Substructural Parametricity

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Abstract

Ordered, linear, and other substructural type systems allow us to expose deep properties of programs at the syntactic level of types. In this paper, we develop a family of unary logical relations that allow us to prove consequences of parametricity for a range of substructural type systems. A key idea is to parameterize the relation by an algebra, which we exemplify with a monoid and commutative monoid to interpret ordered and linear type systems, respectively. We prove the fundamental theorem of logical relations and apply it to deduce extensional properties of inhabitants of certain types. Examples include demonstrating that the ordered types for list append and reversal are inhabited by exactly one function, as are types of some tree traversals. Similarly, the linear type of the identity function on lists is inhabited only by permutations of the input. Our most advanced example shows that the ordered type of the list fold function is inhabited only by the fold function.

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1 Introduction

Substructural type systems and parametric polymorphism are two mechanisms for capturing precise behavioral properties of programs at the type level, enabling powerful static reasoning. The goal of this paper is to give a theoretical account of these mechanisms in combination.

Substructural type systems have been investigated since the advent of linear logic, starting with the seminal paper by Girard and Lafont [11]. Among other applications, with substructural type systems one can avoid garbage collection, update memory in place [20, 21], make message-passing [9, 7] or shared memory concurrency [10, 28] safe, model quantum computation [8], or reason efficiently about imperative programs [19]. Substructural type systems have thus been incorporated into languages that seek to offer such guarantees, such as Rust, Koka, Haskell, Oxidized OCaml, and ProtoQuipper.

Parametricity, originally introduced for System F [35], enables the idea that programs whose types involve universal quantification over type parameters have certain strong semantic properties. This idea supports powerful program reasoning principles such as representation independence across abstraction boundaries [23] and "theorems for free" that can be derived about all inhabitants of certain types, for example that every inhabitant of $\forall\alpha.\ \alpha \to \alpha$ is equivalent to the identity function [38].

The theory of substructural logics and type systems is now relatively well understood, including several ways to integrate substructural and structural type systems [6, 31, 12]. It is therefore somewhat surprising that we do not yet know much about how parametricity and its applications interact with them. The main foray into substructural parametricity is a paper by Zhao et al. [39] that accounts for a polymorphic dual-intuitionistic linear logic. They point out that logical relations on closed terms are problematic because substitution obscures linearity. Their solution was to construct a logical relation on open terms, necessitating the introduction of "semantic typing" judgments that mirror the syntactic type system, which complicates their definition and application.

In this paper, we follow an approach using *constructive resource semantics* in the style of Reed et al. [32, 34, 33] to construct logical relations on *closed terms*. We start with an ordered type system [30, 29, 17], which may be considered the least permissive among substructural type systems and therefore admits a pleasantly minimal definition. However, the construction is generic with respect to certain properties of the resource algebra, which allows us to extend it also to linear and unrestricted types. Consequences of our development include that certain polymorphic types are only inhabited by the polymorphic append and reverse functions on lists. Similarly, certain types are only inhabited by functions that swap or maintain the order of pairs. The most advanced application shows that the ordered type of fold over lists is inhabited only by the fold function.

We conjecture that the three substructural modes we investigate—ordered, linear, and unrestricted—can also be combined in an adjoint framework [6, 12] but leave this to future work. Similarly, we simplify our presentation by defining only a *unary* logical relation since it is sufficient to demonstrate proof-of-concept, but nothing stands in the way of a more general definition (for example, to support representation independence results).

2 A Minimalist Fragment

We start with a small fragment of the Full Lambek Calculus [18, 22], extended with parametric polymorphism [36]. This fragment is sufficient to illustrate the main ideas behind our constructions. For the sake of simplicity we choose a Curry-style formulation of typing, concentrating on properties of untyped terms rather than intrinsically typed terms. This allows the same terms to inhabit ordered, linear, and unrestricted types and thereby focus on semantic rather than syntactic issues.

$$
\begin{array}{llll}
\text{Types} & A, B & ::= & \alpha \mid A \bullet B \mid A \rightarrowtail B \mid A \twoheadrightarrow B \mid \forall \alpha.\, A \\
\text{Expressions} & e & ::= & x \\
 & & \mid & (e_1, e_2) \mid \textbf{match}\; e\; ((x, y) \Rightarrow e') \qquad\qquad (A \bullet B) \\
 & & \mid & \lambda x.\; e \mid e_1\, e_2 \qquad\qquad\qquad\qquad\qquad\quad (A \rightarrowtail B, A \twoheadrightarrow B)
\end{array}
$$

In this fragment, we have $A \bullet B$ (read "A fuse B") which, logically, is a noncommutative conjunction. We have two forms of implication: $A \rightarrowtail B$ (read: "A under B", originally written as $A \backslash B$) which is true if from the hypothesis A *placed at the left end of the antecedents* we can deduce B, and $A \twoheadrightarrow B$ (read: "B over A", originally written as $B \,/\, A$) which is true if from the hypothesis A *placed at the right and of the antecedents* we can prove B. Lambek's original notation was suitable for the sequent calculus and its applications in linguistics, but is less readable for natural deduction and functional programming.

Our basic typing judgment has the form $\Delta \mid \Omega \vdash e : A$ where Δ consists of hypotheses $\alpha\;\mathsf{type}$, and Ω is an *ordered context* $(x_1 : A_1) \ldots (x_n : A_n)$. We make the standard presuppositions that $\Delta \vdash A\;\mathsf{type}$ and $\Delta \vdash A_i\;\mathsf{type}$ for every $x_i : A_i$ in Ω, and that both type variables and term variables are pairwise distinct. The rules are show in Figure 1.

Here are a few example judgments that hold or fail. We elide the context $\Delta = (\alpha\;\mathsf{type}, \beta\;\mathsf{type}, \gamma\;\mathsf{type})$.

$$\frac{}{\Delta \mid x : A \vdash x : A}\ \mathsf{hyp}$$

$$\frac{\Delta \mid \Omega\,(x : A) \vdash e : B}{\Delta \mid \Omega \vdash \lambda x.\, e : A \twoheadrightarrow B}\ {\twoheadrightarrow}I \qquad \frac{\Delta \mid \Omega \vdash e_1 : A \twoheadrightarrow B \quad \Delta \mid \Omega_A \vdash e_2 : A}{\Delta \mid \Omega\,\Omega_A \vdash e_1\, e_2 : B}\ {\twoheadrightarrow}E$$

$$\frac{\Delta \mid (x : A)\,\Omega \vdash e : B}{\Delta \mid \Omega \vdash \lambda x.\, e : A \rightarrowtail B}\ {\rightarrowtail}I \qquad \frac{\Delta \mid \Omega \vdash e_1 : A \rightarrowtail B \quad \Delta \mid \Omega_A \vdash e_2 : A}{\Delta \mid \Omega_A\,\Omega \vdash e_1\, e_2 : B}\ {\rightarrowtail}E$$

$$\frac{\Delta \mid \Omega_A \vdash e_1 : A \quad \Delta \mid \Omega_B \vdash e_2 : B}{\Delta \mid \Omega_A\,\Omega_B \vdash (e_1, e_2) : A \bullet B}\ {\bullet}I \qquad \frac{\Delta \mid \Omega \vdash e : A \bullet B \quad \Delta \mid \Omega_L\,(x : A)\,(y : B)\,\Omega_R \vdash e' : C}{\Delta \mid \Omega_L\,\Omega\,\Omega_R \vdash \mathbf{match}\ e\ ((x, y) \Rightarrow e') : C}\ {\bullet}E$$

$$\frac{\Delta, \alpha\ \mathsf{type} \mid \Omega \vdash e : A}{\Delta \mid \Omega \vdash e : \forall \alpha.\, A}\ \forall I \qquad \frac{\Delta \mid \Omega \vdash e : \forall \alpha. A(\alpha) \quad \Delta \vdash B\ \mathsf{type}}{\Delta \mid \Omega \vdash e : A(B)}\ \forall E$$

Figure 1 Ordered Natural Deduction

$$\begin{array}{llll}
 & \vdash & \lambda x.\, x : \alpha \rightarrowtail \alpha & \\
 & \vdash & \lambda x.\, x : \alpha \twoheadrightarrow \alpha & \\
 & \nvdash & \lambda x.\, \lambda y.\, x : \alpha \twoheadrightarrow (\beta \twoheadrightarrow \alpha) & \text{(no weakening)} \\
 & \nvdash & \lambda x.\, (x, x) : \alpha \twoheadrightarrow (\alpha \bullet \alpha) & \text{(no contraction)} \\
 & \vdash & \lambda x.\, \lambda y.\, (x, y) : \alpha \twoheadrightarrow (\beta \twoheadrightarrow (\alpha \bullet \beta)) & \\
 & \nvdash & \lambda x.\, \lambda y.\, (x, y) : \alpha \rightarrowtail (\beta \rightarrowtail (\alpha \bullet \beta)) & \text{(no exchange)} \\
f : \beta \twoheadrightarrow (\alpha \rightarrowtail \gamma) & \vdash & \lambda x.\, \lambda y.\, (f\, y)\, x : \alpha \rightarrowtail (\beta \twoheadrightarrow \gamma) & \text{(``associativity'')} \\
g : \alpha \rightarrowtail (\beta \twoheadrightarrow \gamma) & \vdash & \lambda y.\, \lambda x.\, (g\, x)\, y : \beta \twoheadrightarrow (\alpha \rightarrowtail \gamma) & \\
g : (\alpha \bullet \beta) \twoheadrightarrow \gamma & \vdash & \lambda x.\, \lambda y.\, g\, (x, y) : \alpha \twoheadrightarrow (\beta \twoheadrightarrow \gamma) & \text{(currying)} \\
f : \alpha \twoheadrightarrow (\beta \twoheadrightarrow \gamma) & \vdash & \lambda p.\, \mathbf{match}\ p\ ((x, y) \Rightarrow f\, x\, y) : (\alpha \bullet \beta) \twoheadrightarrow \gamma & \text{(uncurrying)}
\end{array}$$

The strictures of the typing judgment imply that certain types may be uninhabited, or may be inhabited by terms that are extensionally equivalent to a small number of possibilities. To count the number of linear functions, translate $(A \twoheadrightarrow B)^{\mathsf{L}} = (A \rightarrowtail B)^{\mathsf{L}} = A^{\mathsf{L}} \multimap B^{\mathsf{L}}$ and $(A \bullet B)^{\mathsf{L}} = A^{\mathsf{L}} \otimes B^{\mathsf{L}}$ and similarly for unrestricted functions.

Types	Ordered	Linear	Unrestricted
$\alpha \twoheadrightarrow \alpha$	1	1	1
$\alpha \twoheadrightarrow (\alpha \twoheadrightarrow \alpha)$	0	0	2
$\alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))$	1	2	4
$\alpha \twoheadrightarrow (\alpha \rightarrowtail (\alpha \bullet \alpha))$	1	2	4
$\alpha \twoheadrightarrow (\beta \twoheadrightarrow (\beta \bullet \alpha))$	0	1	1
$\alpha \twoheadrightarrow (\beta \twoheadrightarrow (\alpha \bullet \beta))$	1	1	1

Because our intended application language based on adjoint natural deduction [12] is call-by-value, we can give a straightforward big-step operational semantics [15] relating an expression to its final value. Because this evaluation does not directly interact with or benefit from substructural properties, we show it without further comment in Figure 2. It has the property of preservation that if $\cdot \vdash e : A$ and $e \hookrightarrow v$ then $\cdot \vdash v : A$. Jang et al. give an account [12] that exploits linearity and other substructural properties, although not the lack of exchange.

$$\frac{}{\lambda x.\, e \hookrightarrow \lambda x.\ e} \qquad \frac{e_1 \hookrightarrow \lambda x.\, e_1' \quad e_2 \hookrightarrow v_2 \quad [v_2/x]e_1' \hookrightarrow v}{e_1\, e_2 \hookrightarrow v}$$

$$\frac{e_1 \hookrightarrow v_1 \quad e_2 \hookrightarrow v_2}{(e_1, e_2) \hookrightarrow (v_1, v_2)} \qquad \frac{e \hookrightarrow (v_1, v_2) \quad [v_1/x, v_2/y]e' \hookrightarrow v'}{\mathbf{match}\ e\ ((x,y) \Rightarrow e') \hookrightarrow v'}$$

Figure 2 Big-Step Operational Semantics

3 An Algebraic Logical Predicate

Because of our particular setting, we define two mutually dependent logical predicates: $[\![A]\!]$ for closed expressions and $[A]$ for closed values. In addition, the relation is parameterized by elements from an algebraic domain which may have various properties. For the ordered case, it should be a monoid, for the linear case a commutative monoid. However, the rules themselves do not require this for the pure sets of terms. We use $m \cdot n$ for the binary operation on the monoid, and ϵ for its unit.

Ignoring polymorphism for now, we write $m \Vdash e \in [\![A]\!]$ and $m \Vdash v \in [A]$, which is defined by

$$m \Vdash e \in [\![A]\!] \iff e \hookrightarrow v \wedge m \Vdash v \in [A]$$

$$\begin{array}{lcl} m \Vdash v \in [1] & \iff & m = \epsilon \wedge v = (\,) \\ m \Vdash v \in [A \bullet B] & \iff & \exists m_1, m_2.\, m = m_1 \cdot m_2 \wedge v = (v_1, v_2) \wedge m_1 \Vdash v_1 \in [A] \wedge m_2 \Vdash v_2 \in [B] \\ m \Vdash v \in [A \twoheadrightarrow B] & \iff & \forall k.\, k \Vdash w \in [A] \Longrightarrow m \cdot k \Vdash v\, w \in [\![B]\!] \\ m \Vdash v \in [A \rightarrowtail B] & \iff & \forall k.\, k \Vdash w \in [A] \Longrightarrow k \cdot m \Vdash v\, w \in [\![B]\!] \end{array}$$

We can see how the algebraic structure of the monoid tracks information about order if its operation is not commutative.

The key step, as usual in logical predicates of this nature, is the case for universal quantification and type variables. We map type variables α to relations R_B between monoid elements and values in $[B]$ where B is a closed type. We indicate this mapping from type variables to sets of values S and write it as a superscript on $\Vdash$.

$$\begin{array}{lcl} m \Vdash^S v \in [\alpha] & \iff & m\ S(\alpha)\ v \\ m \Vdash^S v \in [\forall \alpha.\, A(\alpha)] & \iff & \forall B, R_B.\, m \Vdash^{S, \alpha \mapsto R_B} v \in [A(\alpha)] \end{array}$$

The mapping S is just passed through identically in the cases of the relation defined above.

We can already verify some interesting properties. As a first example we show that the logical predicates are nonempty.

▶ **Theorem 1.**

$$\epsilon \Vdash \lambda x.\, \lambda y.\, (x, y) \in [\![\forall \alpha.\, \alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))]\!]$$

Proof. Because the λ-expression is a value, we need to check

$$\epsilon \Vdash \lambda x.\, \lambda y.\, (x, y) \in [\forall \alpha.\, \alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))]$$

By definition, this is true if for an arbitrary A and relation $m\ R_A\ v$ we have

$$\epsilon \Vdash^{\alpha\mapsto R_A} \lambda x.\, \lambda y.\, (x, y) \in [\alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))]$$

Using the definition of the logical predicate for right implication twice and one intermediate step of evaluation, this holds iff

$$m \cdot k \Vdash^{\alpha\mapsto R_A} (\lambda y.\, (v, y))\ w \in [\![\alpha \bullet \alpha]\!]$$

for all m, k with $m \Vdash^{\alpha\mapsto R_A} v$ and $k \Vdash^{\alpha\mapsto R_A} w$. By evaluation, this is true iff

$$m \cdot k \Vdash^{\alpha\mapsto R_A} (v, w) \in [\alpha \bullet \alpha]$$

Now we can apply the definition of $[A \bullet B]$, splitting $m \cdot k$ into m and k and reducing it to

$$m \Vdash^{\alpha\mapsto R_A} v \wedge k \Vdash^{\alpha\mapsto R_A} w$$

Both of these hold because, by assumption, $m\ R_A\ v$ and $k\ R_A\ w$. ◀

More interesting, perhaps, is the reverse.

▶ **Theorem 2.** *If*

$$\epsilon \Vdash e \in [\![\forall\alpha.\, \alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))]\!]$$

then e is extensionally equal to $\lambda x.\, \lambda y.\, (x, y)$. In particular, it can not be $\lambda x.\, \lambda y.\, (y, x)$.

Proof. We choose our monoid to be the free monoid over two generators a and b and we choose an arbitrary closed type A and two elements v and w. Moreoever, we pick R_A relating only $a\ R_A\ v$ and $b\ R_A\ w$.

From the definitions (and skipping over some simple properties regarding evaluation), we obtain

$$a \cdot b \Vdash^{\alpha\mapsto R_A} e\ v\ w \in [\![\alpha \bullet \alpha]\!]$$

By the clauses for $[\![\alpha \bullet \alpha]\!]$, $[\alpha \bullet \alpha]$ and α we conclude that

$$e\ v\ w \hookrightarrow (u_1, u_2)$$

for some values u_1 and u_2 with $a\ R_A\ u_1$ and $b\ R_A\ u_2$. Because the only value related to a is v and the only value related to b is w, we conclude $u_1 = v$ and $u_2 = w$. Therefore

$$e\ v\ w \hookrightarrow (v, w)$$

Since v and w were chosen arbitrarily, we see that e is extensionally equal to $\lambda x.\, \lambda y.\, (x, y)$. ◀

4 The Fundamental Theorem

The fundamental theorem of logical predicates states that every well-typed term is in the predicate. Our relations also include terms that are not well-typed, which can occasionally be useful when one exceeds the limits of static typing.

We need a few standard lemmas, adapted to this case. We only spell out one.

▶ **Lemma 3** (Compositionality). *Define R_A such that $k\ R_A\ w$ iff $k \Vdash w \in [A]$. Then $m \Vdash^{S,\alpha\mapsto R_A} v \in [B(\alpha)]$ iff $m \Vdash^{S} v \in [B(A)]$*

Proof. By induction on $B(\alpha)$. ◀

We would like to prove the fundamental theorem by induction over the structure of the typing derivation. Since our logical relation is defined for closed terms, we need a closing substitution η. We define:

$$\begin{array}{lcl} m \Vdash^S (x \mapsto v) \in [x : A] & \iff & m \Vdash^S v \in [A] \\ m \Vdash^S (\eta_1\, \eta_2) \in [\Omega_1\, \Omega_2] & \iff & \exists m_1, m_2.\, m = m_1 \cdot m_2 \wedge m_1 \Vdash^S \eta_1 \in [\Omega_1] \wedge m_1 \Vdash^S \eta_2 \in [\Omega_2] \\ m \Vdash^S (\cdot) \in [\cdot] & \iff & m = \epsilon \end{array}$$

Due to the associativity of the monoid operation and concatenation of contexts, this constitutes a valid definition.

▶ **Theorem 4** (Fundamental Theorem (purely ordered))**.** *Assume $\Delta \mid \Omega \vdash e : A$, a mapping S with domain Δ, and closing substitution $m \Vdash^S \eta \in [\Omega]$. Then $m \Vdash^S \eta(e) \in [\![A]\!]$.*

Proof. By induction on the structure of the given typing derivation. We show a few cases.
Case:

$$\frac{}{\Delta \mid x : A \vdash x : A}\ \mathsf{hyp}$$

Then $m \Vdash^S \eta(x) \in [A]$ by assumption and definition, and $m \Vdash^S \eta(x) \in [\![A]\!]$ since $\eta(x)$ is a value.

Case:

$$\frac{\Delta \mid \Omega\,(x : A) \vdash e : B}{\Delta \mid \Omega \vdash \lambda x.\, e : A \twoheadrightarrow B}\ \twoheadrightarrow I$$

$m \Vdash^S \eta \in [\Omega]$	Given
$k \Vdash^S v \in [A]$	Assumption (1)
$k \Vdash^S (x \mapsto v) \in [x : A]$	By definition
$m \cdot k \Vdash^S (\eta, x \mapsto v) \in [\Omega\,(x : A)]$	By definition
$m \cdot k \Vdash^S (\eta, x \mapsto v)(e) \in [\![B]\!]$	By ind. hyp.
$m \cdot k \Vdash^S (\eta(\lambda x.\, e))\, v \in [\![B]\!]$	By reverse evaluation, v closed
$m \Vdash^S \eta(\lambda x.\, e) \in [A \twoheadrightarrow B]$	By definition, discharging (1)
$m \Vdash^S \eta(\lambda x.\, e) \in [\![A \twoheadrightarrow B]\!]$	By definition

Case:

$$\frac{\Delta \mid \Omega \vdash e_1 : A \twoheadrightarrow B \quad \Delta \mid \Omega_A \vdash e_2 : A}{\Delta \mid \Omega\, \Omega_A \vdash e_1\, e_2 : B}\ \twoheadrightarrow E$$

$m \Vdash^S \eta \in [\Omega\, \Omega_A]$	Given
$m_1 \Vdash^S \eta_1 \in [\Omega]$ and $m_2 \Vdash^S \eta_2 \in [\Omega_A]$	
for some m_1, m_2, η_1, and η_2 with $m = m_1 \cdot m_2$ and $\eta = \eta_1\, \eta_2$	By definition
$m_1 \Vdash^S \eta_1(e_1) \in [\![A \twoheadrightarrow B]\!]$	By ind. hyp.
$m_2 \Vdash^S \eta_2(e_2) \in [\![A]\!]$	By ind. hyp.
$\eta_1(e_1) \hookrightarrow v_1$ with $m_1 \Vdash^S v_1 \in [A \twoheadrightarrow B]$	By definition
$\eta_2(e_2) \hookrightarrow v_2$ with $m_2 \Vdash^S v_2 \in [A]$	By definition
$m_1 \cdot m_2 \Vdash^S v_1\, v_2 \in [\![B]\!]$	By definition
$(\eta_1\, \eta_2)(e_1\, e_2) = (\eta_1(e_1))\, (\eta_2(e_2))$	By properties of substitution
$m \Vdash^S \eta(e_1\, e_2) \in [\![B]\!]$	Since $m = m_1 \cdot m_2$ and $\eta = (\eta_1\, \eta_2)$

Case:

$$\frac{\Delta, \alpha\ \mathsf{type} \mid \Omega \vdash e : A}{\Delta \mid \Omega \vdash e : \forall \alpha.\, A}\ \forall I$$

$m \Vdash^S \eta \in [\Omega]$	Given
R_B an arbitrary relation $k\ R_B\ v$	Assumption (1)
$m \Vdash^{S,\alpha\mapsto R_B} \eta \in [\Omega]$	Since α fresh
$m \Vdash^{S,\alpha\mapsto R_B} \eta(e) \in [\![A]\!]$	By ind. hyp.
$m \Vdash^S \eta(e) \in [\![\forall\alpha.\, A]\!]$	By definition, discharging (1)

Case:

$$\frac{\Delta \mid \Omega \vdash e : \forall \alpha . A(\alpha) \quad \Delta \vdash B\ \mathsf{type}}{\Delta \mid \Omega \vdash e : A(B)}\ \forall E$$

$m \Vdash^S \eta \in [\Omega]$	Given
$m \Vdash^S \eta(e) \in [\![\forall\alpha.\, A(\alpha)]\!]$	By ind. hyp.
Define $k\ R_B\ v$ iff $k \Vdash^S v \in [B]$	
$m \Vdash^{S,\alpha\mapsto R_B} \eta(e) \in [\![A(\alpha)]\!]$	By definition
$m \Vdash^{S,\alpha\mapsto R_B} v \in [A(\alpha)]$ for $\eta(e) \hookrightarrow v$	By definition
$m \Vdash^S v \in [A(B)]$	By compositionality (Lemma 3)
$m \Vdash^S \eta(e) \in [\![A(B)]\!]$	By definition

◀

Because typing implies that the logical predicate holds, the earlier examples now apply to well-typed terms.

▶ **Theorem 5** ((Theorem 2 revisited)). *If*

$$\cdot \vdash e : \forall \alpha.\, \alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))$$

then e is extensionally equivalent to $\lambda x.\, \lambda y.\, (x, y)$.

Proof. We just note that

$$\epsilon \Vdash e \in [\![\forall \alpha.\, \alpha \twoheadrightarrow (\alpha \twoheadrightarrow (\alpha \bullet \alpha))]\!]$$

since $(\cdot) \in [\cdot]$ and $(\cdot)e = e$ and the empty mapping S suffices without any free type variables. Then we appeal to the reasoning in Theorem 2. ◀

5 Unrestricted Functions

We are interested in properties of functions such as list append or list reversal, or higher-order functions such as fold. This requires inductive types, but the functions on them are not used linearly. For example, append has a recursive call in the case of a nonempty list, but none in the case of an empty list. We could introduce a general modality $!A$ for this purpose. A simpler alternative that is sufficient for our situation is to introduce unrestricted function types $A \to B$ (usually coded as $!A \multimap B$ in linear logic or $!A \twoheadrightarrow B$ in ordered logic). This path has been explored previously [30] with different motivations. There, an *open* logical relation was defined on the negative monomorphic fragment in order to show the existence of canonical forms, a property that is largely independent of ordered typing.

Adding unrestricted functions is rather straightforward in typing by using two kinds of variables: those that are ordered and those unrestricted. Then, in the logical predicate, unrestricted variables must not use any resources, that is, they are assigned the unit element ϵ of the monoid during the definition.

The generalized judgment has the form $\Delta \mid \Gamma \,;\, \Omega \vdash e : A$ where Γ contains type assignments for variables that can be used in an unrestricted (not linear and not ordered) way. All the previous rules are augmented by propagating Γ from the conclusion to all premises. Because our term language is untyped, no extensions are needed there. Similarly, the rules of our dynamics do not need to change.

$$\frac{}{\Delta \mid \Gamma, x : A \,;\, \cdot \vdash x : A}\ \mathsf{hyp}$$

$$\frac{\Delta \mid \Gamma, x : A \,;\, \Omega \vdash e : B}{\Delta \mid \Gamma \,;\, \Omega \vdash \lambda x.\, e : A \to B}\ {\to}I \qquad \frac{\Delta \mid \Gamma \,;\, \Omega \vdash e_1 : A \to B \quad \Delta \mid \Gamma \,;\, \cdot \vdash e_2 : A}{\Delta \mid \Gamma \,;\, \Omega \vdash e_1\, e_2 : B}\ {\to}E$$

Figure 3 Unrestricted functions

We extend the logical predicate using arguments not afforded any resources.

$$m \Vdash v \in [A \to B] \iff \forall w.\, \epsilon \Vdash w \in [A] \Longrightarrow m \Vdash v\, w \in [\![B]\!]$$

The fundamental theorem extends in a straightforward way.

▶ **Theorem 6** (Fundamental Theorem (mixed ordered/unrestricted))**.** *Assume $\Delta \mid \Gamma \,;\, \Omega \vdash e : A$, a mapping S with domain Δ, and two closing substitutions $\epsilon \Vdash^S \theta \in [\Gamma]$ and $m \Vdash^S \eta \in [\Omega]$. Then $m \Vdash^S (\theta \,;\, \eta)(e) \in [\![A]\!]$.*

Proof. By induction on the structure of the given typing derivation. ◀

An interesting side effect of these definitions is that if we omit ordered functions but retain pairs we obtain the "usual" formulation closed logical predicates, including certain consequences of parametricity for the ordinary λ-calculus.

▶ **Theorem 7.** *If*

$$\cdot \vdash e : \forall \alpha.\, \alpha \to (\alpha \to (\alpha \bullet \alpha))$$

then e is extensionally equivalent to one of 4 functions: $\lambda x.\, \lambda y.\, (x, y)$, $\lambda x.\, \lambda y.\, (y, x)$, $\lambda x.\, \lambda y.\, (x, x)$, or $\lambda x.\, \lambda y.\, (y, y)$.

Proof. By the fundamental theorem, we have

$$\epsilon \Vdash e \in [\![\forall \alpha.\, \alpha \to (\alpha \to (\alpha \bullet \alpha))]\!]$$

We use this for an abitrary closed type A with two arbitary values v, and w and relation R_A with $\epsilon\ R_A\ v$ and $\epsilon\ R_A\ w$. Exploiting the definition, we get

$$\epsilon \Vdash^{\alpha \mapsto R_A} e \in [\![\alpha \to (\alpha \to (\alpha \bullet \alpha))]\!]$$

Using the definition of function twice and skipping over some evaluation and reverse evaluation, we obtain

$$\epsilon \Vdash^{\alpha \mapsto R_A} f\, v\, w \in [\![\alpha \bullet \alpha]\!]$$

This means that $f\,v\,w \hookrightarrow (u_1, u_2)$ with $\epsilon\ R_A\ u_1$ and $\epsilon\ R_A\ u_2$. Because of the definition of R_A there are 4 possibilities for (u_1, u_2), namely (v, w), (w, v), (v, v) and (w, w). This in turn means e is extensionally equal to one of the 4 functions shown. ◀

6 Unit, Sums, Twist, and Recursive Types

At this point, we are at a crossroads. Because we would like to prove theorems regarding more complex data structures such as lists, trees, or streams, we could extend the development with general inductive and coinductive types and their recursors. We conjecture that this is possible and leave it to future work. The other path is to work with *purely positive types*, including recursive ones whose values can be directly observed. In this approach, the definition of the logical predicate is quite easy to extend. It becomes a nested inductive definition: either the type becomes smaller or, once we encounter a purely positive type and recursion is possible, from then on the terms become strictly smaller. In this paper we take the latter approach, which excludes coinductive types such as streams from consideration, but still yields many interesting and intuitive consequences.

We take the opportunity to also round out our language with unit, sums, and twist (the symmetric counterpart of fuse). We use a signature defining *equirecursive type names* that may be arbitrarily mutually recursive. Because such type definitions are otherwise closed, they constitute metavariables in the sense of contextual modal type theory [24]. Each type definition $F[\Delta] = A^+$ must be *contractive*, that is, its definiens cannot be be another type name. Moreover, A^+ must be *purely positive*, which is interpreted *inductively*.

$$\begin{array}{llcl}
\text{Types} & A & ::= & \ldots \mid A \circ B \mid \oplus\{\ell : A_\ell\}_{\ell \in L} \mid \mathbf{1} \\
\text{Purely Positive Types} & A^+, B^+ & ::= & A^+ \bullet B^+ \mid A^+ \circ B^+ \mid \mathbf{1} \mid \oplus\{\ell : A_\ell^+\}_{\ell \in L} \mid F[\theta] \\
\text{Type Definitions} & \Sigma & ::= & F[\Delta] = A^+ \mid (\cdot) \mid \Sigma_1, \Sigma_2 \\
\text{Type Substitutions} & \theta & ::= & \alpha \mapsto A^+ \mid (\cdot) \mid \theta_1\,\theta_2
\end{array}$$

The language of expressions does not change much because type names are equirecursive.

$$\begin{array}{llcll}
\text{Expression} & e & ::= & \ldots & \\
& & \mid & k(e) \mid \mathbf{match}\ e\ \{\ell(x_\ell) \Rightarrow e'\}_{\ell \in L} & (\oplus\{\ell : A_\ell\}) \\
& & \mid & (\,) \mid \mathbf{match}\ e\ ((\,) \Rightarrow e') & (\mathbf{1})
\end{array}$$

We add the type $A \circ B$ ("twist"), symmetric to $A \bullet B$, since encoding it as $B \bullet A$ requires rewriting terms, flipping the order of pairs. For $A \circ B$ it is merely the typechecking that changes. This allows more types to be assigned to the same term. We allow silent unfolding of type definitions, so there are no explicit rules for $F[\theta]$.

The logical predicate is also extended in a straightforward manner. We assume the signature Σ is fixed and therefore do not carry it explicitly through the definitions.

$$\begin{array}{lcl}
m \Vdash^S v \in [\mathbf{1}] & \iff & m = \epsilon \wedge v = (\,) \\
m \Vdash^S v \in [A \circ B] & \iff & \exists m_1, m_2.\ m = m_2 \cdot m_1 \wedge v = (v_1, v_2) \\
& & \wedge\ m_1 \Vdash^S v_1 \in [A] \wedge m_2 \Vdash^S v_2 \in [B] \\
m \Vdash^S k(v) \in [\oplus\{\ell : A_\ell\}_{\ell \in L}] & \iff & m \Vdash^S v \in [A_k] \wedge k \in L \\
m \Vdash^S v \in [F[\theta]] & \iff & m \Vdash^S v \in \theta(A^+) \text{ where } F[\Delta] = A^+ \in \Sigma
\end{array}$$

Because we have equirecursive type definitions, the last clause is usually applied silently. The definition of the logical predicate is no longer straightforwardly inductive on the structure of the type. However, we see that for purely positive types (the only ones involved in recursion),

$$\frac{\Delta \mid \Gamma \,;\, \Omega_A \vdash e_1 : A \quad \Delta \mid \Gamma \,;\, \Omega_B \vdash e_2 : B}{\Delta \mid \Gamma \,;\, \Omega_B\, \Omega_A \vdash (e_1, e_2) : A \circ B}\ \circ I$$

$$\frac{\Delta \mid \Gamma \,;\, \Omega \vdash e : A \circ B \quad \Delta \mid \Gamma \,;\, \Omega_L\,(y : B)\,(x : A)\,\Omega_R \vdash e' : C}{\Delta \mid \Gamma \,;\, \Omega_L\, \Omega\, \Omega_R \vdash \mathbf{match}\ e\ ((x, y) \Rightarrow e') : C}\ \circ E$$

$$\frac{}{\Delta \mid \Gamma \,;\, \cdot \vdash (\,) : \mathbf{1}}\ \mathbf{1}I \qquad \frac{\Delta \mid \Gamma \,;\, \Omega \vdash e : A \circ B \quad \Delta \mid \Gamma \,;\, \Omega_L\, \Omega_R \vdash e' : C}{\Delta \mid \Gamma \,;\, \Omega_L\, \Omega\, \Omega_R \vdash \mathbf{match}\ e\ ((\,) \Rightarrow e') : C}\ \mathbf{1}E$$

$$\frac{(k \in L) \quad \Delta \mid \Gamma \,;\, \Omega \vdash e : A_k}{\Delta \mid \Gamma \,;\, \Omega \vdash k(e) : \oplus\{\ell : A_\ell\}_{\ell \in L}}\ \oplus I$$

$$\frac{\Delta \mid \Gamma \,;\, \Omega \vdash e : \oplus\{\ell : A_\ell\}_{\ell \in L} \quad (\Delta \mid \Gamma \,;\, \Omega_L\,(x_\ell : A_\ell)\,\Omega_R \vdash e_\ell : A_\ell) \quad (\forall \ell \in L)}{\Delta \mid \Gamma \,;\, \Omega_L\, \Omega\, \Omega_R \vdash \mathbf{match}\ e\ \{\ell(x_\ell) \Rightarrow e_\ell\}_{\ell \in L} : C}\ \oplus E$$

Figure 4 Ordered Natural Deduction, Extended

$$\frac{}{(\,) \hookrightarrow (\,)} \qquad \frac{e \hookrightarrow (\,) \quad e' \hookrightarrow v'}{\mathbf{match}\ e\ ((\,) \Rightarrow e') \hookrightarrow v'}$$

$$\frac{e \hookrightarrow v}{k(e) \hookrightarrow k(v)} \qquad \frac{e \hookrightarrow k(v) \quad [v/x_k]e_k \hookrightarrow v'}{\mathbf{match}\ e\ \{\ell(x_\ell) \Rightarrow e_\ell\}_{\ell \in L} \hookrightarrow v'}$$

Figure 5 Big-Step Operational Semantics, Extended

the *value* in the definition becomes strictly smaller in each clause if type definitions are contractive. In other words, we now have a nested inductive definition of the logical predicate, first on the type, and when the type is purely positive, on the structure of the value.

We can also add recursion to our expression language with the key proviso that we either restrict ourselves to certain patterns of recursion (for example, primitive recursion), or termination is guaranteed by other external means (for example, using an analysis using sized types [1]). This assumption allows us to maintain the structure of the logical predicate, even if it is no longer a means to prove termination (which we are not interested in for this paper).

▶ **Lemma 8** (Compositionality (including purely positive equirecursive types)). *Define R_A such that $k\ R_A\ w$ iff $k \Vdash w \in [A]$. Then $m \Vdash^{S, \alpha \mapsto R_A} v \in [B(\alpha)]$ iff $m \Vdash^S v \in [B(A)]$.*

Proof. By nested induction on the definition of the logical predicate for $B(\alpha)$, first on the structure of B and second on the structure of the value when a purely positive type $F[\theta]$ has been reached. ◀

▶ **Theorem 9** (Fundamental Theorem (including purely positive recursive types)). *Assume $\Delta \mid \Gamma \,;\, \Omega \vdash e : A$, a mapping S with domain Δ, and two closing substitutions $\epsilon \Vdash^S \theta \in [\Gamma]$ and $m \Vdash^S \eta \in [\Omega]$. Then $m \Vdash^S (\theta \,;\, \eta)(e) \in [\![A]\!]$.*

Proof. By induction on the structure of the given typing derivation. When reasoning about functions and recursion, we need the assumption of termination. ◀

7 Free Theorems for Ordered Lists

We start with some theorems about ordered lists, not unlike those analyzed by Wadler [38], but much sharper due to substructural typing. We define two versions of ordered lists, one that is ordered left-to-right and one that is ordered right-to-left. Both of these use exactly the same representation; just their typing is different.

$$\begin{aligned} \mathit{llist}\ \alpha &= \oplus\{\underline{\mathsf{nil}} : \mathbf{1}, \underline{\mathsf{cons}} : \alpha \bullet \mathit{llist}\ \alpha\} \\ \mathit{rlist}\ \alpha &= \oplus\{\underline{\mathsf{nil}} : \mathbf{1}, \underline{\mathsf{cons}} : \alpha \circ \mathit{rlist}\ \alpha\} \end{aligned}$$

The following will be a useful lemma about ordered lists.

▶ **Lemma 10** (Ordered Lists).

$$\begin{aligned} m \Vdash^S v \in [\mathit{llist}\ \alpha] \quad &\Longleftrightarrow \quad m = \epsilon \wedge v = \underline{\mathsf{nil}}\,(\,) \\ &\qquad \vee \exists m_1, m_2.\, m = m_1 \cdot m_2 \wedge v = \underline{\mathsf{cons}}\,(v_1, v_2) \\ &\qquad\quad \wedge m_1\ S(\alpha)\ v_1 \wedge m_2 \Vdash v_2 \in [\mathit{llist}\ \alpha] \\ m \Vdash^S v \in [\mathit{rlist}\ \alpha] \quad &\Longleftrightarrow \quad m = \epsilon \wedge v = \underline{\mathsf{nil}}\,(\,) \\ &\qquad \vee \exists m_1, m_2.\, m = m_2 \cdot m_1 \wedge v = \underline{\mathsf{cons}}\,(v_1, v_2) \\ &\qquad\quad \wedge m_1\ S(\alpha)\ v_1 \wedge m_2 \Vdash v_2 \in [\mathit{rlist}\ \alpha] \end{aligned}$$

Proof. By unrolling the definitions of the logical predicate and the equirecursive nature of the definition of lists. ◀

For the applications, we abbreviate lists, writing $[v_1, \ldots, v_n]$ for $\underline{\mathsf{cons}}(v_1, \ldots, \underline{\mathsf{cons}}(v_n, \underline{\mathsf{nil}}\,(\,)))$.

$$\begin{aligned} m \Vdash^{\alpha \mapsto R_A} v \in [\mathit{llist}\ \alpha] &\Longleftrightarrow m = m_1 \cdots m_n, v = [v_1, \ldots, v_n] \text{ where } m_i\ R_A\ v_i \text{ (for some } m_i, v_i) \\ m \Vdash^{\alpha \mapsto R_A} v \in [\mathit{rlist}\ \alpha] &\Longleftrightarrow m = m_n \cdots m_1, v = [v_1, \ldots, v_n] \text{ where } m_i\ R_A\ v_i \text{ (for some } m_i, v_i) \end{aligned}$$

Now we state a first property of lists that follows as a consequence of our parameterized logical predicate.

▶ **Theorem 11.** *If* $\cdot \vdash f : \forall \alpha.\, \mathit{llist}\ \alpha \twoheadrightarrow \mathit{llist}\ \alpha$ *then* f *is extensionally equal to the identity function on lists.*

Proof. By the fundamental theorem, we have

$$\epsilon \Vdash f \in [\forall \alpha.\, \mathit{llist}\ \alpha \twoheadrightarrow \mathit{llist}\ \alpha]$$

To construct a relation R_A we pick an arbitary closed type A. For the monoid, we pick the one freely generated by $a_1, a_2, \ldots$ and define

$$m\ R_A\ v \Longleftrightarrow m = a_i \wedge v = v_i$$

for arbitrary elements v_i. By definition, we obtain

$$\epsilon \Vdash^S f \in [\mathit{llist}\ \alpha \twoheadrightarrow \mathit{llist}\ \alpha]$$

Again by definition, that's the case iff

$$\forall m, v.\, m \Vdash^S v \in [\mathit{llist}\ \alpha] \Longrightarrow \epsilon \cdot m \Vdash^S f\ v \in [\![\mathit{llist}\ \alpha]\!]$$

Here, $\epsilon \cdot m = m$, by the monoid laws. Therefore $f\,v \hookrightarrow w$ and

$$\forall m, v.\, m \Vdash^S v \in [\mathit{llist}\ \alpha] \Longrightarrow m \Vdash^S w \in [\mathit{llist}\ \alpha]$$

We use this for $m = a_1 \cdots a_n$ and $v = [v_1, \ldots, v_n]$. By our lemma about lists and the arbitrary nature of A and v_i we conclude that $w = v$. ◀

By similar reasoning we can obtain the following properties.

▶ **Theorem 12.**
1. *If $f : \forall\alpha.\mathit{rlist}\ \alpha \twoheadrightarrow \mathit{rlist}\ \alpha$ then f is extensionally equal to the identity function.*
2. *If $f : \forall\alpha.\mathit{rlist}\ \alpha \twoheadrightarrow \mathit{llist}\ \alpha$ then f is extensionally equal to the list reversal function.*
3. *If $f : \forall\alpha.\mathit{llist}\ \alpha \twoheadrightarrow \mathit{rlist}\ \alpha$ then f is extensionally equal to the list reversal function.*

Proof. By very similar reasoning to the one in Theorem 11. ◀

But can we deduce properties of higher-order functions using ordered parametricity? We show one primary example; others such as *map* follow directly from it or similarly.

Unlike the usual or even linear parametricity, the type of *fold* guarantees that it must be *the* fold function! Note that the combining function and initial element are unrestricted arguments (one is called for every list element, and one is called only for the empty list), but that the combining function's arguments are ordered.

▶ **Theorem 13.** *If*

$$\cdot \vdash f : \forall\alpha.\forall\beta.\,(\alpha \bullet \beta \twoheadrightarrow \beta) \to \beta \to \mathit{llist}\ \alpha \twoheadrightarrow \beta$$

then f extensionally equal to the fold function, that is,

$$f\ g\ b\,[v_1, v_2, \ldots, v_n] = g(v_1, g(v_2, \ldots, g(v_n, b)))$$

Proof. We use the free monoid over constructors $a_1, a_2, \ldots$. Furthermore, given a type A with arbitrary elements v_i we define the relation R_A by

$$m\ R_A\ v \Longleftrightarrow m = a_i \wedge v = v_i \text{ for some } i$$

Since the type involves another quantified type β, we need to define a second relation R_B where

$$m\ R_B\ d \Longleftrightarrow m = a_{i_1} \cdots a_{i_k} \wedge d = g(v_{i_1}, g(v_{i_2}, \ldots, g(v_{i_k}, b)))$$

With these relations and the definition on of the logical predicate we get the following two properties.
1. $\forall m_1, m_2, v, d.\, m_1\ R_A\ v \wedge m_2\ R_B\ d \Longrightarrow m_1 \cdot m_2\ R_B\ g(v, d)$
2. $\epsilon\ R_B\ g$

Since

$$a_1 \cdots a_n \Vdash^{\alpha \mapsto R_A} [v_1, \ldots, v_n] \in [\mathit{llist}\ \alpha]$$

we can use the second and iterate the first property to conclude that

$$a_1 \cdots a_n\ R_B\ w \qquad \text{for } f\ g\ b\,[v_1, \ldots, v_n] \hookrightarrow w$$

By definition of R_B, this yields

$$f\ g\ b\,[v_1, \ldots, v_n] = g(v_1, \ldots g(v_n, b))$$

in the sense that both sides evaluate to w. Because functions and values were chosen arbitrarily, this expresses the desired extensional equality. ◀

8 Free Theorems Regarding Trees

Consider

$$\begin{aligned}
lxrtree\ \alpha &= \oplus\{\underline{\mathsf{leaf}} : \mathbf{1}, \underline{\mathsf{cons}} : lxrtree\ \alpha \bullet \alpha \bullet lxrtree\ \alpha\} \\
xlrtree\ \alpha &= \oplus\{\underline{\mathsf{leaf}} : \mathbf{1}, \underline{\mathsf{cons}} : (xlrtree\ \alpha \circ \alpha) \bullet xlrtree\ \alpha\} \\
lrxtree\ \alpha &= \oplus\{\underline{\mathsf{leaf}} : \mathbf{1}, \underline{\mathsf{cons}} : lrxtree\ \alpha \bullet (\alpha \circ xlrtree\ \alpha)\}
\end{aligned}$$

Here are a few free theorems regarding such trees. Further variations exist.

▶ **Theorem 14.**

1. *If $f : \forall\alpha.\, lxrtree\ \alpha \twoheadrightarrow llist\ \alpha$ then $f\ t$ lists the elements of t following an* inorder traversal.
2. *If $f : \forall\alpha.\, xlrtree\ \alpha \twoheadrightarrow llist\ \alpha$ then $f\ t$ lists the elements of t following a* preorder traversal.
3. *If $f : \forall\alpha.\, lrxtree\ \alpha \twoheadrightarrow llist\ \alpha$ then $f\ t$ lists the elements of t following a* postorder traversal.

Proof. Trees, like lists, are purely positive types. As such, we can prove an analogue of Lemma 10. We only show one of them, writing t for tree values.

$$\begin{aligned}
m \Vdash^S t \in [lxrtree\ \alpha] \quad \Longleftrightarrow \quad & m = \epsilon \wedge t = \underline{\mathsf{leaf}}(\,) \\
& \vee \exists m_1, k, m_2.\, m = m_1 \cdot k \cdot m_2 \wedge v = \underline{\mathsf{node}}(t_1, v, t_2) \\
& \quad \wedge m_1 \Vdash^S t_1 \in [lxrtree\ \alpha] \wedge k\ S(\alpha)\ v \wedge m_2 \Vdash^S t_2 \in [lxrtree\ \alpha]
\end{aligned}$$

◀

9 From Ordered to Linear Types

Exploring parametricity for *linear* types instead of ordered ones is now a rather straightforward change. We conflate the left and right implication into a single implication, and similarly for conjunction.

ordered	linear	structural	values
$B \mathbin{/} A$			
	$A \multimap B$	$A \to B$	$\lambda x.\, e$
$A \rightarrowtail B$			
$A \bullet B$			
	$A \otimes B$	$A \times B$	(v_1, v_2)
$A \circ B$			
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$(\,)$
$\oplus\{\ell : A_\ell\}$	$\oplus\{\ell : A_\ell\}$	$\oplus\{\ell : A_\ell\}$	$\ell(v)$

We see that in the transition from the linear to the structural case, no further connectives collapse. That's because we would still distinguish eager pairs ($A \times B$) from lazy records that we have elided from our development since they do not introduce any fundamentally new ideas.

From the point of view of typing, the easiest change is to just permit the silent rule of exchange

$$\frac{\Delta \mid \Gamma\,;\, \Omega_L\, (y : B)\, (x : A)\, \Omega_R \vdash e : C}{\Delta \mid \Gamma\,;\, \Omega_L\, (x : A)\, (y : B)\, \Omega_R \vdash e : C}\ \mathsf{exchange}$$

The more typical change is to replace context concatenation $\Omega_L\, \Omega_R$ with context merge $\Omega_L \bowtie \Omega_R$ which allows arbitrary interleavings of the hypotheses.

Our definition of the logical predicates remains that same, except that we assume that the algebraic structure parameterizing our definitions is a *commutative monoid*. This immediately validates the rules of exchange and the fundamental theorem goes through as before.

The results of exploiting the fundamental theorem to obtain parametricity results are no longer as sharp. For example:

▶ **Theorem 15.** *If* $\cdot \vdash e : \forall \alpha.\, \alpha \multimap \alpha \multimap \alpha \otimes \alpha$ *then* f *is extensionally equal to* $\lambda x.\, \lambda y.\, (x, y)$ *or* $\lambda x.\, \lambda y.\, (y, x)$.

Proof. By the fundamental theorem, we have

$$\epsilon \Vdash e \in [\![\forall \alpha.\, \alpha \multimap \alpha \multimap \alpha \otimes \alpha]\!]$$

Therefore $e \hookrightarrow f$ and

$$\epsilon \Vdash f \in [\forall \alpha.\, \alpha \multimap \alpha \multimap \alpha \otimes \alpha]$$

We use a free commutative monoid with two generators, a and b, arbitrary values v and w such that $a \; R \; v$ and $b \; R \; w$. By the fundamental theorem:

$$\epsilon \Vdash^{\alpha \mapsto R} f \in [\alpha \multimap \alpha \multimap \alpha \otimes \alpha]$$

Applying this function to v and w, we obtain that $f \, v \, w \hookrightarrow p$ and

$$a \cdot b \Vdash^{\alpha \mapsto R} p \in [\alpha \otimes \alpha]$$

This is true, again by definition, if for some m and k and p_1 and p_2 we have

$$m \cdot k = a \cdot b \wedge p = (p_1, p_2) \wedge m \Vdash^{\alpha \mapsto R} p_1 \in [\alpha] \wedge k \Vdash^{\alpha \mapsto R} p_2 \in [\alpha]$$

Further applying definitions, we get that for some m, k, p_1, and p_2, we have

$$m \cdot k = a \cdot b \wedge m \; R \; p_1 \wedge k \; R \; p_2$$

There are 4 ways that $a \cdot b$ could be decomposed into $m \cdot k$, but the definition of R leaves only two possibilities: $m = a$, $k = b$, $p_1 = v$ and $p_2 = w$ or $m = b$, $k = a$, $p_1 = w$ and $p_2 = v$. Summarizing: either

$$e \, v \, w \hookrightarrow (v, w)$$

or

$$e \, v \, w \hookrightarrow (w, v)$$

which expresses that e is extensionally equal to $\lambda x.\, \lambda y.\, (x, y)$ or $\lambda x.\, \lambda y.\, (y, x)$. ◀

▶ **Theorem 16.** *If* $\cdot \vdash e : \forall \alpha. \mathit{list}\, \alpha \multimap \mathit{list}\, \alpha$ *then* e *is extensionally equal to a permutation of the list elements.*

Proof. As in the proof of the related ordered theorem, we apply the fundamental theorem and then the definition for arbitrary values v_i with $a_i \; R \; v_i$ where $\alpha \mapsto R$, and the commutative monoid is freely generated from $a_1, a_2, \ldots$.

Taking analogous steps to the ordered case, we conclude that $a_1 \cdots a_n = m_1 \cdots m_n$ modulo commutative (and associativity, as always) where each m_i is a unique a_j. ◀

In the unrestricted case where various algebraic elements are fixed to be ϵ, we can only obtain that every element of the output list must be a member of the input list, because those elements are in $\epsilon \; R \; v_i$. We do not write out the details of this straightforward adaptation of foregoing proofs.

10 Related Work

The most directly related work is Zhao et al.'s [40] open logical relation for parametricity for a dual intuitionistic-linear polymorphic lambda calculus. In this work, they define an *open* logical relation that includes an analog of typing contexts in the semantic model. While our development follows a similar structure, our resource algebraic account allows us to eliminate spurious typechecking premises in definitions and permits a more flexible range of substructural type systems.

Ahmed, Fluet, and Morrisett [3] introduce a logical relation for substructural state via step-indexing, followed by [4] a *linear language with locations* (L3) defined by a Kripke-style logical relation to account for a language with mutable storage. However, the underlying languages in these developments do not support parametric polymorphism. Ahmed, Dreyer, and Rossberg later provide a logical relations account of a System F-based language supporting imperative state update, and they demonstrate representation independence results for this system [2]. The languages modeled in this body of work represent a specific point in the design space with respect to imperative state update and references, as opposed to our more general schema for substructural types in a functional setting. However, Kripke-style logical relations that model a store as a partial commutative monoid have some parallels to our development, and drawing out a more precise relationship between these systems represents an interesting path of future work.

Finally, there are a few developments that start from different settings but develop semantics with similar properties. Pérez et al. develop logical relations for linear session types [26, 27] to establish normalization results, but there is no account of parametricity. The Iris system for program reasoning via higher-order separation logic incorporates a semantic model initially based on monoids [14], which is later extended to more general resource algebras [13]. Their parameterization over resource algebras seems to work similarly to ours, but towards the goal of program verification rather than type-based reasoning. The use of "resource semantics" more generally to account for the semantics of substructural logics extends at least to Kamide [16] and the logic of bunched implications [25], and similar ideas have recently gained traction in the context of graded modal type systems [37].

11 Conclusion

We have provided an account of substructural parametricity including ordered, linear, and unrestricted disciplines. The fewer structural properties are supported, the more precise the characterization of a function's behavior from its type. We have also implemented an ordered type checker using a bidirectional type system with so-called additive contexts [5], but the details are beyond the scope of this paper. Suffice it to say that all the functions such as append, reverse, tree traversals, and fold can actually be implemented in a variety of ways and are therefore not vacuous theorems.

The most immediate item of future work is to support general inductive and coinductive types instead of purely positive recursive types. This would allow a new class of applications, including (productive) stream processing and object-oriented program patterns. We also envision an adjoint combination of different substructural type systems [12], extended to include exchange among the explicit structural rules.

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