Assignment 2
Ordered Proofs

15-836: Substructural Logics
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Due Thursday, September 14, 2023
80 points

In this assignment we explore proofs, both from inside and outside the logic. Inside the logic, you are asked to prove or refute some propositions. Outside the logic, you are asked to fill in some cases in the proofs of the admissibility of cut or demonstrate that certain proposed rules are incorrect.

You should hand in two files:

- `hw2.pdf` with parts (or all) of the solutions Problems 1 and 2 as explained below. This is required.
- `hw2.sml` with the proof terms for propositions you believe are true. This file is autograded, with a nominal score. If you have difficulties with the starter code or Standard ML, you are welcome to add the sequent derivations and proof terms to `hw2.pdf`, but of course you will not get the benefit of any autograding.

The handout has the following structure:

- `src/`, the implementation of structural, linear, and ordered proof checker (you only need the ordered one for this assignment)
- `starter/`, the starter code for this assignment, including a template for `hw2.sml`
- `hw2.pdf`, this assignment spec
- `*.{tex,sty}`, LaTeX source, macros, and style files

1 Ordered Proofs (40 pts)

As explained in Lecture 4, the usual logical connectives of structural logic split into multiple versions when structural rules are removed. Just as in lecture, all the logics we consider are automatically intuitionistic. We have the following table.

structural	linear	ordered	pronunciation
$A \supset B$	$A \multimap B$	$A \setminus B$	A under B
		$B \mathbin{/} A$	B over A
$A \wedge B$	$A \otimes B$	$A \bullet B$	A fuse B
		$A \circ B$	A twist B
	$A \mathbin{\&} B$	$A \mathbin{\&} B$	A with B
$A \vee B$	$A \oplus B$	$A \oplus B$	A plus B
$\top$	1	1	one
	$\top$	$\top$	top
$\bot$	0	0	zero

Each of the following proposed laws represents two valid Boolean equivalences (as you may check via truth tables). However, only some of them hold in (intuitionistic) structural logic, and only some of those have a counterpart in ordered logic. We are only interested in those that carry over in both directions; if one direction fails we say the law fails.

In each case, either (a) indicate that you do not believe the proposed law holds in (intuitionistic) structural logic (you do not need to prove that), or (b) indicate that you believe it holds in structural logic but it has no true ordered counterpart (again, you don't need to prove that), or (c) give an ordered counterpart and its proof. If you see more than one valid counterpart, you may pick an arbitrary one. In `hw2.pdf`, just indicate (a), (b), or (c) and in case of (c) add the proof term to `hw2.sml`. If you do not with to test your proof, you can also add it here, but please give it in two forms: as a sequent calculus derivation (without proof terms) and separately write out the proof term in the notation from lecture.

Task 1 $A \wedge A \dashv\vdash A$

Task 2 $A \supset (B \wedge C) \dashv\vdash (A \supset B) \wedge (A \supset C)$

Task 3 $(A \wedge B) \supset C \dashv\vdash A \supset (B \supset C)$

Task 4 $(A \supset B) \wedge C \dashv\vdash (C \supset A) \supset (B \wedge C)$

Task 5 $(A \wedge B) \vee C \dashv\vdash (A \vee C) \wedge (B \vee C)$

2 Admissibility of Cut (40 pts)

Recall that a proposed rule is admissible if whenever all premises hold, so do all conclusions. The admissibility of cut in ordered logic (without cut) can be written in the form

$$\frac{\begin{array}{c}\mathcal{D}\\ \Omega \vdash A\end{array} \qquad \begin{array}{c}\mathcal{E}\\ \Omega_L\ A\ \Omega_R \vdash C\end{array}}{\Omega_L\ \Omega\ \Omega_R \vdash C}\ \mathsf{cut}_A$$

Task 6 *Show the case where $\mathcal{D}$ infers $A = A_1 \mathbin{/} A_2$ with rule $/R$ and $\mathcal{E}$ infers A with $/L$ by reducing the cut to one or more cut of smaller formulas.*

Task 7 *Show all cases where $\mathcal{D}$ is arbitrary and $\mathcal{E}$ ends in $\bullet R$ (so $C = C_1 \bullet C_2$ for some C_1 and C_2). You should not distinguish cases on what $\mathcal{D}$ might be.*

For parsing purposes, Lambek used what amounts to a rule of the form

$$\frac{\Omega_L \; B \; \Omega_R \vdash C}{\Omega_L \; A \; (A \setminus B) \; \Omega_R \vdash C} \setminus L^*$$

Task 8 *Show that this rule is admissible in the cut-free sequent calculus. You may use the admissibility of cut and identity as you see fit.*

Task 9 *If we* replaced *the usual rule of $\setminus L$ by this new rule $\setminus L^*$, some desirable property of ordered logic would fail. Sample candidates would be cut elimination or identity elimination. Indicate which property would fail and give a counterexample.*