FINDING SMALL SIMPLE CYCLE SEPARATORS FOR 2-CONNECTED PLANAR GRAPHS.

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ABSTRACT

We show that every 2-connected triangulated planar graph with n vertices has a simple cycle C of length at most $4\sqrt{n}$ which separates the interior vertices A from the exterior vertices B such that neither A nor B contains more than 2/3n vertices. The method also gives a linear time algorithm for finding the simple cycle. In general, if the maximum face size is d then we exhibit a cycle C as above of size at most $2\sqrt{2d \cdot n}$.

1. Introduction

Many computational problems on graphs can be performed more efficiently on planar graphs. One basic technique used on planar graphs is "divide-and-conquer." Here one uses the fact that every planar graph has a set of vertices B of size $O(\sqrt{n})$ which separates the vertices A from the vertices C where A, B, C is a partition of the vertires and the size of A and C < 2n/3 [LT 79]. The set B is called a $O(\sqrt{n})$ separator. Two, now classical, applications of a separator are layouts for VLSI [Le 80, Va 81] and nested dissection in numerical analysis [LRT 79].

Some applications require that the separator B have further properties. The planar flow algorithm of Johnson and Venhatrsan [JV 83] required that the separator be a collection of non-nesting cycles. The algorithm can be further simplified if the separator is a simple cycle which we shall exhibit. The work of Dolev, Leighton and Trickey (DLT 83] can also be simplified by using a separator which is a simple cycle.

Some applications may require that the separator be a subset of edges. If $B \subseteq V$ is a separator of a graph G = (V,E) of size b and the maximum vertex degree is d then it follows that the edges at B form a separator of size $d \cdot b$. We may also want to require that the separator consists of edges which form a simple incision. We formally capture this notion bv asking for a simple cycle C In G*, the geometric dual of C, which separates the faces of G*. This motivates a natural generalization of the problem of separators consisting of vertices and separators consisting of edges. Here. we assign weights to the faces and vertices of G so that the combined weights sum to 1. We say that C is a weighted separator if the weight of the interior $\leq$ 2/3 and the weight of the exterior $\leq$ 2/3. We now state the main theorem of the paper.

Theorem 1. If G is an embedded 2-connected planar graph, # is an assignment of weights to the

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vertices and faces that sum to 1, and no face has weight $> 2/3$ then there exists a simple cycle separator of G of size $2\sqrt{2 \cdot n}$ constructible in linear time.

In the special case when G is triangulated, that is $d = 3$, we get a separator of size $4\sqrt{n}$. This is comparable to separators of size $\sqrt{8}\sqrt{n}$ of Lipton and Tarjan [LT 79] and of size $\sqrt{6}\sqrt{n}$ of Djidjev [Dj 82]. Their separators in general are not simple cycles. However, they did not require their graphs be 2-connected.

Theorem 1 is false if the hypothesis that G is 2-connected is dropped. A tree is a simple example. We next observe a simple generalization of the previous theorem which eliminates the need for the 2-connected hypothesis.

Theorem 2. If G is an embedded planar graph and $\#$ is an assignment of weights which sums to 1, such that nonsimple faces have weight zero and no face has weight $> 2/3$, then either there exist a vertex which is a weighted separator or there exist a simple cycle separator of size at most $2\sqrt{2d \cdot n}$.

We will modify the embedding of G by rearranging the 2-connected components. As in Theorem 1 the separator is constructible in linear time.

We first show that Theorem 1 implies Theorem 2. Let G be an embedded graph as in the hypothesis of Theorem 2.

Since a face is simple if and only if all its edges are in the same 2-connected component the faces with nonzero weight can be associated with a unique 2-connected component. Thus, every 2-connected component has a unique associated weight. It follows that either there is a vertex which is a weighted separator of the 2-connected components or there exists a unique proper 2-connected component H (not a simple edge) such that each subtree of components common to H has weight $< 1/3$. Let T be such a subtree with vertex of attachment **x**. Since H is a proper component there are at least 2 faces, F_1 and F_2, common to x in the embedding of H. If the weight on either F_1 or F_2 is $> 1/3$ we shall pick that face as the separator and embed T in the other face. Otherwise, we can discard T and add its weight to either the weight of F_1 or F_2. Continuing in this manner we can reduce the question of separators to a 2-connected graph with weights and then apply Theorem 1. Note that the size of faces only decreases. □

2. Preliminaries.

There are many formal definitions and many intuitive definitions of graphs "drawn" or embedded in the plane. Following Edmonds, Lehman, Tutte, and many others, we make the following formal definition: Let **G** be an undirected graph. We view each edge of G as two directed edges or darts. An embedding will simply be a description of the cyclic orderings of the darts radiating from each vertex. Formally, let $Sym(E)$ denote all permutations of the darts of G.

Definition: The permutation $\phi \in Sym(E)$ is an embedding of G if:

1) $Tail(e) = Tail(\phi(e))$ for any $e \in E$,
2) ϕ restricted to the darts at $v \in V$ is a cyclic permutation.

To specify the faces of this embedding consider the permutation R such that $R(e)$ is the reflection of the dart e. Now, successive application of ϕ will traverse the darts radiating from a vertex, in say, a clockwise order. On the other hand, the permutation $\phi^*=\phi\cdot R$ will traverse the darts of the boundary of a face in counterclockwise order. We say that ϕ is a planar embedding if the number of faces of the embedding, say f, satisfies Euler's formula:

$$f - e + v = 2$$

We shall not distinguish between a face and its boundary of counterclockwise oriented darts. Given two boundaries of two distinct faces, F and F', the boundary of their union will be equal to F + F' where e + R(e) = 0. By a path in C we shall mean the darts on the path. It follows easily, in this formal model, that any simply cycle C has a well-defined interior int(C) (the faces, vertices, and edges to the left of C) and a well-defined exterior ext(C) (the faces, vertices, and edges to the right of C).

Let C be a simply cycle of the embedded planar graph G. We next define a natural breath first search into the exterior of C. Let EF be the faces of G in the exterior of C which share a vertex or an edge with C, R(e) ≡ e. Consider the sum C' = C + CF where FEEF. We next show that C' can be written as a disjoint sum of simple cycles such that their exteriors are also disjoint. Let EF* be the subgraph in the geometric dual of G induced by the faces in ext(C'). Further, let L be the faces in a connected component of EF*. Consider D the boundary of the union of the faces of L, D = ΣF for FEL. It follows that R(D), the reflection of all the darts of D, is contained in C'.

We need only show that D is a simple cycle.

Lemma 3. The graph D as described above is a simple cycle.

Proof: We note that the boundary of L consists of a collection of cycles. Since, given a dart on the boundary of L, there is a unique successor and a unique predecessor. If D is not simple it can be decomposed into 2 or more simple cycles. Suppose that D is not simple. Let e and e' be two darts of D on distinct simple cycles, say, C_1 and C_2. Since the regions defined by L and C' are connected there are two vertex disjoint paths, one in the interior of L and other in the interior of C' which only share a point on e and a point on e'. These two paths form a cycle T on the surface that crosses over C_1 in a fundamental way. Thus T and C_1 form a graph of genus 1. This is a contradiction. Thus we may conclude that D is simple. □

Thus, the unsearched region decomposes into a collection of connected regions each with a boundary consistinq of a simple cycle. We shall call C' the next level out from C and each R(D) a branch of C.

3. Finding a subgraph of small diameter.

The algorithm consists of two passes. In the first pass, outlined in this section, we find a subgraph H which has $O(\sqrt{n})$ diameter and $O(\sqrt{n})$ face size. The second pass will find a separator contained in H. The planar embedding of H will be the one induced by the embedding of G, the original graph. The weight on a vertex of H will equal the weight assigned in G. A face F of H will have weight equal to the sum of the weights of faces and vertices of G which are embedded in F. This weight will be called the induced weight on F. We give the main theorem of this section.

Theorem 4. If G is a 2-connected embedded planar graph with weights on its faces and vertices which sum to 1, no face weight > 2/3 and the maximum face size is d, then there exist a 2-connected subgraph H with spanning tree T satisfying:

1) The diameter of T plus maximum size of any face of H is at most $2\sqrt{2d \cdot n}$
2) The maximum induced weight on any face of H is $\leq 2/3$.

Proof: Note that G is 2-connected if and only if every face of G is simple. Let G satisfy the hypothesis of the theorem and F be some face of G. Further, let # be an assignment of weights also satisfying the hypothesis.

We start by constructing a breath first search of the levels from F as defined in the preliminaries. Namely, we construct the next level out from F and decompose it into branches. For each branch we again construct its branches. This gives us a tree of branches with root F. Note that by starting from the leaves of this tree we can compute the induced weight on the interior and exterior of each branch in linear time.

Let C be the first branch such that #(int(C)) < 1/3, the interior of C is the side containing F. Further, let B be the ℓ_1^{th} ancestor of C such that $d\ell_1 + \text{size}(B) \leq \sqrt{2d \cdot n}$. Such a B must exist since otherwise the i^{th} ancestor B_i of C must have size $> b_i = \sqrt{2d \cdot n} - d \cdot i$ for $0 \leq i \leq \sqrt{2n/d}$. Now, the B_i's have disjoint vertices and therefore the sum over the b_i's must be < n. By a straightforward calculation the sum of the bi's is larger than n which is a contradiction.

Let $C = B_0, \ldots, B_{\ell_1} = B$ be the ancestors of C up to B. Consider the subgraph H' obtained from G by deleting 1) the exterior of any branch of B_1 thru B_{ℓ_1} which is distinct from $B_1, \ldots, B_{\ell_1 - 1}$ plus 2) the interior of B. Note that we have deleted the exterior of C. The subgraph H' has small diameter and induced face weights $\leq 2/3$ but the face sizes may be too large. For each face of H' construct the next level out until the maximum number of levels constructed ℓ_2 and the maximum branch size f satisfies $d \cdot \ell_2 + f \leq \sqrt{2d \cdot n}$. By similar arguments as used above this procedure will terminate. The subgraph H will be G minus the exteriors of these branches. We call the portion of G added onto a face of H' a cap. We next construct the spanning tree T.

Note that if D is a simple cycle and x is a vertex on the next level out from D then the distance from x to D can be at most d/2 since they must share a face of size $\leq$ d. Thus a breach first search from any point on B in H' will generate paths of length at most $(d \cdot \ell_1 + |B|)/2$. By similar arguments, any point in a cap is at most $d \cdot \ell_2 / 2$ away from H'. Thus, H has a spanning tree of diameter $d(\ell_1 + \ell_2) + |B|$. Adding in the maximum face size we get $d \cdot \ell_1 + |B| + d \cdot \ell_2 + f < 2\sqrt{2d \cdot n}$ from the inequalities above.

4. Finding a separator in a graph of small diameter.

By the last section we can find a subgraph of radius $O(\sqrt{d \cdot n})$. Here we find a small simple Cycle which is a separator. The main theorem of this section is:

Theorem 5. If G_ϕ is a 2-connected embedded planar graph with spanning tree T then there exist a simple cycle weight separator of size at most dia+S, where dia= the diameter of T, S the maximum face size, and no face weight > 2/3.

Proof: The proof will consist of a sequence of successive approximations that will converge to a cycle that is a weighted separator. Let e be any non-tree edge and C the induced simple cycle in the spanning tree T.

If C is not a weighted separator then, without loss of generality, we may assume that the weight of the interior of C_e > *213*. Let F be the face common to e on the interior of C_e. Further, let $e_1,\ldots,e_k$ be the non-tree edges on F distinct from e. Note that $k \geq 1$. For if k = 0 then F would be the interior of C_e since F is simple. This contradicts the facts: 2/3 > #(F) = #(int(C_e)) > *213*. We now partition int(C_e).

Let C_i be the cycle induced by e_i such that int(C_i) is contained in int(C_e). Thus the regions int(C_1),...,int(C_k),int(F) are a partition of int(C) up to vertices and edges. We first reduce the problem to the case when #(ext(F)), #(ext(C_1)), ..., #(ext(C_k)) > *213* as follows: (*)

1) If #(int(C_i)) > 2/3 for some $1 \leq i \leq k$ then set e to e_i and repeat.

2) If #(ext(F)) $\leq$ 2/3 then F is a weighted simple cycle separator of size $\leq$ S.

3. If #(ext(C_i)) $\leq$ 2/3 for some $1 < i < k$ then C_i is a weighted simple cycle separator of size $\leq$ dia+1.

Given condition (*) we shall construct the separator from F plus some of the C_i's. But we must do this in such a way that the cycle is simple. We introduce a partial order on the C_i's.

Let x and y be the end points of the edge e. Since F is a simple cycle, if we remove e from F we obtain a simple path from x to y on F. Let $x=x_1$, $\ldots,x_t=y$ be the vertices on the path in the order they appear. Given any cycle C_i it will have a vertex of minimum index and one of maximum index in $\{x_1,\ldots,x_t\}$. We shall call these vertices the left most and right most vertices of C_i respectively. We say C_i domains C_j, $i \neq j$, if $i_\ell \leq j_\ell \leq j_r \leq i_r$, where i_ℓ and i_r are the indices of the left most and the right most vertices of C_i and similarly for j_ℓ and j_r. Using the fact that the graph is planar we get a forest *on* the C_i's by adding a directed edge from C_i to C_j if C_i domains C_j and there is no k such that C_i domains C_k and C_k domains C_j. By adding C_e we get a directed tree. If C_i and C_j have the same parent then C_i is left of C_j if $i_r < j_\ell$.

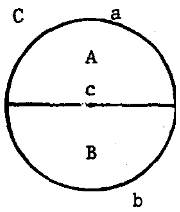
We associate with each region C_i the union of all regions domained by it, i.e., $\bar{C}_i=\{\Sigma C_j | C_i$ domains C_j or $i=j\}$. Similar to the fact that trees have a separator consisting of a single vertex we get:

Lemma 6. Either a) there exists an i such that F+$\bar{C}_i$ is a weighted separator or b) there exists an i such that #(int(F+$\bar{C}_i$)) > *2/3* and for all j, such that $\bar{C}_j$ is a child of $\bar{C}_i$, #(ext(F+$\bar{C}_j$)) > 2/3.

Note that $\bar{C}_i$ forms a simple cycle which intersects F on some interval of F. Thus, F+$\bar{C}_i$ will consist of an interval of F plus and interval of $\bar{C}_i$ which are disjoint except at the end points . Since the interval of $\bar{C}_i$ is contained in T the size of F+$\bar{C}$ is at most dia+S. We will assume that [illegible]isfies condition b) of Lemma 6 for the [illegible]nder of the proof of Theorem 5.

[illegible]et $D_1, \ldots, D_t$ be the children of $\bar{C}_i$. We say [illegible] **left** of D_j if the vertices of D_i on F are [illegible] of the vertices of D_j on F. We partition the [illegible] into those that are left of e_i and those that [illegible] right of e_i. We shall successively add either [illegible] left most D_i if it is left of e_i or the right [illegible]st D_i if it right of e_i. Let $D=D_i$ be such a D_i. [illegible] must show that $\#(int(F+D)) < 2/3$. We know that [illegible](int(F)), #(int(D)) < 1/3. But, $F \cap D$ will also be [illegible]n the interior of F+D. We shall use the stronger [illegible]act that #(ext(F) > 2/3.

Lemma: If G is an embedded graph, A and B are faces, #(ext(A)) > 2/3, and #(int(B)) < 1/3 then #(A+B) < 2/3.



Proof: Let A and B satisfy the hypothesis. Let $a=A-(A \cap B)$, $b=B-(A \cap B)$, $c=A \cap B$, and $C=\partial(A+B)$. The figure may help keep track of the notation. The lemma will follow if we show that #(b)+#(int C) ≥ 1/3. Now ext(A) is the disjoint union of int(B), b, and int(C). Thus #(int(B))+#(b)+#(int(C)) > 2/3. Since #(int(B)) < 1/3, we get that #(b)+#(int(C)) ≥ 1/3. □

Using the last lemma we can simply pick $D_1, \ldots, D_j$ for some j such that $F'=F+D_1+\ldots+D_j$ is a separator. We must show that F' is simple and of small size. We state without proof the following simple lemma.

Lemma: If $D_1, \ldots, D_j$ are consecutive and all left (right) of e_i then $F+D_1+, \ldots, +D_j$ is simple and consists of an interval from F plus a simple path in T, the spanning tree.

Thus, the new region will consist of F plus consecutive elements from the left of e_i and consecutive elements from the right of e_i. Its boundary will consist of two paths from the tree plus 2 paths from F. Thus, the size of this region is at most 2dia+S. Actually these two paths in the tree can be joined to form one simple path in T. Thus the size ≤ dia+S. □

Conclusions

In this paper we have concentrated on worst case separators. That is, an algorithm which finds a relatively small separator when the smallest separator is relatively large. It is open whether there is a polynomial time algorithm which finds the optimal separator for planar graphs. It is easy to show that there is always an optimal separator which consists of non-nesting simple cycles if the graph is triangulated. We say a simple cycle C is a separator of ratio k if size(C)/min{I(int(C)),#(ext(C))}=k. Question: IS finding an optimal ratio separator for planar graphs polynomial time computable?

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