Weak versions of extended resolution

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Resolution

Refutes a propositional formula in conjunctive normal form (i.e., a set of clauses) by using the single rule

$$\frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

to derive the empty clause.

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Throughout this talk, "proof" $\equiv$ "refutation."

Example: resolution proof

$$\Gamma = (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y}) \wedge (y \vee z)$$

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Tree-like:

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Sequence-like:

$$\overline{x} \vee \overline{z},\ \overline{y} \vee z,\ x \vee y \vee \overline{z},\ x \vee \overline{y},\ y \vee z,\ z,\ x \vee y,\ x,\ \overline{x},\ \bot$$

Extended resolution (ER)

At any step, derive

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where p, q are arbitrary literals and x is a *new* variable.

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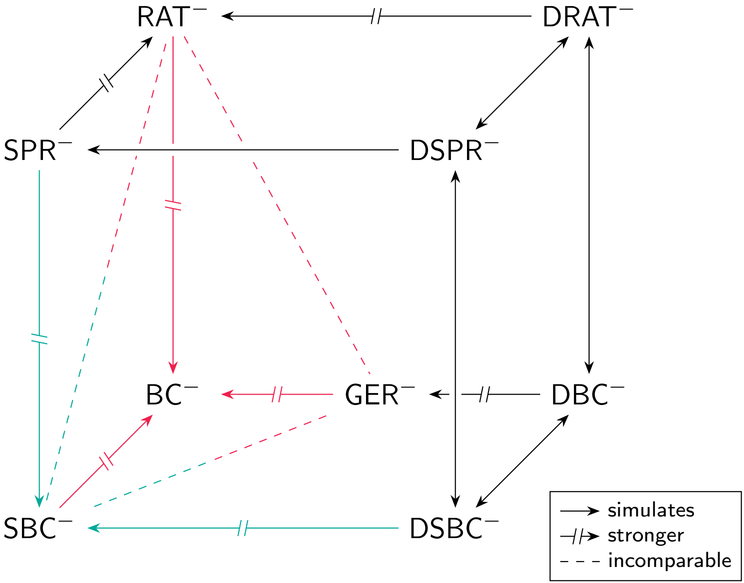
This talk: Relative strengths of different weak versions of ER

Results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- Pigeonhole principle
- Bit pigeonhole principle
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- Clique-coloring principle
- Tseitin tautologies
- OR-ification, XOR-ification, lifting with indexing gadgets

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Without loss of generality, pigeon $n + 1$ is mapped to hole n...

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Lemma

A clause C is redundant with respect to a formula Γ if and only if there exists a partial assignment τ such that

$$\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\tau.$$

Syntactic criteria for redundancy

Definition (Blocked clause)

A clause $C = x \vee C'$ is *blocked* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ,

$$C' \vee D' \text{ is tautological.}$$

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Example (Blocked clause)

$$C = x \lor y \lor \neg z$$
$$\Gamma = (\neg x \lor \neg y) \land (\neg x \lor z) \land (y \lor z)$$

C is blocked for x with respect to Γ.

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Claim. If assignment τ sets $x = 1$, then $\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\tau$.

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Clauses of the form $\neg x \lor D'$ might be falsified by $\alpha \circ \tau$, but there is some $y \in C'$ such that $\neg y \in D'$ and $\alpha \circ \tau$ still sets y to $\alpha(y) = 0$.

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Definition (Resolution asymmetric tautology*)

A clause $C = x \vee C'$ is a *RAT* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ,

$$C' \vee D' \text{ is tautological or subsumed by } \Gamma.$$

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Definition (Set-blocked clause)

A clause $C = L \vee C'$ is an *SBC* for L with respect to a formula Γ if, for every clause D in Γ such that $D \cap \neg L \neq \varnothing$ and $D \cap L = \varnothing$,

$$\big(C \setminus L\big) \vee \big(D \setminus \neg L\big) \text{ is tautological.}$$

The proof systems

- ER — resolution + extension
- BC — resolution + blocked clause addition
- BC^{-} — BC without new variables
- DBC — BC with deletion
- DBC^{-} — BC with deletion and without new variables

  ⋮

- SPR — resolution + "SBC $\times$ RAT" addition

  ⋮

Effectively*, BC^- simulates ER

Lemma

Suppose that a formula Γ has an ER proof of size m and that $X = (y \vee x_1 \vee \cdots \vee x_m) \wedge y$ has no variables in common with Γ. Then $\Gamma \wedge X$ has a BC^- proof of size $O(m)$.

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Lemma

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Proof. Consider a use of the extension rule in the ER proof that introduces $x_i \leftrightarrow p \wedge q$. WLOG, the literals p and q are not new.

Add the clauses

$$\neg x_i \vee \neg y \vee p \qquad \neg x_i \vee \neg y \vee q \qquad x_i \vee \neg p \vee \neg q$$

in sequence as blocked clauses. Resolve against y. □

Guarded extension variables

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.
Recall $X = (y \vee x_1 \vee \cdots \vee x_m) \wedge y$, which made $\Gamma \wedge X$ easy for BC^-.

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Idea: Guard the variables by clauses to make them hard to access.
P will somehow use the included variables to simulate the ER proof.
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Example

With respect to the formula $(\neg x \vee y) \wedge (\neg x \vee \neg y)$, any clause blocked for x has to include both $\neg y$ and y.

Example: lower bound

Lemma

$f(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \left[(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma) \right]$ is no easier than Γ for BC^-.

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1. View a BC^- proof of $f(\Gamma)$ as a resolution proof of $f(\Gamma) \wedge \Delta$, where Δ is derived by a sequence of blocked clause additions.

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3. Since $\Gamma \subseteq f(\Gamma)$, every clause in Δ is in particular blocked wrt Γ.
4. For the assignment $\alpha(x_i) = 1$, we have $(f(\Gamma) \wedge \Delta)|_\alpha = \Gamma \wedge \Delta'$, where Δ' is possible to derive from Γ in BC^-.

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$f(\Gamma) \coloneqq \Gamma \wedge \bigwedge_{i=1}^{m} \big[(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma)\big]$ is no easier than Γ for BC^-.

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5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.

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5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.
6. If $\Gamma \wedge \Delta'$ is easy for resolution, then Γ is easy for BC^-. $\square$

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

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$\mathsf{GER}^- \not\geq \mathsf{RAT}^-$
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$$f(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \left[(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma) \right]$$

$f(\Gamma)$ is easy for RAT^- (regardless of whether Γ is).

$f(\Gamma)$ is at least as hard as Γ for GER^- and SBC^-.

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

$\mathsf{RAT}^- \not\geq \mathsf{GER}^-$
$\mathsf{RAT}^- \not\geq \mathsf{SBC}^-$

$$g(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \left[(\neg x_i \vee y_i) \wedge (x_i \vee \neg y_i) \right]$$

$g(\Gamma)$ is easy for both GER^- and SBC^- (for different reasons).

$g(\Gamma)$ is at least as hard as Γ for RAT^-.

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

$\mathrm{SBC}^- \not\geq \mathrm{GER}^-$

$$h_s(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{s} \left[(x_i \vee y_j \vee \neg z_j) \wedge (\neg x_i \vee y_j \vee \neg z_j) \right]$$
$$\wedge \bigwedge_{j=1}^{s} \left[(\neg y_j \vee z_j) \wedge (y_j \vee \Gamma) \wedge (\neg z_j \vee \Gamma) \right]$$

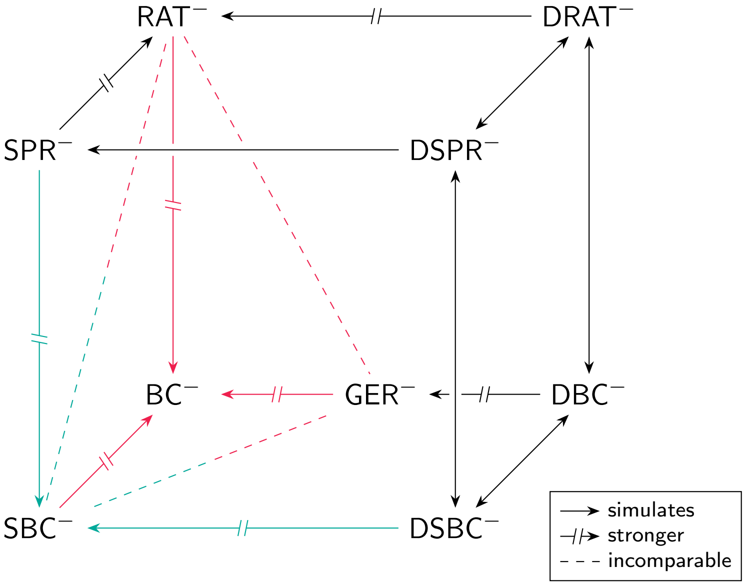
$h_s(\Gamma)$ is easy for GER^- (regardless of whether Γ is).

$h_s(\Gamma)$ is at least as hard as Γ for SBC^- with suitable choice of s.

Open questions

- Lower bounds for SPR^- and above
- Separations using "natural" principles
- Any subsystem of Frege above resolution that DBC^- simulates
- Other uses of the high-level idea in proof complexity

Results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