:
- The first customer enters a restaurant, and picks a table.



## Chinese restaurant process

- The distribution over partitions can be described in terms of the following restaurant metaphor:
- The first customer enters a restaurant, and picks a table.
- The $n^{th}$ customer enters the restaurant. He sits at an existing table with probability $m_k/(n-1+\alpha)$, where $m_k$ is the number of people sat at table $k$. He starts a new table with probability $\alpha/(n-1+\alpha)$.



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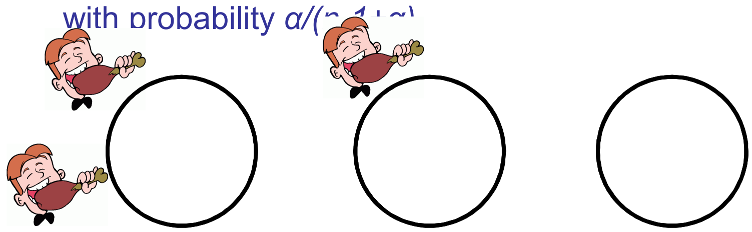

 

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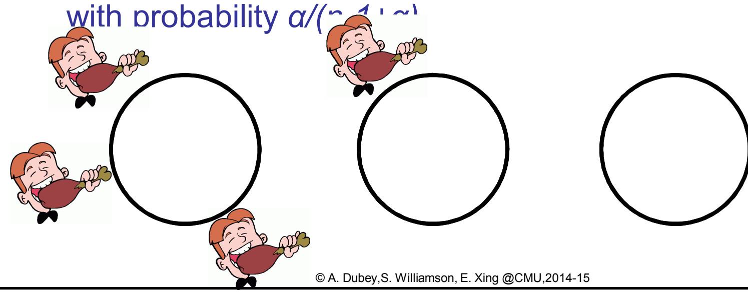




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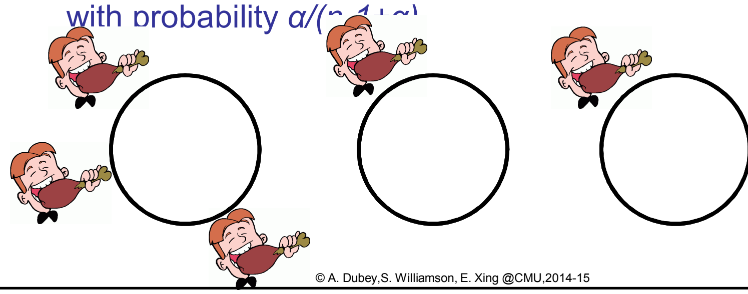

## Exchangeability

- An interesting fact: the distribution over the clustering of the first *N* customers *does not depend on the order in which they arrived*.
- Homework: Prove to yourself that this is true.
- However, the customers are not independent – they tend to sit at popular tables.
- We say that distributions like this are *exchangeable*.
- De Finetti's theorem: If a sequence of observations is exchangeable, there must exist a distribution given which they are iid.
- The customers in the CRP are iid given the underlying Dirichlet process – by integrating out the DP, they become dependent.

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## Stick breaking construction

- We can represent samples from the Dirichlet process exactly.
- Imagine a stick of length 1, representing total probability.
- For k=1,2,…
  - Sample a beta(1,α) random variable $b_k$.
  - Break off a fraction $b_k$ of the stick. This is the $k^{th}$ atom size
  - Sample a random location for this atom.
  - Recurse on the remaining stick.

$$G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}$$

$$\pi_k := b_k \prod_{j=1}^{k-1} (1 - b_k)$$

$$b_k \sim \mathrm{Beta}(1, \alpha)$$

[Sethuraman, 1994]

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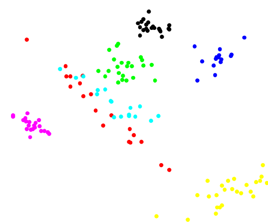
# Inference in the DP mixture model

$$G := \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k} \sim \mathrm{DP}(\alpha, H)$$

$$\phi_n \sim G$$

$$x_n \sim f(\phi_n)$$



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# Inference: Collapsed sampler

- We can integrate out $G$ to get the CRP.
- Reminder: Observations in the CRP are exchangeable.
- Corollary: When sampling any data point, we can always rearrange the ordering so that it is the last data point.
- Let $z_n$ be the cluster allocation of the $n$th data point.
- Let $K$ be the total number of instantiated clusters.
- Then

$$p(z_n = k | x_n, z_{-n}, \phi_{1:K}) \propto \begin{cases} m_k f(x_n|\phi_k) & k \leq K \\ \alpha \int_\Omega f(x_n|\phi) H(d\phi) & k = K+1 \end{cases}$$

- If we use a conjugate prior for the likelihood, we can often integrate out the cluster parameters

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## Problems with the collapsed sampler

- We are only updating one data point at a time.
- Imagine two “true” clusters are merged into a single cluster – a single data point is unlikely to “break away”.
- Getting to the true distribution involves going through low probability states ➔ mixing can be slow.
- If the likelihood is not conjugate, integrating out parameter values for new features can be difficult.
- Neal [2000] offers a variety of algorithms.
- Alternative: Instantiate the latent measure.

## Inference: Blocked Gibbs sampler

- Rather than integrate out *G*, we can instantiate it.
- Problem: *G* is infinite-dimensional.
- Solution: Approximate it with a truncated stick-breaking process:

$$G^K := \sum_{k=1}^{K} \pi_k \delta_{\theta_k}$$

$$\pi_k = b_k \prod_{j=1}^{k-1} (1 - b_j)$$

$$b_k \sim \mathrm{Beta}(1, \alpha), k = 1, \dots, K-1$$

$$b_K = 1$$

# Inference: Blocked Gibbs sampler

- Sampling the cluster indicators:

$$p(z_n = k|\text{rest}) \propto \pi_k f(x_n|\theta_k)$$

- Sampling the stick breaking variables:
  - We can think of the stick breaking process as a sequence of binary decisions.
  - Choose $z_n = 1$ with probability $b_1$.
  - If $z_n \neq 1$, choose $z_n = 2$ with probability $b_2$.
  - *etc..*

$$b_k|\text{rest} \sim \text{Beta}\left(1 + m_k, \alpha + \sum_{j=k+1}^{K} m_j\right)$$




# Inference: Slice sampler

- Problem with batch sampler: Fixed truncation introduces error.
- Idea:
  - Introduce *random truncation*.
  - If we marginalize over the random truncation, we recover the full model.
- Introduce a uniform random variable $u_n$ for each data point.
- Sample indicator $z_n$ according to

$$p(z_n = k|\text{rest}) = I(\pi_k > u_n) f(x_n|\theta_k)$$

- Only a **finite** number of possible values.

## Inference: Slice sampler

- The conditional distribution for $u_n$ is just:

$$u_n | \text{rest} \sim \text{Uniform}[0, \pi_{z_n}]$$

- Conditioned on the $u_n$ and the $z_n$, the $\pi_k$ can be sampled according to the block Gibbs sampler.
- Only need to represent a finite number $K$ of components such that

$$1 - \sum_{k=1}^{K} \pi_k < min(u_n)$$

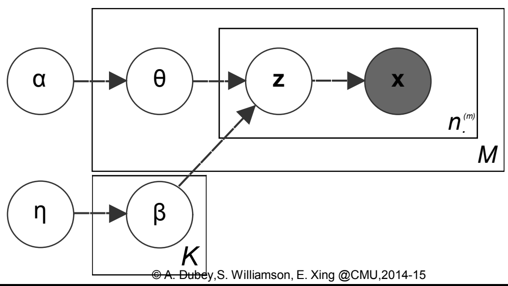
## Topic models

- Topic models describe documents using a distribution over features.
- Each feature is a distribution over words
- Each document is represented as a collection of words (usually unordered – “bag of words” assumption).
- The words within a document are distributed according to a document-specific mixture model
  - Each word in a document is associated with a feature.
- The features are shared between documents.
- The features learned tend to give high probability to semantically related words – “topics”

# Latent Dirichlet allocation

- For each topic $k=1,\dots,K$
  - Sample a distribution over words, $\beta \sim \mathrm{Dir}(\eta_1,\dots,\eta_V)$
- For each document $m=1,\dots,M$
  - Sample a distribution over topics, $\theta_m \sim \mathrm{Dir}(\alpha_1,\dots,\alpha_K)$
  - For each word $n=1,\dots,N_m$
    - Sample a topic $z_{mn} \sim \mathrm{Discrete}(\theta_m)$
    - Sample a word $w_{mk} \sim \mathrm{Discrete}(\beta_z)$




Blei et al, 2002

# "Topics" found by LDA

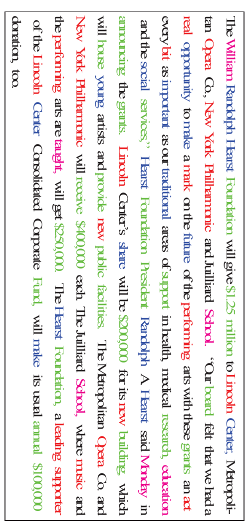

The William Randolph Hearst Foundation will give $1.25 million to Lincoln Center, Metropolitan Opera Co., New York Philharmonic and Juilliard School. "Our board felt that we had a real opportunity to make a mark on the future of the performing arts with these grants an act every bit as important as our traditional areas of support in health, medical research, education and the social services," Hearst Foundation President Randolph A. Hearst said Monday in announcing the grants. Lincoln Center's share will be $200,000 for its new building, which will house young artists and provide new public facilities. The Metropolitan Opera Co. and New York Philharmonic will receive $400,000 each. The Juilliard School, where music and the performing arts are taught, will get $250,000. The Hearst Foundation, a leading supporter of the Lincoln Center Consolidated Corporate Fund, will make its usual annual $100,000 donation, too.

Image from Blei et al, 2002

## Constructing a topic model with infinitely many topics

- LDA: Each distribution is associated with a distribution over *K* topics.
- Problem: How to choose the number of topics?
- Solution:
  - Infinitely many topics!
  - Replace the Dirichlet distribution over topics with a Dirichlet process!
- Problem: We want to make sure the topics are *shared* between documents




## Sharing topics

- In LDA, we have *M* independent samples from a Dirichlet distribution.
- The weights are different, but the topics are fixed to be the same.
- If we replace the Dirichlet distributions with Dirichlet processes, each atom of each Dirichlet process will pick a topic *independently* of the other topics.

## Sharing topics

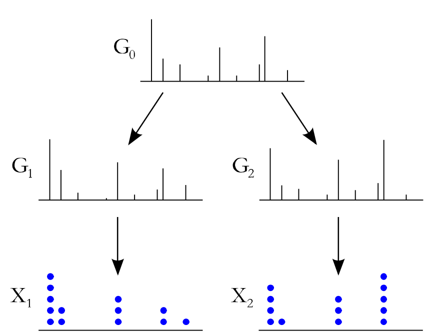
- Because the base measure is *continuous*, we have zero probability of picking the same topic twice.
- If we want to pick the same topic twice, we need to use a *discrete* base measure.
- For example, if we chose the base measure to be $H = \sum_{k=1}^{K} \alpha_k \delta_{\beta_k}$ then we would have LDA again.
- We want there to be an infinite number of topics, so we want an *infinite, discrete* base measure.
- We want the location of the topics to be random, so we want an *infinite, discrete, random* base measure.




## Hierarchical Dirichlet Process (Teh et al, 2006)

- Solution: Sample the base measure from a Dirichlet process!

$$G_0 \sim \mathrm{DP}(\gamma, H)$$
$$G_m \sim \mathrm{DP}(\alpha, G_0)$$



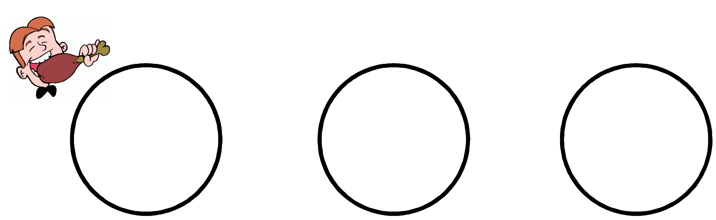
## Chinese restaurant franchise

- Imagine a *franchise* of restaurants, serving an infinitely large, global menu.
- Each table in each restaurant orders a single dish.
- Let $n_{rt}$ be the number of customers in restaurant *r* sitting at table *t.*
- Let $m_{rd}$ be the number of tables in restaurant *r* serving dish *d*.
- Let $m_{.d}$ be the number of tables, across *all* restaurants, serving dish *d.*



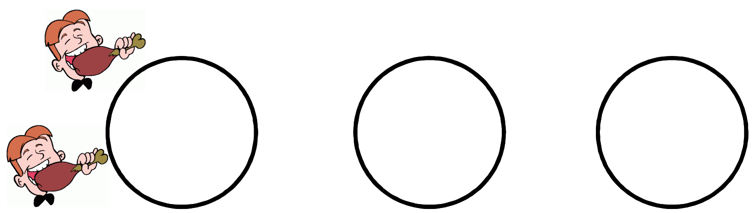

## Chinese restaurant franchise

- Customers enter the restaurants, and sit at tables according to the Chinese restaurant process
  - The first customer enters a restaurant, and picks a table.



## Chinese restaurant franchise

- Customers enter the restaurants, and sit at tables according to the Chinese restaurant process
  - The first customer enters a restaurant, and picks a table.
  - The $n^{th}$ customer enters the restaurant. He sits at an existing table with probability $m_k/(n-1+\alpha)$, where $m_k$ is the number of people sat at table $k$. He starts a new table with probability $\alpha/(n-1+\alpha)$.



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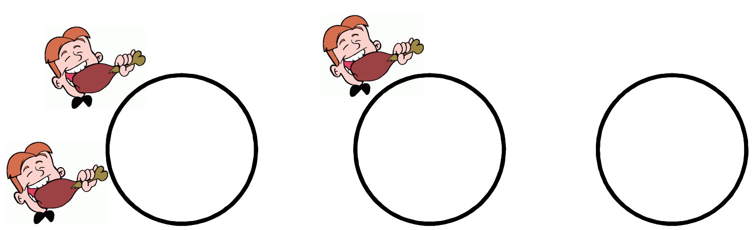

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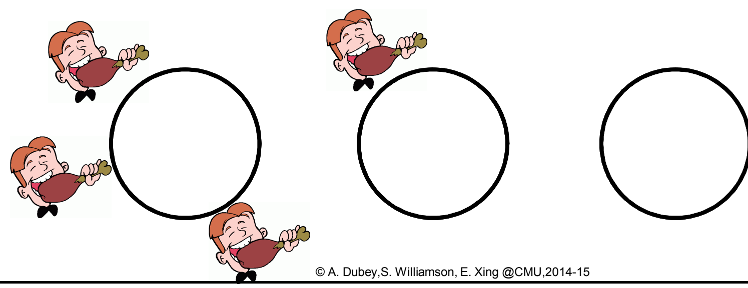





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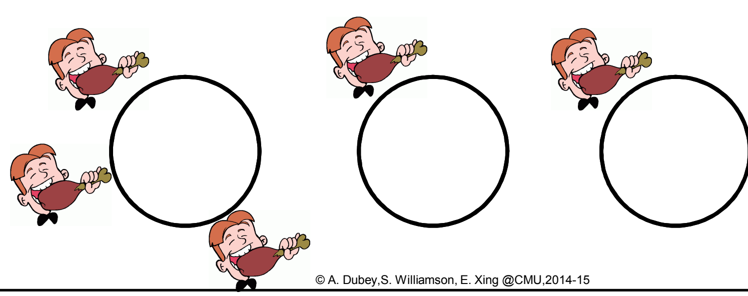
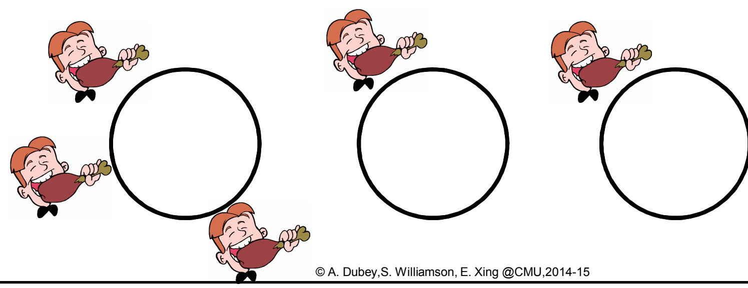


## Chinese restaurant franchise

- Each *table* in each restaurant picks a *dish*, with probability proportional to the number of times it has been served across *all* restaurants.

$$p(\text{table } t \text{ chooses dish } d|\text{previous tables}) = \begin{cases} \frac{m_d}{T+\gamma} & \text{for an existing table} \\ \frac{\gamma}{T+\gamma} & \text{for a new table} \end{cases}$$







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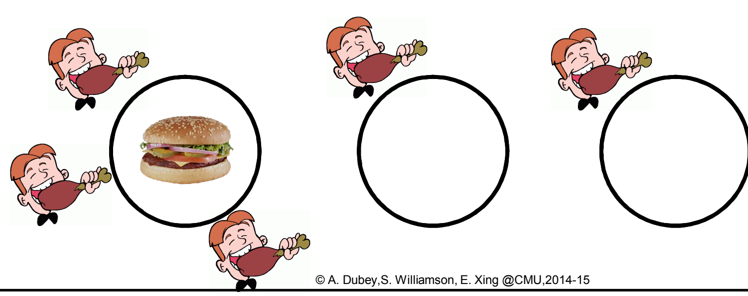

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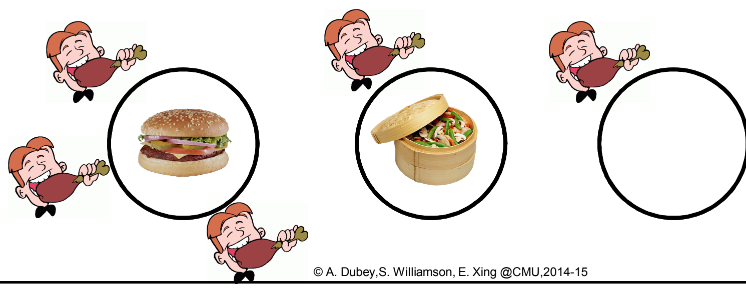




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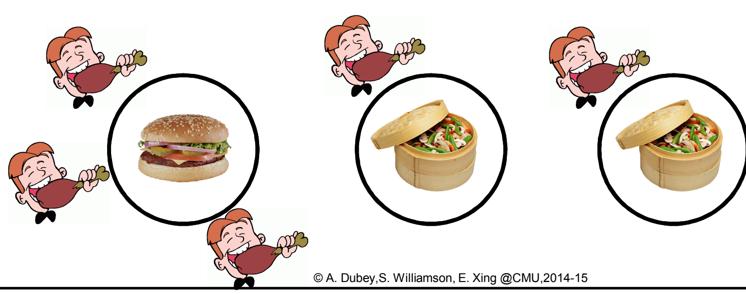

# An infinite topic model

- Restaurants = documents; dishes = topics.
- Let *H* be a V-dimensional Dirichlet distribution, so a sample from *H* is a distribution over a vocabulary of *V* words.
- Sample a global distribution over topics,

$$G_0 := \sum_{k=1}^{\infty} \pi_k \delta_{\beta_k} \sim \mathrm{DP}(\alpha, H)$$

- For each document *m=1,…,M*
  - Sample a distribution over topics, $G_m \sim DP(\gamma, G_0)$.
  - For each word $n=1,\ldots,N_m$
    - Sample a topic $\phi_{mn} \sim \mathrm{Discrete}(G_0)$.
    - Sample a word $w_{mk} \sim \mathrm{Discrete}(\phi_{mn})$.




# The “right” number of topics