Lecture Notes on Focusing

15-836: Substructural Logics
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1 Introduction

In this lecture we first revisit cut elimination by adapting it for adjoint logic. Then we introduce Andreoli's focusing [1992], a calculus for proof construction that exploits the negative and positive nature of the connectives to an extreme. Andreoli calls them asynchronous and synchronous connectives, and investigates *classical linear logic*, but his work seems nevertheless the beginning of the study of polarity. This was later adapted to various other logics [Liang and Miller, 2009], including adjoint logic [Pruiksma et al., 2018].

2 Cut Elimination Revisited

So far, we have studied cut elimination just in the ordered case, in some ways the most particular form. The argument for the linear connectives does not differ much, but the structural cases introduce some new considerations. Since we have introduced adjoint logic with explicit rules for weakening and contraction (mode permitting), we examine it in this case.

Recall the rule of cut, in this case formulated as a (hopefully) admissible property.

$$\frac{\Delta \geq m \geq r \quad \begin{array}{c}\mathcal{D} \\ \Delta \vdash A_m\end{array} \quad \begin{array}{c}\mathcal{E} \\ \Delta', A_m \vdash C_r\end{array}}{\Delta, \Delta' \vdash C_r}\ \mathsf{cut}_{A_m}$$

The case we are interested in here is where A_m is the result of contraction.

$$\dfrac{\Delta \geq m \geq r \quad \overset{\mathcal{D}}{\Delta \vdash A_m} \quad \dfrac{\mathsf{C} \in \sigma(m) \quad \overset{\mathcal{E}'}{\Delta', A_m, A_m \vdash C_r}}{\Delta', A_m \vdash C_r}\,\mathsf{contract}}{\Delta, \Delta' \vdash C_r}\,\mathsf{cut}_{A_m}$$

A natural reduction is the cut out the two copies of A_m in sequence.

$$\longrightarrow \quad \dfrac{\Delta \geq m \geq r \quad \overset{\mathcal{D}}{\Delta \vdash A_m} \quad \dfrac{\Delta \geq m \geq r \quad \overset{\mathcal{D}}{\Delta \vdash A_m} \quad \overset{\mathcal{E}'}{\Delta', A_m, A_m \vdash C_r}}{\Delta, \Delta', A_m \vdash C_r}\,\mathsf{cut}_{A_m}}{\Delta, \Delta, \Delta' \vdash C_r}\,\mathsf{cut}_{A_m}$$

First we see that the conditions that $\Delta \geq m \geq r$ are known from the derivation before the reduction, so they are satisfied. Second, we observe that this reduction *does not preserve the conclusion* because there are now two copies of Δ!

Fortunately, the monotonicity condition on the structural properties comes to the rescue. We know $\mathsf{C} \in \sigma(m)$ from the given derivation, and we also know $\Delta \geq m$. This means for every $B_\ell \in \Delta$ we have $\ell \geq m$ and therefore $\mathsf{C} \in \sigma(\ell)$. This is sufficient to apply contraction to all antecedents in Δ.

$$\longrightarrow \quad \dfrac{\dfrac{\Delta \geq m \geq r \quad \overset{\mathcal{D}}{\Delta \vdash A_m} \quad \dfrac{\Delta \geq m \geq r \quad \overset{\mathcal{D}}{\Delta \vdash A_m} \quad \overset{\mathcal{E}'}{\Delta', A_m, A_m \vdash C_r}}{\Delta, \Delta', A_m \vdash C_r}\,\mathsf{cut}_{A_m}}{\Delta, \Delta, \Delta' \vdash C_r}\,\mathsf{cut}_{A_m}}{\Delta, \Delta' \vdash C_r}\,\mathsf{contract}^*$$

One bullet dodged!

Now the second bullet. The upper of the two cuts is a valid appeal to the induction hypothesis since the cut formula A_m remains the same, $\mathcal{D}$ remains the same, and $\mathcal{E}' < \mathcal{E}$. So the upper cut yields a derivation $\mathcal{F}$ of $\Delta, \Delta', A_m \vdash C_r$. Unfortunately, $\mathcal{F}$ may be much larger that $\mathcal{E}$, and the cut formula A_m is still the same, so we cannot appeal to our induction hypothesis!

While this particular reduction may have some merit if used in a process dynamics, it actually does not lead to a correct proof of cut admissibility.

At this point there are two main options. One is to rewrite the sequent calculus for adjoint logic so that contraction is implicit, as in our early formulas of structural logic. That is, antecedents subject to contraction are treated as a set. A second option is to generalize the rule of cut to eliminate multiple antecedents at once. We call this *multicut* even if this name is not universally agreed upon. We follow this latter path because it illustrates a general technique when an inference system becomes more complex: we "absorb" additional rules into the cut by making it more general. Of course, this means that in some way we have to start from scratch and restart our overall proof, but the hope is we can follow the general inductive structure from before.

Our new, more general rule uses the notation $(A)^n$ to mean n copies of A.

$$\frac{\Delta \geq m \geq r \quad n \in \mu(m) \quad \Delta \vdash A_m \quad \Delta', (A_m)^n \vdash C_r}{\Delta, \Delta' \vdash C_r}\ \mathsf{multicut}_{A_m}$$

The new condition $n \in \mu(m)$ ensures that neither more nor fewer copies of A_m are cut out than allowed by the mode m. It is defined by

$$\begin{array}{lcll} \mu(m) & = & \{1\} & \text{if } \sigma(m) = \{\,\} \\ \mu(m) & = & \{0, 1\} & \text{if } \sigma(m) = \{\mathsf{W}\} \\ \mu(m) & = & \{1, 2, \ldots\} & \text{if } \sigma(m) = \{\mathsf{C}\} \\ \mu(m) & = & \{0, 1, 2, \ldots\} & \text{if } \sigma(m) = \{\mathsf{W}, \mathsf{C}\} \end{array}$$

An interesting observation is that weakening and contraction are special cases of multicut. For contraction, the condition that $2 \in \mu(m)$ implies that $\mathsf{C} \in \sigma(m)$.

$$\frac{2 \in \mu(m) \quad \dfrac{}{A_m \vdash A_m}\ \mathsf{id} \quad \Delta', (A_m)^2 \vdash C_r}{\Delta', A_m \vdash C_r}\ \mathsf{multicut}$$

Weakening uses the odd special case where $n = 0$. The condition $0 \in \mu(m)$ implies that $\mathsf{W} \in \sigma(m)$.

$$\frac{m \geq r \quad 0 \in \mu(m) \quad \dfrac{}{A_m \vdash A_m}\ \mathsf{id} \quad \Delta', (A_m)^0 \vdash C_r}{\Delta', A_m \vdash C_r}\ \mathsf{multicut}$$

The rule of contraction is now a trivial case in the proof of admissibility of multicut. In fact, due to the admissibility of contraction it wouldn't even be necessary any more. But if we allow it, there must be at least one copy of A_m in the conclusion, which we capture by writing $(A)^{n+1}$.

$$\frac{\Delta \geq m \geq r \quad n+1 \in \mu(m) \quad \begin{array}{c}\mathcal{D}\\ \Delta \vdash A_m\end{array} \quad \dfrac{\mathsf{C} \in \sigma(m) \quad \begin{array}{c}\mathcal{E}'\\ \Delta', (A_m)^{n+2} \vdash C_r\end{array}}{\Delta', (A_m)^{n+1} \vdash C_r}\ \mathsf{contract}}{\Delta, \Delta' \vdash C_r}\ \mathsf{multicut}$$

$$\longrightarrow \quad \frac{\Delta \geq m \geq r \quad n+2 \in \mu(m) \quad \begin{array}{c}\mathcal{D}\\ \Delta \vdash A_m\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Delta, (A)^{n+2} \vdash C_r\end{array}}{\Delta, \Delta' \vdash C_r}\ \mathsf{multicut}$$

The condition that $n+2 \in \mu(m)$ follows from $\mathsf{C} \in \sigma(m)$.

The price for generalizing cut has to be paid somewhere, if not in contraction. We show only one such case, which illustrates the new form of the principal cases. We omit some conditions on dependence and multiplicity for the sake of brevity.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta \vdash A_m\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Delta \vdash B_m\end{matrix}}{\Delta \vdash A_m \mathbin{\&} B_m}\ {\&}R \qquad \dfrac{\begin{matrix}\mathcal{E}' \\ \Delta', (A_m \mathbin{\&} B_m)^n, A_m \vdash C_r\end{matrix}}{\Delta', (A_m \mathbin{\&} B_m)^{n+1} \vdash C_r}\ {\&}L_1}{\Delta, \Delta' \vdash C_r}\ \mathsf{multicut}_{A_m \& B_m}$$

$$\longrightarrow \quad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta \vdash A_m\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta \vdash A_m\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Delta \vdash B_m\end{matrix}}{\Delta \vdash A_m \mathbin{\&} B_m}\ {\&}R \qquad \begin{matrix}\mathcal{E}' \\ \Delta, (A_m \mathbin{\&} B_m)^n, A_m \vdash C_r\end{matrix}}{\Delta, \Delta', A_m \vdash C_r}\ \mathsf{multicut}_{A_m \& B_m}}{\Delta, \Delta, \Delta' \vdash C_r}\ \mathsf{multicut}_{A_m}$$

What saves us here is that the upper of the two multicuts has the same proposition ($A_m \mathbin{\&} B_m$) and the same first premise $\mathcal{D}$, but a smaller second premise $\mathcal{E}' < \mathcal{E}$. The lower multicut has a potentially much larger second premise, but is only on A_m so is smaller by our lexicographic induction ordering (first on the structure of the cut formula, and then on the structure of the left and right derivations).

We see that with multicut, every left rule applies to one of $n+1$ copies of a principal proposition, after which are n copies remaining.

In the case of weakening (or cut of zero propositions) we can directly construct a proof of the conclusion with using $\mathcal{D}$.

$$\dfrac{\Delta \geq m \geq r \quad 0 \in \mu(m) \quad \begin{matrix}\mathcal{D} \\ \Delta \vdash A_m\end{matrix} \quad \begin{matrix}\mathcal{E} \\ \Delta', (A_m)^0 \vdash C_r\end{matrix}}{\Delta, \Delta' \vdash C_r}\ \mathsf{multicut}_{A_m}$$

$$\longrightarrow \quad \dfrac{\begin{matrix}\mathcal{E} \\ \Delta', (A_m)^0 \vdash C_r\end{matrix}}{\Delta, \Delta' \vdash C_r}\ \mathsf{weaken}^*$$

For the correctness we see that $0 \in \mu(m)$ implies that $\mathsf{W} \in \sigma(m)$, and since $\Delta \geq m$ we also have $\mathsf{W} \in \sigma(\ell)$ for every B_ℓ in Δ. Furthermore, $(A_m)^0$ means zero copies of A_m.

3 Inversion

We have talked about right and left invertibility of connectives a lot in this course. Not only is it important for proof search, but it also affects the operational interpretation. For example, channels of invertible type will receive under our message-passing interpretation.

We now want to refine the proof system for the sequent calculus so that inversion is *forced*, that is, inversion *must* be applied during search. This is not immediately relevant to the computational interpretation of proof reduction since it limits program expression.

It is a recurring theme of this course that we express ideas via rules of inference. We will do so here. Fundamentally, the idea is to take away all choice during proof search as long as invertible rules apply. We also remind ourselves of positive and negative proposition.

$$
\begin{array}{lll}
\text{Negatives} & A_m^-, B_m^- & ::= \quad P_m^- \mid A_m \to B_m \mid A_m \mathbin{\&} B_m \mid \top \mid {\uparrow}_k^m A_k \; [\mid \langle P_m^+ \rangle] \\
\text{Positives} & A_m^+, B_m^+ & ::= \quad P_m^+ \mid A_m \times B_m \mid \mathbf{1} \mid A_m + B_m \mid \mathbf{0} \mid {\downarrow}_m^\ell A_\ell \; [\mid \langle P_m^- \rangle] \\
\text{Propositions} & A_m, B_m & ::= \quad A_m^- \mid B_m^-
\end{array}
$$

We have not fully *polarized* the propositions which would mean to continue with negative or positive subformulas until there is an explicit change of polarity via a polarity-changing modality. We do this to reduce syntactic complexity since we already use shifts for different purpose (namely switching between modes). We explain *suspended atomic propositions* $\langle P_m^- \rangle$ and $\langle P_m^+ \rangle$ later.

We walk through the inference rules the way you might discover them. We start with negative propositions in the succedent. We write $\xrightarrow{\mathsf{IR}} A_m$ for a judgment that forces inversion to be applied to the succedent until it is no longer possible. We have omitted the antecedents until we see what they might need to look like.

$$
\frac{\xrightarrow{\mathsf{IR}} A_m \qquad \xrightarrow{\mathsf{IR}} B_m}{\xrightarrow{\mathsf{IR}} A_m \mathbin{\&} B_m} \;\&R
\qquad\qquad
\frac{}{\xrightarrow{\mathsf{IR}} \top} \;\top R
$$

$$
\frac{A_m \xrightarrow{\mathsf{IR}} B_m}{\xrightarrow{\mathsf{IR}} A_m \to B_m} \;{\to}R
$$

We see that $\to R$ introduces an antecedent that may or may not be invertible. We want to make sure that there is no choice so we force right inversion and accumulate antecedents until right rules can no longer be applied. The accumulator is *ordered* (rather than linear) so we can process it deterministically during the left inversion phase. We also add the upshift.

$$\frac{\Omega \xrightarrow{\mathsf{IR}} A_m \quad \Omega \xrightarrow{\mathsf{IR}} B_m}{\Omega \xrightarrow{\mathsf{IR}} A_m \,\&\, B_m}\ \&R \qquad \frac{}{\Omega \xrightarrow{\mathsf{IR}} \top}\ \top R$$

$$\frac{A_m\, \Omega \xrightarrow{\mathsf{IR}} B_m}{\Omega \xrightarrow{\mathsf{IR}} A_m \rightarrow B_m}\ {\rightarrow}R \qquad \frac{\Omega \xrightarrow{\mathsf{IR}} A_k}{\Omega \xrightarrow{\mathsf{IR}} {\uparrow}_k^m A_k}\ {\uparrow}R$$

At this point we miss negative atoms P_m^- and positive propositions A_m^+. In both cases, the right inversion phase comes to an end and we switch to perform possible inversions on Ω. For negative atoms, in a way, we *should* still invert but we can't since there are no subformulas. So we *suspend* this proposition and count it, for the purpose of our judgment, as a negative proposition. It is important to recognize that the angle brackets $\langle P_m^- \rangle$ are only a syntactic ("judgmental") marker and not a propositional modality.

$$\frac{\Omega \xrightarrow{\mathsf{IL}} C_m^+}{\Omega \xrightarrow{\mathsf{IR}} C_m^+}\ \mathsf{IL/IR}^+ \qquad \frac{\Omega \xrightarrow{\mathsf{IL}} \langle P_m^- \rangle}{\Omega \xrightarrow{\mathsf{IR}} P_m^-}\ \mathsf{IL/IR}^*$$

Now left inversion peels off left invertible propositions from the left end of Ω. These are, of course, the positive propositions.

$$\frac{A_m\, B_m\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}{(A_m \times B_m)\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ {\times}L \qquad \frac{\Omega \xrightarrow{\mathsf{IL}} C_r^+}{\mathbf{1}\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ \mathbf{1}L$$

$$\frac{A_m\, \Omega \xrightarrow{\mathsf{IL}} C_r^+ \quad B_m\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}{(A_m + B_m)\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ {+}L \qquad \frac{}{\mathbf{0}\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ \mathbf{0}L$$

What happens when we reach a negative proposition or a positive atom? We have to postpone dealing with them because we want to force inversion, so we need another linear zone Δ^- consisting only of negative propositions (including positive atoms). We move negative propositions into Δ' when they pop up in Ω.

$$\frac{\Delta^-, A_m^- \,;\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}{\Delta^- \,;\, A_m^-\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ \mathsf{IL}^+ \qquad \frac{\Delta^-, \langle P_m^+ \rangle \,;\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}{\Delta^- \,;\, P_m^+\, \Omega \xrightarrow{\mathsf{IL}} C_r^+}\ \mathsf{IL}^*$$

Now we have to add Δ^- to all the earlier rules and propagate them unchanged from conclusion to premise. Sigh.

When the ordered antecedents become empty the inversion phase comes to an end and we have to make an actual choice. We represent this judgment as $\Delta^- \xrightarrow{\mathsf{C}} C_r^+$.

$$\frac{\Delta^- \xrightarrow{\mathsf{C}} C_r^+}{\Delta^- \,;\, \cdot \xrightarrow{\mathsf{IL}} C_r^+}\ \mathsf{C/IL}$$

4 Chaining

Once the inversion phase is over, we could just make a choice of applying a left rule to a proposition in Δ^- or a right rule to C_r^+. This would be captured by recapping all the noninvertible rules for positive propositions on the right and negative propositions on the left.

Nondeterminism is further drastically reduced if we *focus* on a proposition and then chain the rules on this particular proposition until we switch back to an invertible one.

Unfortunately, I made a significant error in lecture in that the rules I showed only work as given in the case where there are no structural rules are allowed. Or, to put it another way, if all modes are linear and do not allow weakening or contraction, only exchange (which is always implicit). We show these rules and fix our mistake in a future lecture.

For the remainder of this section, all modes must be linear.

The first step is to select a negative antecedent or the succedent for focus. We indicate focus by using [square brackets]. Note that only a single proposition can be in focus in a given sequent.

$$\frac{\Delta^- \xrightarrow{\mathsf{FR}} [C_m^+]}{\Delta^- \xrightarrow{\mathsf{C}} C_m^+}\ \mathsf{FR/C} \qquad \frac{\Delta^- \,;\, [A^-] \xrightarrow{\mathsf{FL}} C_m^+}{\Delta^-, A^- \xrightarrow{\mathsf{C}} C_m^+}\ \mathsf{FL/C}$$

We start with the right rules. They are the usual right rules, but they retain focus on the subformulas.

$$\frac{\Delta_1^- \xrightarrow{\mathsf{FR}} [A_m] \quad \Delta_2^- \xrightarrow{\mathsf{FR}} [B_m]}{\Delta_1^-, \Delta_2^- \xrightarrow{\mathsf{FR}} [A_m \times B_m]} \times R \qquad \frac{}{\cdot \xrightarrow{\mathsf{FR}} [\mathbf{1}]} \mathbf{1}R$$

$$\frac{\Delta^- \xrightarrow{\mathsf{FR}} [A_m]}{\Delta^- \xrightarrow{\mathsf{FR}} [A_m + B_m]} +R_1 \quad \frac{\Delta^- \xrightarrow{\mathsf{FR}} [B_m]}{\Delta^- \xrightarrow{\mathsf{FR}} [A_m + B_m]} +R_2 \qquad \text{no } \mathbf{0}R \text{ rule}$$

$$\frac{\Delta^- \geq \ell \quad \Delta^- \xrightarrow{\mathsf{FR}} [A_\ell]}{\Delta^- \xrightarrow{\mathsf{FR}} [{\downarrow}_m^\ell A_\ell]} {\downarrow}R$$

$$\frac{\Delta^- \, ; \, \cdot \xrightarrow{\mathsf{IR}} A_m^-}{\Delta^- \xrightarrow{\mathsf{FR}} [A_m^-]} \mathsf{FR}^- \qquad \frac{}{\langle P_m^+ \rangle \xrightarrow{\mathsf{FR}} [P_m^+]} \mathsf{id}^+$$

The left rules are also the usual left rules, but the principal formula must be in focus.

$$\frac{\Delta_1^- \geq m \quad \Delta_1^- \xrightarrow{\mathsf{FR}} [A_m] \quad \Delta_2^- \, ; \, [B_m] \xrightarrow{\mathsf{FL}} C_r^+}{\Delta_1^-, \Delta_2^- \, ; \, [A_m \to B_m] \xrightarrow{\mathsf{FL}} C_r^+} \to L$$

$$\frac{\Delta^- \, ; \, [A_m] \xrightarrow{\mathsf{FL}} C_r^+}{\Delta^- \, ; \, [A_m \mathbin{\&} B_m] \xrightarrow{\mathsf{FL}} C_r^+} \mathbin{\&}L_1 \quad \frac{\Delta^- \, ; \, [B_m] \xrightarrow{\mathsf{FL}} C_r^+}{\Delta^- \, ; \, [A_m \mathbin{\&} B_m] \xrightarrow{\mathsf{FL}} C_r^+} \mathbin{\&}L_2 \qquad \text{no } \top L \text{ rule}$$

$$\frac{k \geq r \quad \Delta^- \, ; \, [A_k] \xrightarrow{\mathsf{FL}} C_r^+}{\Delta^- \, ; \, [{\uparrow}_k^m A_k] \xrightarrow{\mathsf{FL}} C_r^+} {\uparrow}L$$

$$\frac{\Delta^- \, ; \, A_m^+ \xrightarrow{\mathsf{IL}} C_r^+}{\Delta^- \, ; \, [A_m^+] \xrightarrow{\mathsf{FL}} C_r^+} \mathsf{FL}^+ \qquad \frac{}{\Delta^- \, ; \, [P_m^-] \xrightarrow{\mathsf{FL}} \langle P_m^- \rangle} \mathsf{id}^-$$

The noninvertible rules constitute a phase of *chaining*. Taken together with *inversion* this proof search strategy is called *focusing*.

5 A Simple Example

As a simple example, consider

$$(P_m \to Q_m) \to ((Q_m \to R_m) \to (P_m \to R_m))$$

We can assign any polarity to the atoms we like and we choose all of them to be positive. The we start with right inversion until we hit the positive atom.

$$\dfrac{\begin{array}{c}\vdots\\ \cdot\,;\, P_m^+\ (Q_m^+ \to R_m^+)\ (P_m^+ \to Q_m^+) \xrightarrow{\mathsf{IR}} R_m^+\end{array}}{\cdot\,;\,\cdot \xrightarrow{\mathsf{IR}} (P_m^+ \to Q_m^+) \to ((Q_m^+ \to R_m^+) \to (P_m^+ \to R_m^+))} \to\!R \times 3$$

Now we switch to left inversion until we complete inversion and reach a choice.

$$\dfrac{\dfrac{\dfrac{\langle P_m^+\rangle, Q_m^+ \to R_m^+, P_m^+ \to Q_m^+ \xrightarrow{\mathsf{C}} R_m^+}{\cdot\,;\, P_m^+\ (Q_m^+ \to R_m^+)\ (P_m^+ \to Q_m^+) \xrightarrow{\mathsf{IL}} R_m^+}\ \cdots}{\cdot\,;\, P_m^+\ (Q_m^+ \to R_m^+)\ (P_m^+ \to Q_m^+) \xrightarrow{\mathsf{IR}} R_m^+}\ \mathsf{IL/IR}}{\cdot\,;\,\cdot \xrightarrow{\mathsf{IR}} (P_m^+ \to Q_m^+) \to ((Q_m^+ \to R_m^+) \to (P_m^+ \to R_m^+))} \to\!R \times 3$$

This is a critical point in the search.

1. We cannot focus on R_m^- because $\langle R_m^+\rangle$ is not among the antecedents.
2. We cannot focus on $Q_m^+ \to R_m^+$ because Q_m^+ is not among the antecedents.
3. We cannot focus on $\langle P_m^+\rangle$ because it is a suspended atom.

The only choice that remains is to focus on $P_m^+ \to Q_m^+$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle P_m^+\rangle\,;\,\cdot \xrightarrow{\mathsf{FR}} [P_m^+]}\ \mathsf{id}^+ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \langle Q_m^+\rangle, Q_m^+ \to R_m^+ \xrightarrow{\mathsf{C}} R_m^+\end{array}}{\langle Q_m^+\rangle, Q_m^+ \to R_m^+\,;\,\cdot \xrightarrow{\mathsf{IL}} R_m^+}\ \mathsf{C/IL}}{Q_m^+ \to R_m^+\,;\, Q_m^+ \xrightarrow{\mathsf{IL}} R_m^+}\ \mathsf{IL}^*}{Q_m^+ \to R_m^+\,;\,[Q_m^+] \xrightarrow{\mathsf{FL}} R_m^+}\ \mathsf{IL/FL}}{}}{\langle P_m^+\rangle, Q_m^+ \to R_m^+\,;\,[P_m^+ \to Q_m^+] \xrightarrow{\mathsf{FL}} R_m^+}\ \to\!L}{\langle P_m^+\rangle, Q_m^+ \to R_m^+, P_m^+ \to Q_m^+ \xrightarrow{\mathsf{C}} R_m^+}\ \mathsf{FL/C}}{\cdot\,;\, P_m^+\ (Q_m^+ \to R_m^+)\ (P_m^+ \to Q_m^+) \xrightarrow{\mathsf{IL}} R_m^+}\ \cdots}{\cdot\,;\, P_m^+\ (Q_m^+ \to R_m^+)\ (P_m^+ \to Q_m^+) \xrightarrow{\mathsf{IR}} R_m^+}\ \mathsf{IL/IR}}{\cdot\,;\,\cdot \xrightarrow{\mathsf{IR}} (P_m^+ \to Q_m^+) \to ((Q_m^+ \to R_m^+) \to (P_m^+ \to R_m^+))} \to\!R \times 3$$

At this point once again the only possibility is to focus on $Q_m^+ \to R_m^+$, after which we can focus on R_m^+ in the succedent.

Even though it looks complex, there is just one proof and (excepting shallow backtracking) only one way to construct this proof.

The mistake I made in lecture, by the way, is that I claimed the FL/C rule was the only one where weakening and contraction came into play. There are actually several others, so we postpone a full discussion to a future lecture.

References

Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):197–347, 1992.

Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, November 2009.

Klaas Pruiksma, William Chargin, Frank Pfenning, and Jason Reed. Adjoint logic. Unpublished manuscript, April 2018. URL `http://www.cs.cmu.edu/~fp/papers/adjoint18b.pdf`.