Regular Groups of Automorphisms of Cubic Graphs

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Background

Tutte [14] made the following definition which is crucial in understanding symmetric cubic graphs. We should call a pair consisting of a connected cubic graph G and a group of automorphisms of G an *object* denoted (G, A).

Definition Given an object (G, A) we say A is *s-transitive* (*s-regular*) over G if A is (sharply) transitive over paths of length s.

In [14] Tutte showed that:

Theorem 1 *If (G, A) is an object and A is s-transitive but not $(s + 1)$-transitive then A is s-regular.*

The other major result of Tutte, on which all that is to follow is based, is

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Theorem 2 [13] *If (G, A) is as in Theorem 1 and A is s-regular then* $1 \leq s \leq 5$.

A very natural infinite cubic graph is the infinite tree of valence 3 which we will denote by Γ_3. The full automorphism group of Γ_3 is ω-transitive but we shall see that there are s-regular groups acting on Γ_3 for $1 \leq s \leq 5$. An interesting corollary to Tutte's two theorems is

Corollary *If (Γ_3, A) is an object and A is 6-transitive then A is ω-transitive.*

1. Vertex Fixers and Edge Stabilizers

Let (G, A) be an s-regular object, $u_0, \ldots, u_s$ a simple path in G, $A(u_0, \ldots, u_j)$ the subgroup of A which fixes $u_0, \ldots, u_j$, $0 \leq j \leq s$. The s-regularity of A implies that $|A(u_0, \ldots, u_j)| = 2^{s-j}$ for $1 \leq j \leq s$ and $|A(u_0)| = 3 \cdot 2^{s-1}$. Thus the cardinality of $A(u_0)$ is dependent only on s. In fact, up to isomorphism $A(u_0)$ is only dependent on s.

Theorem 3 *If (G, A) is an s-regular object and u is a vertex of G then $A(u)$ is unique independent of G and A.*

Proof See Propositions 2–5 in [6]. ■

We shall denote this group by X_s.

As a corollary to Theorem 3 we get that for each s the edge fixers are unique. Let H_s denote the edge fixers. Let $\{u, v\}$ be an edge in G, (G, A) as before, and let $A[u, v]$ be the subgroup of A which need only stabilize $\{u, v\}$. Since $s \geq 1$, $A(u, v)$ is a subgroup of index 2 in $A[u, v]$ and there exist elements in $A[u, v]$ which flip the edge $\{u, v\}$. We say that A is of *type* s' if there exists an involution $\alpha \in A$ which flips an edge. Otherwise we say A is of *type* s''.

Theorem 4 *If (G, A) is an s-regular object and if s is odd then the edge stabilizer is unique and of type s'; on the other hand, if s is even then the edge stabilizer is either of type s' or type s'' and otherwise unique.*

Proof See Propositions 2–5 in [6]. ■

Let $Y_{s'}$ $(Y_{s''})$ denote the edge stabilizer of type s' (s''). The group $Y_{s''}$ is defined only when s is even.

2. Amalgams and Amalgamated Products

An amalgam is an ordered pair (X, Y) consisting of two groups X and Y, such that multiplication in X and Y coincide on $X \cap Y$ and $X \cap Y$ is a group. Given an arc (u, v) in G we can in a natural way form the amalgam

$(A(u), A[u, v])$ where $A(u) \cap A[u, v] = A(u, v)$. By the proof of the last three theorems we get

Theorem 5 *The amalgam of an arc is unique up to s and the type.*

Proof See Proposition 10 in [6]. ■

We shall denote this amalgam by $(X_s, Y_{s'})$ or $(X_s, Y_{s''})$, depending on the type.

For each amalgam we can construct a unique group the amalgamated product, i.e.,

$$A_{s'} = X_s \mathop{*}_{H_s} Y_{s'}, \qquad A_{s''} = X_s \mathop{*}_{H_s} Y_{s''}.$$

The importance of A_s is twofold. Not only shall we see that A_s acts s-regularly on Γ_3 but if (G, A) is an s-regular object and A_s is of the right type, then there exists a natural surjective homomorphism from A_s to A.

Given A_s and X_s, Y_s we can construct a graph $G_s = (V, E)$ as follows:

$$V = A_s/X_s \quad \text{(the left cosets of } X_s\text{)},$$
$$E = \{\{ux_s, vx_s\} \mid u^{-1}v \in X_s y X_s\},$$

where $y \in Y_s - X_s$.

We list a few facts:

Theorem 6 (1) *G_s is Γ_3*; (2) *A_s acts on G_s by left multiplication*; *and* (3) (Γ_3, A_s) *is an s-regular object.*

Proof See Proposition 11 in [6]. ■

Let (G, A) be an s-regular object and A_s be of the same type as A and g the canonical group homomorphism from A_s to A then we can define a graph homomorphism f: $\alpha X_s \mapsto g(\alpha)u$, $\alpha \in A_s$ and u the implicit vertex in G. Now the pair (f, g) satisfies certain properties which we now define.

Definition A covering morphism is a pair (f, g): $(G, A) \to (G', A')$ such that:

(1) $g: A \to A'$ onto group homomorphism;
(2) $f: G \to G'$ onto graph homomorphism;
(3) f is locally 1–1 (neighbors of a vertex are sent to distinct vertices); and
(4) the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ G' & \xrightarrow[g(\alpha)]{} & G' \end{array}$$

commutes, i.e., $g(\alpha) \cdot f = f \cdot \alpha$ for all $\alpha \in A$.

With one more definition we can state the key idea for the diagrams. A subgroup of A_s is said to be *small* (in A_s) if $K \lhd A_s$, $K \cap X_s = K \cap Y_s = 1$, and $(A_s\colon KX_s) > 2$.

Theorem 7 *Let G be a connected cubic graph and A an s-regular subgroup of* Aut(G), $1 \leq s \leq 5$. *Let* $A_s = A_s'$ *or* A_s'' *be of the same type as A. Then there exists a covering morphism* $(f, g)\colon (\Gamma_3, A_s) \to (G, A)$ *and* ker g *is small in* A_s.

Conversely, let K be a small subgroup of A_s *and put* $A_K = A_s/K$. *Let* G_K *be the graph whose vertex-set is* A_s/KX_s *and in which two vertices* uKX_s *and* vKX_s *are adjacent iff* $u^{-1}v \in KX_s yX_s$. *Then* G_K *is a connected cubic graph,* A_K *is an s-regular subgroup of* Aut(G_K), *and* A_K *is of the same type as* A_s. *There is a covering morphism* $(f, g)\colon (\Gamma_3, A_s) \to (G_K, A_K)$ *where* $g\colon A_s \to A_K$ *is the canonical map and* $f\colon \Gamma_3 \to G_K$ *is defined by* $f(uX) = uKX$.

Proof See Theorem 1 in [6]. ■

If (f, g) is a covering morphism the kernel of (f, g) will be defined to be the kernel of g. We have from Theorem 6 that every s-regular object generates a small normal subgroup of A_s, i.e., the ker(f, g), and every small normal subgroup of A_s generates s-regular objects. The next two theorems tell us how many normal subgroups correspond to a given cubic graph.

Theorem 8 *Let* (G, A) *be an s-regular object,* $s = 3$ *or* 5; *then all covering morphisms* $(f, g)\colon (\Gamma_3, A_s) \to (G, A)$ *have the same kernel.*

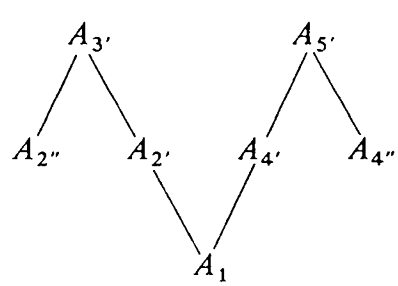
Proof See Theorem 2 in [6]. ■

Theorem 9 *Let* (G, A) *be an s-regular object,* $s = 1, 2, 4$, *and* (Γ_s, A_s) *being the same type as* (G, A); *then over all covering morphisms from* (Γ_s, A_s) *to* (G, A) *there are either one or two kernels and if there is exactly one kernel then* G *is* $(s+1)$-*transitive.*

Proof See Theorem 8 in [6]. ■

Before presenting the diagram, we need a few more results about the subgroup structure of the A_s's.

Theorem 10 *The subgroup structure for the* A_s*'s is indicated by the following diagram*:



3. Some Subgroups of A_3

Some subgroups of $A_{3'}$ are shown in Fig. 1 and we shall describe them now. The lines indicate normal subgroups and the number indicates the index. Starting with A_3 it has three subgroups of index 2, namely, $A_{2'}$ and $A_{2''}$ and the even subgroup of $A_{3'}$, the subgroup which preserves the bipartition on Γ_3 denoted $A_{3'}^{+}$. The group $A_{2'}$ contains two copies of A_1 and its even subgroup $A_2{}^{+}$ which equals $A_{2'} \cap A_3{}^{+} \cap A_{2''}$. The even subgroup of $A_{1'}$ is $A_{1'} \cap A_2{}^{+}$ and $A_{1'}$ contains a normal subgroup of index 3, $A_{0'}$ which is vertex-regular on Γ_3. Now $A_{1'}^{+}$ contains two copies of two even subgroups of vertex-regular groups on Γ_3. The subgroups $K_{4'}$, $K_{4''}$ and $Q_{3'}$, $Q_{3''}$ correspond to the two copies of the graph K_4 and the cube, respectively. The graph $K_{3,3}$ corresponds to a unique normal subgroup of A_3 since it is 3-regular. Now N corresponds to a 3-regular graph which is a 12-fold covering of Q_3 and a 16-fold covering of $K_{3,3}$. The subgroup $P \lhd A_{3'}$ corresponds to Petersen's graph; D', $D'' \lhd A_{2'}$ correspond to dodecahedron; and $Z \lhd A_3$ corresponds to Desargue's graph. Finally $K \lhd A_{3'}$ corresponds to the vertex primitive 3-regular graph on 28 vertices $G(28)$.

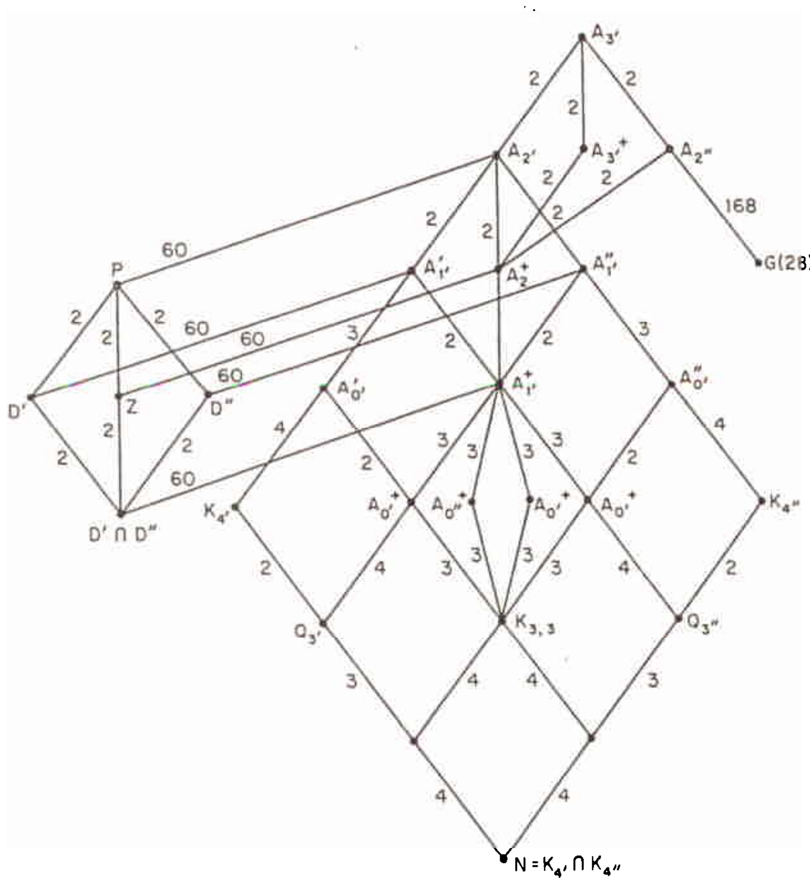


Figure 1 Partial subgroup structure of A_3.

By simply noting where the subgroup of a graph lies in the diagram a large amount of information can be read off. As an example, $K_{3,3}$ is contained in $A_{2'}$ and $A_{2''}$; therefore, Aut($K_{3,3}$) contains subgroups of type 2′ and type 2″. Similarly, Aut($K_{3,3}$) contains two 1-regular subgroups and at least two 0-regular subgroups: $K_{3,3}$ must be bipartite. Now Petersen's graph has no type 2″ or type 1′ actions since it is not contained in $A_{2''}$ nor $A_{1'}$. The fact that Petersen's graph contains no 1-regular subgroup follows by the fact that Petersen's graph is not bipartite and the following proposition:

Proposition *Let* (G, A) *be a* 3-*regular object*; *then the following hold*:

(1) *if* A *contains a* 1-*regular subgroup then* G *is bipartite and* A *contains two* 2-*regular subgroups and two* 1-*regular subgroups*;
(2) *if* A *contains two* 2-*regular subgroups then* G *is bipartite.*

Proof See Proposition 26 in [6]. ■

4. Some Subgroups of $A_{5'}$

Some subgroups of $A_{5'}$ are shown in Fig. 2, and we shall describe them now. We have $A_{4'}^{+} = A_{4'} \cap A_{4''}$. T is the normal subgroup of $A_{5'}$ which

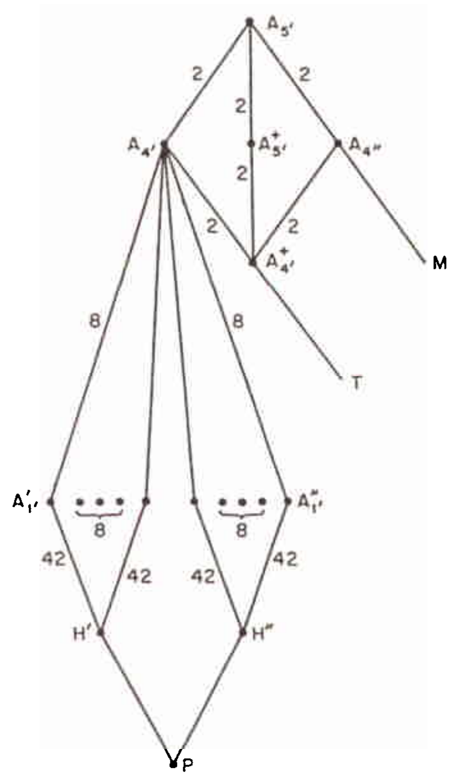


Figure 2 Partial subgroup structure of A_5.

corresponds to Tutte's 8-cage. As is well known, we have $A_{4'}/T \approx A_6$, $A^{+}_{5'}/T \approx S_6$, and $A_{5'}/T \approx \mathrm{Aut}(S_6)$. Now $M \lhd A_s$ corresponds to the unique finite primitive 5-regular object ($G(234)$, $\mathrm{Aut}(SL_3(3))$). We have $A_{4''}/M \approx SL_3(3)$ and $A_{5'}/M \approx \mathrm{Aut}(SL_3(3))$.

Now $A_{4'}$ contains 16 copies of $A_{1'}$. They are all conjugate in $A_{5'}$ and fall into two conjugacy classes of size 8 in $A_{4'}$. Let $A_{1'}$ and $A_{1''}$ denote members of each of the two conjugacy classes. The intersection of the $A_{1'}$'s and the intersection of the $A_{1''}$'s correspond to the two copies of Heawood's graph.

Theorem 11 *Heawood's graph is the unique minimal graph which is 4-regular and whose automorphism group contains a 1-regular subgroup. So, every 4-regular object with a 1-regular subgroup is a covering of Heawood's graph.*

Proof See Proposition 29 of [6]. ■

Finally, $P = H' \cap H''$ corresponds to the unique minimal 5-regular graph such that $\mathrm{Aut}(G)$ contains a 1-regular group.

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