Lecture Notes on Resource Semantics

15-836: Substructural Logics
Frank Pfenning

Lecture 19
November 16, 2023

1 Introduction

When researchers use the term "*programming language semantics*" they usually either mean a *dynamic semantics* (also referred to as an *operational semantics*) or a *denotational semantics*, that is, an interpretation into a mathematical domain. Meaning is given to programs either by how they execute or by what they denote in some abstract domain. These are both respectable and valuable notions to study. As an intuitionist, though, I find it difficult myself to follow the domain-theoretic approach since it is mostly carried out in classical mathematics.

In logic, a similar divide exists. Semantics (usually credited to Tarski [1956]) interprets the syntax of logic in classical mathematics. In contrast, proof theory studies the notions of truth and consequence via the structure of proofs. Much (but certainly not all) of proof theory is carried out using constructive means, as exemplified by Gentzen's [1935] proof of cut elimination.

One thing I can do, as an intuitionist, is to interpret one constructive language in another and thereby gain a deeper understanding. For example, in Assignment 5 you are asked to interpret affine logic in linear logic. Previously, we have shown (following Girard [1987]) how to interpret (structural) intuitionistic logic in linear logic.

Can we go the other way? Is there, for example, an interpretation of linear logic in (structural) intuitionistic logic? A first answer to the question is "*No!*". For example, we know via a variety of different proofs that structural intuitionistic logic is decidable. Take, for example, the inverse method from the previous lecture. Because of the presence of contraction, the space of possible sequents we can derive is finite and saturation is assured. On the other hand, linear logic that includes the exponential ($!A$) or the adjoint modalities $\uparrow\downarrow A$) is *undecidable* since it can interpret Minsky machines [Lincoln et al., 1992]. This may be surprising, since linear logic *without* the exponential is decidable. As an example proof, in backward search for a cut-free sequent proof, each premise of each rule is smaller than the conclusion in a simple multiset ordering based on counting the number of connectives.

Since we cannot interpret an undecidable problem in a decidable one, what is left? We can look for an interpretation into an intuitionistic *predicate calculus*, which is certainly also undecidable. This is indeed illuminating since it clarifies a *resource interpretation* of substructural logics that is also applicable to others such as ordered logic.

We start with a preliminary study by giving a *structural* set of rules in which the use of antecedents as resources is explicit. We then follow it by an explicit translation, following Reed and Pfenning [2010]. Related material is also in Reed's Ph.D. Thesis [Reed, 2009].

2 A Sequent Calculus with Explicit Resources

In this formulation of substructural logics we think of each antecedent as a resource and the succedent as a goal to be achieved using an explicit stated collection of resources. We write

$$A_1[\alpha_1], \ldots, A_n[\alpha_n] \vdash C[p]$$

where p is a combination of the resource variables $\alpha_1, \ldots, \alpha_n$.

For the moment, we focus on linear logic. We write $p * q$ for the combination of the resources denoted by p and q, and ϵ for the absence of resources.

$P[\alpha], Q[\beta] \vdash P \otimes Q[\alpha * \beta]$	holds
$P[\alpha], Q[\beta] \vdash P \otimes Q[\beta * \alpha]$	holds
$P[\alpha], Q[\beta] \vdash P \otimes Q[\alpha]$	does not hold
$P[\alpha], Q[\beta], R[\gamma] \vdash P \otimes Q[\alpha * \beta]$	holds
$P[\alpha], Q[\beta], R[\alpha] \vdash P \otimes Q[\alpha * \beta]$	holds?

We can argue if the last one should be allowed, because α does not stand for a unique resource but for two different ones. In our first system we make a presupposition that resource labelings are unique and the last sequent would not be well-formed. In Section 4 we will generalize our point of view. We will stick to the "no repetition in p" mantra because we are not interested in this lecture to capture that a specific resource may be used a fixed number of times.

Since we are modeling linear logic, resource combination $p * q$ should satisfy the following laws:

Associativity	$(p * q) * r = p * (q * r)$
Unit	$\epsilon * p = p = p * \epsilon$
Commutativity	$p * q = q * p$

We can recognize this as a *commutative monoid* which, not surprisingly, is also the structure of the antecedents in linear logic. Because we have no other equations, we can also say it is the *free commutative monoid* over the variables denoted by α, β, etc.

Before we give the rules, it is easy to conjecture how we might, for example, capture *ordered logic*: we drop commutativity of the resource combination operator. For affine logic, we would need a partial order to capture that some resources may not need to be used.

Starting on formal rules, the identity is easy with weakening implicit.

$$\frac{}{\Gamma, A[\alpha] \vdash A[\alpha]}\ \mathsf{id}$$

The cut rule is a bit tricky because of the asymmetry between antecedents and succedents. But we are used to that by now. We start by writing the rule *without the resources* on the way to deriving what they should be.

$$\frac{\Gamma \vdash A[\quad] \quad \Gamma, A[\quad] \vdash C[\quad]}{\Gamma \vdash C[\quad]}\ \mathsf{cut}$$

We know that the antecedent $A[\quad]$ should be labeled with a fresh resource variable, and that the proof of C in the second premise should be able to use it.

$$\frac{\Gamma \vdash A[\quad] \quad \Gamma, A[\alpha] \vdash C[\quad * \alpha]}{\Gamma \vdash C[\quad]}\ \mathsf{cut}$$

Next we know that the proof of A in the first premise will use some resources q. These will be required for C as well.

$$\frac{\Gamma \vdash A[q] \quad \Gamma, A[\alpha] \vdash C[\quad * \alpha]}{\Gamma \vdash C[\quad * q]}\ \mathsf{cut}$$

Finally, we see that there may be some resources p besides q that are required in the second premise.

$$\frac{\Gamma \vdash A[q] \quad \Gamma, A[\alpha] \vdash C[p * \alpha]}{\Gamma \vdash C[p * q]}\ \mathsf{cut}^{\alpha}$$

Of course, they can't be used in the first premise. We also annotate the rule with α to remind ourselves that α in the premise must be fresh, that is, not already occur in Γ or $p * q$.

Based on similar considerations we get the following rules for conjunction.

$$\frac{\Gamma \vdash A[p] \quad \Gamma \vdash B[q]}{\Gamma \vdash A \otimes B[p * q]}\ {\otimes}R \qquad \frac{\Gamma, A \otimes B[\gamma], A[\alpha], B[\beta] \vdash C[p * \alpha * \beta]}{\Gamma, A \otimes B[\gamma] \vdash C[p * \gamma]}\ {\otimes}L^{\alpha,\beta}$$

Because our logic is structural we retain a copy of $A \otimes B[\gamma]$. However, if there is no repetition in the resources of the succedent, we can no longer use it in the premise—it is carried along but no longer usable.

Besides a theorem to come (we can go back and forth between linear logic and resource logic), we also check identity expansion and cut reduction.

$$\dfrac{}{\Gamma, A \otimes B[\gamma] \vdash A \otimes B[\gamma]}\ \mathsf{id}_{A\otimes B}$$

$$\longrightarrow_E \quad \dfrac{\dfrac{\dfrac{}{\Gamma, A \otimes B[\gamma], A[\alpha] \vdash A[\alpha]}\ \mathsf{id}_A \quad \dfrac{}{\Gamma, A \otimes B[\gamma], B[\beta] \vdash B[\beta]}\ \mathsf{id}_B}{\Gamma, A \otimes B[\gamma], A[\alpha], B[\beta] \vdash A \otimes B[\alpha * \beta]}\ \otimes R}{\Gamma, A \otimes B[\gamma] \vdash A \otimes B[\gamma]}\ \otimes L^{\alpha,\beta}$$

and

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A[p]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B[q]\end{array}}{\Gamma \vdash A \otimes B[p * q]}\ \otimes R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, A \otimes B[\gamma], A[\alpha], B[\beta] \vdash C[\alpha * \beta * r]\end{array}}{\Gamma, A \otimes B[\gamma] \vdash C[\gamma * r]}\ \otimes L^{\alpha,\beta}}{\Gamma \vdash C[p * q * r]}\ \mathsf{cut}^{\gamma}_{A\otimes B}$$

Now γ is fresh in the cut, so it does not appear in $\alpha * \beta * r$. So we can apply *strengthening* to eliminate it from $\mathcal{E}'$. Then we can have two appeals to cut at smaller propositions.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A[p]\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B[q]\end{array}}{\Gamma \vdash A \otimes B[p * q]}\ \otimes R \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, A \otimes B[\gamma], A[\alpha], B[\beta] \vdash C[\alpha * \beta * r]\end{array}}{\Gamma, A \otimes B[\gamma] \vdash C[\gamma * r]}\ \otimes L^{\alpha,\beta}}{\Gamma \vdash C[p * q * r]}\ \mathsf{cut}^{\gamma}_{A\otimes B}$$

$$\longrightarrow_R \quad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash B[q]\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A[p]\end{array}}{\Gamma, B[\beta] \vdash A[p]}\ \mathsf{weaken} \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, A \otimes B[\gamma], A[\alpha], B[\beta] \vdash C[\alpha * \beta * r]\end{array}}{\Gamma, A[\alpha], B[\beta] \vdash C[\alpha * \beta * r]}\ \mathsf{strengthen}}{\Gamma, B[\beta] \vdash C[p * \beta * r]}\ \mathsf{cut}^{\alpha}_{A}}{\Gamma \vdash C[p * q * r]}\ \mathsf{cut}^{\beta}_{B}$$

There is an implicit use of weakening to equalize the antecedents in the two premises in the cut on A.

Lemma 1 (Strengthening and Weakening)

(i) *If* $\Gamma, A[\alpha] \vdash C[p]$ *where* α *is not in* p*, then* $\Gamma \vdash C[p]$ *with the same derivation.*

(ii) *If* $\Gamma \vdash C[p]$ *then* $\Gamma, A[\alpha] \vdash C[p]$ *when* α *is not in* p*, with the same derivation.*

Because they follow largely similar considerations, we just show the rules for implication and external choice. Note how in external choice the same resources are required for both branches. The left rule for disjunction will have a similar property.

$$\frac{\Gamma, A[\alpha] \vdash B[p * \alpha]}{\Gamma \vdash A \multimap B[p]} \multimap R^{\alpha} \quad \frac{\Gamma, A \multimap B[\gamma] \vdash A[p] \quad \Gamma, A \multimap B[\gamma], B[\beta] \vdash C[q * \beta]}{\Gamma, A \multimap B[\gamma] \vdash C[p * q * \gamma]} \multimap L^{\beta}$$

$$\frac{\Gamma \vdash A[p] \quad \Gamma \vdash B[p]}{\Gamma \vdash A \mathbin{\&} B[p]} \mathbin{\&} R$$

$$\frac{\Gamma, A \mathbin{\&} B[\gamma], A[\alpha] \vdash [p * \alpha]}{\Gamma, A \mathbin{\&} B[\gamma] \vdash C[p * \gamma]} \mathbin{\&} L_1^{\alpha} \qquad \frac{\Gamma, A \mathbin{\&} B[\gamma], B[\beta] \vdash [p * \beta]}{\Gamma, A \mathbin{\&} B[\gamma] \vdash C[p * \gamma]} \mathbin{\&} L_2^{\beta}$$

The system has been carefully designed to preserve the structure of proofs as much as possible. This makes the proof of adequacy relatively easy.

The first direction states that if $A_1, \ldots, A_n \vdash A$ then $A_1[\alpha_1], \ldots, A_n[\alpha_n] \vdash A[\alpha_1 * \cdots * \alpha_n]$ where for distinct variable α_i. This is proved by induction over the given derivation, taking advantage of weakening to add on the antecedents that remain in the premises of the target calculus.

For the other direction, we go from $\Gamma \vdash A[p]$ to $\Gamma|_p \vdash A$, where $\Gamma|_p$ is the restriction of Γ to the resource variables in p. For this direction, we assume that p does not have any repeated resource variables—if it did, we would not be able to model the structural proof with a linear one.

3 Adding Validity

Perhaps surprisingly, we already have all the expressive power we need in the target calculus to model at least $!A = \downarrow\uparrow A$. The key idea is that a structural proposition A_{S} corresponds to a new kind of antecedent $A[\epsilon]$. This expresses that A can be proved without the use of any resources, which is precisely our definition of validity in linear logic!

We restrict ourselves to the single structural proposition $\uparrow A_{\mathsf{L}}$ but we believe it should easily extend to allow further structural propositions (after all, our target calculus is structural).

We look at the rules for the shifts of the mixed linear/nonlinear logic and derive the corresponding resource-aware rules.

We start with the right rule for $\downarrow A_{\mathsf{S}}$, which is not invertible since $\downarrow$ is a positive connective.

$$\frac{\Delta_{\mathsf{S}} \vdash A_{\mathsf{S}}}{\Delta_{\mathsf{S}} \vdash \downarrow A_{\mathsf{S}}} \downarrow R \qquad \frac{\Gamma \vdash A[\epsilon]}{\Gamma \vdash \downarrow A[\epsilon]} \downarrow R$$

The restriction that Δ consists only of structural propositions is represented here by the fact that $\downarrow A$ must be true without any resource (ϵ). If there were usable resources in Γ, the would show up in the resources for the succedent.

In the left rule (which is invertible), α must be an available resource, but that the only requirement. In the mixed linear/nonlinear system this means the succedent must be linear.

$$\frac{\Delta, A_{\mathsf{S}} \vdash C_{\mathsf{L}}}{\Delta, \downarrow A_{\mathsf{S}} \vdash C_{\mathsf{L}}} \downarrow L \qquad \frac{\Gamma, \downarrow A[\alpha], A[\epsilon] \vdash C[p]}{\Gamma, \downarrow A[\alpha] \vdash C[p * \alpha]} \downarrow L$$

Next, we come to $\uparrow R$. In mixed linear/nonlinear logic, the presupposition of independence ensures that the antecedents in the conclusion are all structural, indicated by writing Δ_{S}. In the corresponding rule in resource logic there may be antecedents $B[\beta]$, but they cannot be used because the the succedent has the empty set of resources ϵ.

$$\frac{\Delta_{\mathsf{S}} \vdash A_{\mathsf{L}}}{\Delta_{\mathsf{S}} \vdash \uparrow A_{\mathsf{L}}} \uparrow R \qquad \frac{\Gamma \vdash A[\epsilon]}{\Gamma \vdash \uparrow A[\epsilon]} \uparrow R$$

The left rule codifies the idea that if $\uparrow A[\epsilon]$ we can obtain a copy of the resource A without using any resources. After all, it doesn't cost any resources to do so!

$$\frac{\Delta, \uparrow A_{\mathsf{L}}, A_{\mathsf{L}} \vdash C_{\mathsf{L}}}{\Delta, \uparrow A_{\mathsf{L}} \vdash C_{\mathsf{L}}} \uparrow L \qquad \frac{\Gamma, \uparrow A[\epsilon], A[\alpha] \vdash C[p * \alpha]}{\Gamma, \uparrow A[\epsilon] \vdash C[p]} \uparrow L^{\alpha}$$

At this point in lecture we were concerned about $\uparrow R$ and $\uparrow L$. Because $\uparrow$ is negative, the right rule should be invertible and the left rule should not. We therefore applied our identity expansion test. First, the correct proof that starts with the right rule.

$$\cfrac{\cfrac{\cfrac{}{\Gamma, \uparrow A[\epsilon], A[\alpha] \vdash A[\alpha]} \mathsf{id}_A}{\Gamma, \uparrow A[\epsilon] \vdash A[\epsilon]} \uparrow L^{\alpha}}{\Gamma, \uparrow A[\epsilon] \vdash \uparrow A[\epsilon]} \uparrow R$$

We would not expect the opposite order to work out, which would be evidence that $\uparrow L$ is not invertible.

$$\frac{\begin{array}{c} ?? \\ \uparrow A[\epsilon], A[\alpha] \vdash \uparrow A[\alpha] \end{array}}{\uparrow A[\epsilon] \vdash \uparrow A[\epsilon]} \uparrow L^{\alpha}$$

Luckily we do get stuck here, because we can not apply the $\uparrow R$ rule. If A is atomic, the only option is to apply $\uparrow L$ again and again, obtaining many copies of A but never being able to apply the right rule.

In order to account for this we have to update our adequacy proof for the translation. In the first direction:

(i) If $\Delta_{\mathsf{S}}, \Delta_{\mathsf{L}} \vdash A_{\mathsf{L}}$ then $\Delta_{\mathsf{S}}[\epsilon], \Delta_{\mathsf{L}}[\overline{\alpha}] \vdash A_{\mathsf{L}}[\overline{\alpha}]$

(ii) If $\Delta_{\mathsf{S}} \vdash A_{\mathsf{S}}$ then $\Delta_{\mathsf{S}}[\epsilon] \vdash A_{\mathsf{S}}[\epsilon]$

This direction takes advantage of weakening in resource logic (Lemma 1).

For the other direction, we have:

If $\Gamma \vdash A[p]$ with $\Gamma = \Gamma_1[\epsilon], \Gamma_2[\overline{\alpha}]$ then $\Gamma|_p \vdash A$.

Here, $\Gamma|_p$ restricts Γ to antecedents $B[\epsilon]$ and $B[\alpha_i]$ for α_i in p. For this direction we use strengthening which applies due to our presuppositions on antecedents and resource terms.

4 Untethering

All left rules for linear antecedents refer to the resources annotating the succedent. We say the left rules are *tethered* to the succedent. This is by design, since we would like to model the rules of linear logic as closely as possible. This feature makes it somewhat difficult to turn the system into a *translation* from propositional linear logic to intuitionistic logic.

As a step in the direction of a translation we'll *untether* the rules for the *negative connectives*. We don't investigate this system in its own right, just using it for intuition. The first step is untethering is to allow complex resource terms for antecedents. We think of $A[p]$ as saying that the justification of the antecedent A requires resources p.

The identity is straightforward; it is where the resources among the antecedents are eventually tied to the succedent.

$$\frac{}{\Gamma, A[p] \vdash A[p]}\ \mathsf{id}$$

Next consider implication. While the right rule is unchanged, the new left rule is untethered.

$$\frac{\Gamma, A[\alpha] \vdash B[p * \alpha]}{\Gamma \vdash A \multimap B[p]}\ {\multimap}R^{\alpha} \qquad \frac{\Gamma \vdash A[q] \quad \Gamma, B[p * q] \vdash C[r]}{\Gamma, A \multimap B[p] \vdash C[r]}\ {\multimap}L$$

We can read the left rule as: "*if the implication $A \multimap B$ requires resources p and A requires resources q, then B requires resources $p * q$*". Thinking about the cut reduction (in a new form), we would instantiate the α in ${\multimap}R^{\alpha}$ with q so we can reduce the cut between ${\multimap}R$ and ${\multimap}L$.

External choice can be untethered more directly.

$$\frac{\Gamma \vdash A[p] \quad \Gamma \vdash B[p]}{\Gamma \vdash A \mathbin{\&} B[p]}\ \&R$$

$$\frac{\Gamma, A \mathbin{\&} B[p], A[p] \vdash C[r]}{\Gamma, A \mathbin{\&} B[p] \vdash C[r]}\ \&L_1 \qquad \frac{\Gamma, A \mathbin{\&} B[p], B[p] \vdash C[r]}{\Gamma, A \mathbin{\&} B[p] \vdash C[r]}\ \&L_2$$

It is interesting to consider what happens if, say, $p = \alpha$. In that case, after the $\& L_1$ rule, we have both $A \& B[\alpha]$ and $A[\alpha]$ among the antecedents. But eventually the connection to the resources in the succedent will have to made, and then only one of these two can be used. We could even take it a step further, applying $\& L_2$ so we have both $A[\alpha]$ and $B[\alpha]$ among the antecedents, but only one of them can be used because of resource constraints. From these considerations is should be clear that the structure of the proofs no longer matches up so directly.

We now turn the idea of untethering into a translation $A \mathbin{@} p$.

$$\begin{array}{lcl} (P) \mathbin{@} p & = & P(p) \\ (A \multimap B) \mathbin{@} p & = & \forall \alpha. (A \mathbin{@} \alpha) \supset (B \mathbin{@} (p * \alpha)) \\ (A \& B) \mathbin{@} p & = & (A \mathbin{@} p) \wedge (B \mathbin{@} p) \end{array}$$

Recall also the equations on resource terms:

Associativity	$(p * q) * r = p * (q * r)$
Unit	$\epsilon * p = p = p * \epsilon$
Commutativity	$p * q = q * p$

As a simple example, $(P \& Q) \multimap P \mathbin{@} \epsilon$ is translated to $\forall \alpha. P(\alpha) \wedge Q(\alpha) \supset P(\alpha)$ (which is true). $P \multimap (Q \multimap P) \mathbin{@} \epsilon$ is $\forall \alpha. P(\alpha) \supset \forall \beta. Q(\beta) \supset P(\alpha * \beta)$ which is not true.

The adequacy theorem states that $\cdot \vdash A$ if and only if $\cdot \vdash A \mathbin{@} \epsilon$ in first-order intuitionistic logic with the stated laws of equality. These laws, plus reflexivity, symmetry, transitivity, and congruence can be written as axioms in the predicate calculus to complete this translation. The proof roughly models the untethered rules given earlier in this section.

You can find an extension to the positive connectives and a proof of adequacy in an unpublished paper by Reed and Pfenning [2010]. This paper further shows that the same recipe applies to ordered logic, and that one can organize the translation so that focusing phases are preserved across the translation. The close relationship to *hybrid logic* is further explored by Reed [2009].

References

Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935. English translation in M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131, North-Holland, 1969.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

Patrick Lincoln, John Mitchell, Andre Scedrov, and Natarajan Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56:239–311, April 1992.

Jason C. Reed. *A Hybrid Logical Framework*. PhD thesis, Carnegie Mellon University, September 2009. Available as Technical Report CMU-CS-09-155.

Jason C. Reed and Frank Pfenning. Focus-preserving embeddings of substructural logics in intuitionistic logic. Unpublished Manuscript, January 2010. URL http://www.cs.cmu.edu/~fp/papers/substruct10.pdf.

Alfred Tarski. The concept of truth in formalized languages. In John Corcoran and J. H. Woodger, editors, *Logic, Semantics, Metamathematics*, pages 152–278. Clarendon Press, Oxford, 1956. Translation of a paper from 1931.