Separators in Two and Three Dimensions

Gary L. Miller*
School of Computer Science
Carnegie Mellon University &
Dept of Computer Science
University of Southern California

William Thurston
Dept of Mathematics
Princeton University

Abstract

We show that every graph that is the 1-skeleton of a simplicial complex K in 3-dimensions has a separator of size $O(c^{2/3} + \bar{v})$, where c is the number of 3-simplexes in K and $\bar{v}$ is the number of 0-simplexes on the boundary of K, if every 3-simplex has bounded aspect-ratio. This is natural generalization of the separator results for planar graphs, such as the Lipton and Tarjan planar separator theorem. We also show that a family of separators of size $O(c^{2/3})$ exists and is constructible. Using this family of separators we get an $O(n^2)$ time algorithm for solving linear systems that arise from the finite element method. In particular, we solve linear systems in $O(n^2)$ time where the underlying graph is the 1-skeleton of a simplicial complex having bounded aspect-ratio and small boundary All the constructions work in RNC with a reasonably small number of processors.

1 Introduction

The Divide-and-Conquer paradigm is fundamental for a large number of both sequential as well as parallel algorithm design. Divide-and-Conquer can give both fast and efficient algorithms. For graph algorithms and numerical analysis the efficiency of the algorithm is determined by the size and quality of the separator used in the algorithm.

Definition 1.1 *A subset of vertices B of a graph with n vertices is said to δ-**separate** if the remaining vertices can be partitioned into 2 sets A and C such that there are no edges from A to C, $|A|$, $|C| \leq \delta \cdot n$. The subset B is an $f(n)$-**separator** if there exist a constant $6 < 1$ such that B δ-separates and $|B| \leq f(n)$.*

Two of the most important classes of graphs with small separators have been trees and planar graphs. It is well known that a tree has a single vertex separator that 2/3-separates. Another natural class of graphs with small separators are the planar graphs. Lipton and Tarjan showed that any planar graph has $\sqrt{8} \cdot n$-separator that 2/3-separates, [LT79]. They gave a linear time algorithm to find this separator. There have been many extensions of this work, [Mil86, GM87, GM, Dji82, Gaz86]. All these results only consider planar graphs. There have also been several results on finding separators for graphs of a given genus, [GHT82, HM86]. Many applications of these separators exist, [Lei83, FJ86] One of the main applications of these separators is the finite element method, [LRT79, PR85a, PR85b, GT87]. We show that our separators do not generate too much fill-in and, therefore, can also be used for solving these linear systems in $O(n^2)$ sequential time or $O(\log^2 n \log\log n)$ time using $O(n^2)$ processors on a PRAM. Thus, we show that n^2 direct

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methods exist for the finite element method in the 3-dimensions (possibly the most important dimension). The algorithms we present for finding these family are randomized. But linear system solvers are otherwise deterministic.

To motivate our result we view the planar separator theorem as a statement about 2-dimensional simplicial complexes. We first give a few definitions.

Definition 1.2
A k-dimensional simplex **(k-simplex)** *is the convex hull of $k+1$ affinely independent points in $\Re^d$ space. A simplicial complex is a collection of simplexes closed under subsimplex and intersection. A* **k-complex** *K is a simplicial complex such that* **for** *every k'-simplex in K, $k' \leq k$.*

Thus, a 3-complex is a collection of tetrahedra or cells (3-simplexes), triangular patches or faces (2-simplexes), edges (l-simplexes), and vertices (O-simplexes). The **k-skeleton** of a simplicial complex K is the k-complex consisting of all k'-simplexes in K for $k' \leq k$. Thus, the l-skeleton of a 2-complex in the plane can be viewed as a graph that is planar. On the other hand, by Fáry's Theorem we know that every planar graph can be embedded in the plane such that each edge maps to a straight line, [Tho80a]. Thus, if G is a triangulated planar graph then it can be embedded in a l-skeleton of a 2-complex in the plane. Thus we can view the planar separator theorem as statements about the l-skeletons in 2-dimensions. The main goal of this paper is to show that under reasonable assumptions small separators exist and can be found for l-skeletons of 3-complexes embedded 3-dimensions.

We next discuss the restriction we place on the 3-complex. It is not hard to see that any graph can be embedded in 3-dimensions. In particular, one can show that the complete graph can be embedded in the l-skeleton of a 3-complex in $\Re^3$. The 3-complex will have $O(n^2)$ 3-simplexes. We can accomodate this example by allowing the size of the separator to be a function of the number of 3-simplexes.

This restriction alone is not sufficient to insure the existence of small separators. In Section 6 we exhibit a 3-complex such that its l-skeleton has only separators for size $\geq t \cdot c/\log c$, where c is the number of 3-simplexes and t is some fixed constant. The 3-simplexes in the example are long and thin. For many applications we can restrict our attention to those complexes where the simplexes have bounded aspect-ratio. There are many equivalent definitions of the aspect-ratio of a simplex such as: all angles have minimum size, the number of simplexes that can share a point is bounded below, and ratio between the diameter of the circumscribing sphere and diameter of the inscribing sphere is bounded. We have picked the following definition:

Definition 1.3 *The* **diameter** *$Dia(S)$ of a k-simplex S is the maximum distance between any pair* **for** *points in S. While the* **aspect-ratio** *equals*

$$\alpha = \frac{Dia(S)}{\sqrt[k]{|S|}} \tag{1}$$

where $|S|$ denotes the k-dimensional volume of S.

There is one other restriction that we need to complete the list. A simplicial complex need not contain all of space. We assume that the k-complex is contained in $\mathcal{R}^l$ for $l \geq k$. **A point** is a vector in R'. The **underlying space** of **K** is all those points lying in some simplex of K.

Definition 1.4 A *simplex S of K is* **exterior** *if S is contained in the boundary of the underlying space of K. A simplex of K is* **interior** *if it is not exterior.*

Our separator will "separate" any subset of vertices. For many applications one only needs to separate interior vertices. For example, in the finite element method vertices on the boundary correspond to points with known values and thus they need not be separated. For other applications one would need to introduce dummy 3-simplexes around the boundary. We use this idea when we show how to find a family of separators.

We can now state our first separator theorem. A k-complex K is **a-stable** if every k-simplex has aspect-ratio at most a.

Theorem 1.5 *If K is a a-stable 3-complex in $\Re^3$ with c 3-simplexes and $\bar{v}$ exterior vertices, then the*

1-skeleton has a $(t(\alpha)\cdot c^{2/3}+\bar{v})$*-separator that* 4/5*-separates where* $t(\alpha)$ *is a constant only depending on* α.

We shall find the separator in two steps. In the first step we solve a continuous version of the problem. While in the second, we show how to use a solution to the first problem to solve the separator problem for simplicial complexes.

Let $f(x)$ be a real valued nonnegative function defined on $\Re^d$ such that all integrals exist. We think of f as the cost function. The total cost of the system is

$$\text{Total-Cost} = \int_{v\in\Re^d} (f(v))^d (dv)^d < \infty \qquad (2)$$

Note that in d-dimensions we are integrating over f^d. This definition seems rather nonintuitive. But in fact this definition is crucial for our application.

If S is a $(d-1)$-sphere then the cost of S will be

$$Cost(S) = \int_{v\in S} (f(v))^{d-1} (dv)^{d-1}. \qquad (3)$$

Theorem 1.6 *If f is a cost function on* $\Re^d$ *and Q is a set of n distinct points in* $\Re^d$ *then there exists a* $(d-1)$*-sphere in* $\Re^d$ *that separates the points of Q not on S into two sets, the interior and exterior of S, each of size at most* $\frac{d}{d+1}n$*, and the* $Cost(S) = O((\textit{Total-Cost})^{\frac{d-1}{d}})$.

This last theorem can be strengthened by replacing the set of points Q by almost any mass function such that the mass at any point is at most $\frac{d}{d+1}n$ where n is the total mass. The above theorem will suffice for our application. We discuss generalizations in the full paper.

The general outline of the algorithm for finding the separator for Theorem 1.5 is as follows:

1. Define a cost function that is constant in each 3-simplex such that the cost of each 3-simplex is one.
2. Find a 2-sphere SP that separates the vertices and has cost $O(c^{2/3})$ by Theorem 1.6.
3. "Pull" the sphere SP back to a 2-complex in K that also separates the vertices of K.

2 Separating Density Functions

Our goal in this section is to prove Theorem 1.6 by finding a sphere that separates a set of points and whose cost is small. As in the introduction let f be a "nice" nonnegative real valued function on $\Re^d$. Piecewise linear will suffice for our application. Let Q be a set of n points in $\Re^d$. We shall separate Q with a plane by first mapping the points Q onto the unit d-sphere in d+ 1 dimensions and then separating the points with a hyperplane. We let SP_d denote the unit d-sphere in $\Re^{d+1}$ centered at the origin.

Recall that a map is conformal if it preserves angles. Clearly, rigid motions and dilations (v goes to av for $a\in\Re$) are conformal maps. A nontrivial conformal map is the stereographic projection which maps $\Re^d$ plus infinity onto SP_d. Since the composition of any two conformal maps is conformal, we may apply any combination of the above maps and obtain a conformal map. If F is a conformal map then we can also map the cost function f to a new cost function such that costs are preserved in every dimension simultaneously. We will not need to compute this cost function. In fact in our application the cost function only occurs in the proof and not in the actual algorithm.

Thus, we may assume that the cost function f is defined on SP_d, the points Q are also on the unit sphere, and our goal is to find a hyperplane **P** in $\Re^{d+1}$ which separates Q and the cost of $P\cap S = O((\text{Total-Cost})^{\frac{d}{d+1}})$.

It is well known that there exists a point v in the sphere SP_d such that every hyperplane containing v is a separator, these points are called centerpoints.

Definition 2.1 *The point* $p\in\Re^d$ *is a* **centerpoint** *for a set of n points* $Q\subset\Re^d$*, if every hyperplane P, not containing p contains at most* $\frac{d}{d+1}n$ *point of Q.*

It follows from Brouwer's fixed point theorem [Bir59] and also Helly's theorem [Ede87] that centerpoints always exist. In Section 5 we discuss known and new algorithms for finding centerpoints and "almost" centerpoints.

We next conformally move the centerpoint to the origin. First we observe that a conformal map of SP_d can be extended to all points of $\Re^d$ such that lines go to lines. This extension is not in general conformal on the unit ball. However, it is not hard to see that, in this extension, any point interior to the sphere can be moved to the center by one rotation followed by a dilation.

WLOG, we may assume that any hyperplane containing the origin is a separator of the points of Q and that cost function f is defined on the sphere. We next show that, on the average, the cost of a hyperplane containing the origin intersected with the unit sphere (a great circle) is small. Let $x \in SP_d$. We denote by GC_x the oriented great circle in SP_d that is centered at x. We define the expected cost of a great circle to be

$$\text{avg} = \frac{\int_{x \in SP_d} g(x)(dx)^d}{V_d} \tag{4}$$

where V_d is the d-dimensional volume of SP_d and $g(z) = \int_{w \in GC_z} f^{d-1}(w)(dw)^{d-1}$. This integral can be rewritten as

$$\begin{aligned} \text{avg} \cdot V_d &= \int_{x \in SP_d} g(x)(dx)^d \\ &= V_{d-1} \int_{x \in SP_d} f^{d-1}(x)(dx)^d \end{aligned} \tag{5}$$

2.1 Using Cauchy-Schwarta Inequality

In this subsection we bound the size of an average great circle.

Let the kth moment be

$$M_k = \int_{x \in SP_d} f^k(x)(ds)^d. \tag{6}$$

Observe that M_d = Total-Cost and $M_0 = V_d$. We use the following form of the Cauchy-Schartz's inequality:

$$\left(\int u(x)v(x)dx\right)^2 \leq \int (u(x))^2 dx \cdot \int (v(x))^2 dx \tag{7}$$

Setting u to f and v to the constant function 1 we get:

$$\left(\int_{x \in SP_d} f(x) \cdot 1(dx)^d\right)^2 \leq \int_{x \in SP_d} (f(x))^2 (dx)^d \int_{x \in SP_d} 1(dx)^d \tag{8}$$

Rewriting this in terms of moments we have $(M_1)^2 \leq M_2 M_0$. Using inequalities of this form we get:

Theorem 2.2 $(M_{d-1})^d \leq (M_d)^{d-1} M_0$

We rewrite equations **4** and **5** in terms of moments; avg $= \frac{V_{d-1}}{V_d} M_{d-1}$. Combining this with Theorem 2.2 we get:

Theorem 2.3 *The average cost of a great circle is at most*

$$\frac{V_{d-1}}{(V_d)^{\frac{d-1}{d}}} (M_d)^{\frac{d-1}{d}}. \tag{9}$$

For d=3, avg $\leq \frac{2^{4/3}}{\sqrt[3]{\pi}}$*(Total-Cost)*$^{2/3}$.

Thus, we see that picking a random great circle, after conformally mapping the centerpoint to the origin, gives a separator of cost 2/3 power of the total cost for 3-dimensions.

3 Separating a Simplicial Complex with a Sphere

In this section, we show how to separate a k-dimensional simplicial complex by a $k-1$-dimensional subcomplex given that we have a $k-1$-dimensional sphere which separates the vertices of the complex. We also restrict ourselves to $k = 2, 3$, two and three dimensions.

3.1 Definition of a Sufficient Half-Simplex and Related Bounds

Let SP be a $k-1$-dimensional sphere which intersects some of the cells of K. We assume that P does not contain any vertices from ***K*.** We next describe a 2-dimensional simplicial complex of K derived from SP which partitions the cells of K and the vertices of K.

A **half-k-simplex** in $\Re^d$ is the intersection of a closed d-ball (closed complement of a d-ball) and a k-simplex, the ball may be a half-space. A half-simplex is **trivial** if it empty or the whole simplex. The **complement** of a half-simplex is the closure of its set-theoretic complement. A half-simplex is **major** if its volume is $\geq$ its complement, otherwise it is **minor.**

Definition 3.1 *A half-k-simplex is* **sufficient** *if it contains a major half-(k − 1)-simplex. The half-(k − 1)-simplex may be a complete simplex.*

Observe that a half-simplex or its complement must be sufficient. It is possible the a half-simplex and it complement may both be sufficient.

We need the following technical lemma which we state separately for 2 and 3 dimensions.

In two dimensions we will use the notion of an 7-split.

Definition 3.2 *We say that S_1 and S_2 is an γ-split of S if*

$$\gamma = \min_{i=1,2} \max_{E \subseteq S} \frac{|E_i|}{|E|} \tag{10}$$

where E_i is the half-edge of E in S_i.

Thus, if S_1 and S_2 is a partition of S into 2 sufficient half-cells then they are a 1/2-split. An **orthogonal projection** is a linear map from @ onto $\Re^2$ in the natural way.

Lemma 3.3 *If a line L a is γ-split of a 2-simplex F then there exists an 'orthogonal projection P such that $|P(F)| \leq |P(F \cap L)| \leq |F \cap L|$*

Proof: Suppose that L is a 7-split of F half-faces F_1 and $F2$. The line L must intersect two of the edges E and E' of F. Assume that the vertex V common to E and E' is in F_1. Since L is a 7-split of F, we know $7 = \max\{|E_1|/|E|, |E'_1|/|E'|\}$. WLOG we may assume that $7 = |E'_1|/|E'|$. If we let P be the orthogonal projection of $\Re^2$ onto a line normal to E then $|P(F \cap L)| = \gamma \cdot |P(E')| \geq |P(F)|$. □

Lemma 3.4 *Suppose that F is a 2-simplex, and F_1 and F_2 is a division of F into two sufficient half-tsimplexes then the length 1 in the intersection of F_1 and F_2 satisfies the inequality $\frac{|F|}{4 \cdot Dia(F)} \leq l$.*

Proof: Let F be a 2-simplex and C be a circle which intersects the boundary of F. Since F can intersect the boundary of F in at most *6* points $F \cap C$ will consist of $t \leq 3$ curved segments. We replace the t curved segments with t nonintersecting straight line segments $L_1, \ldots, L_t$ without increasing the length of the intersection. We view each of the line segments as splitting F. Thus we have γ_i-splits for $1 \leq i \leq t$. Since the half-face exterior to C is sufficient some half-edge E is major. Let $L_1, L_{t'}$ be the line segments common to E, $t' = 1, 2$. It follows that $\gamma_1 + \gamma_{t'} \geq 1/2$. Thus by Lemma 3.3 there exists orthogonal projections P_1 and P_2 such that $\gamma_1 \cdot |P_1(F)| + \gamma_{t'} \cdot |P_{t'}(F)| \leq |L_1| + |L_{t'}|$ I $|L \cap F|$. Since $2|P(F)| \cdot Dia(F) \geq |F|$. We have

$$\frac{|F|}{4Dia(F)} \leq \frac{|F|}{2Dia(F)}(\gamma_1 + \gamma_{t'}) \leq |L \cap F| \tag{11}$$

□

The similar bound in 3 dimensions is more complicated. The result follows using the theory of isoperimetric inequalities and was proved with the help of Fred Almgren.

Lemma 3.5 *Suppose that S is a 3-simplex, and S_1 and S_2 is a division of S into two sufficient half-cells then there exists a constant τ, independent of a, such that the area A in the intersection of S_1 and S_2 satisfies the inequality*

$$\frac{\tau \cdot |S|}{Dia(S)} \leq A. \tag{12}$$

We start by proving a special case of Lemma 3.5 where S is the symmetric 3-simplex, d = $Dia(S)$, and $Cut \subset S$ is a 2-dimensional incision of S that separates, B, the boundary of S into two parts B_1 and B_2. We let $b = \min\{|B_1|, |B_2|\}$ and $\gamma = b/|B|$.

Lemma 3.6 *The incision Cut satisfies $|Cut| \geq \tau_1 b$ for some fixed constant τ_1. If S has unit volume instead of diameter d then $|Cut| \geq (3^{4/3}/\pi)\gamma$.*

Proof: Suppose S and Cut are as above. We assume that the barycentre, see [Gib77] lies at the origin. Let S_δ for $0 < \delta < 1$ be the simplex obtained from S by multiplying every element in S

by 6. Consider the following integral

$$\int_{x \in S_\delta} \int_{y \in Cut} \frac{d(x,z)}{d(x,y)} \theta(x,z)(dy)^2(dx)^3 \geq |S_\delta| \cdot b \tag{13}$$

where $d(x,y)$ is the Euclidean distance from x to y. The point z is on the boundary of S obtain by projecting a ray R starting from x through y, for $x \neq y$. $\theta(x,z)$ is the cotangent of the angle that R makes with the boundary of S, for z not in a subsimplex of S. We substitute d for $d(x,z)$ since it bounded by the diameter of S. Observe that $\theta(x,z) \leq 1/\sqrt{24}(1-6)$. Reversing the order of integration and making the above substitutions we get:

$$\int_{y \in Cut} \frac{d}{\sqrt{24}(1-\delta)} \int_{x \in S_\delta} \frac{1}{d(x,y)}(dx)^3(dy)^2 \geq |S_\delta| \cdot b \tag{14}$$

It is not hard to see that the integral over $x \in S_\delta$ is a maximum when y is at the origin. Thus, by a straightforward calculation this integral is at most $\pi d^2 \delta^2$. The volume $|S_\delta| = d^3\delta^3/\sqrt{2}$. Making these substitutions and simplifying we get:

$$|Cut| \geq \frac{\sqrt{12}\delta(1-\delta)}{\pi} b. \tag{15}$$

Setting $\delta = 1/2$ we get $|Cut| \geq \frac{\sqrt{3}}{2\pi} b$. Thus, τ_1 is at least $\frac{\sqrt{3}}{2\pi}$. If S has unit volume then the surface area of S is $2 \cdot 3^{5/6}$. Combining these two facts gives the second part of the lemma. □

To prove Lemma 3.5 let S be a 3-simplex satisfying the hypothesis of Lemma 3.5 and S' the symmetric 3-simplex of unit volume. We may pick a linear transformation T which maps S' onto S such that the largest eigenvalue λ_1 of T is the diameter d of S. Let *Cut* be the preimage of $S_1 \cap S_2$ in S', i.e., $Cut = T^{-1}(S_1 \cap S_2)$. Since S_1 and S_2 are both sufficient the constant 7 for the *Cut* must be $7 \geq 1/8$. Thus $|Cut| \geq \tau_2/8$. The area

$$\begin{aligned} |T(Cut)| &\geq \frac{det(T)}{\lambda_1}|Cut| = \frac{|S|}{Dia(S)}|Cut| \\ &\geq \frac{|S|}{Dia(S)} \cdot \frac{\tau_2}{8}. \end{aligned} \tag{16}$$

We need one further estimate of the size of a cut. We say an edge E in a 3-complex K is **exposed** with respect to a sphere SP if:

1. E is internal to K.
2. E and SP are disjoint.
3. The complement of every half-cell containing E is sufficient.

Lemma 3.7 *If E is an exposed edge with respect to a sphere SP then there exists a 3-simplex S containing E such that* $\rho|E| \cdot |S|/Dia(S) < |SP \cap S|$ *for some constant p.*

Proof: Let E be an exposed edge of length e. There are two cases depending on whether or not there exists a 3-simplex S containing E and a point x on E such that the distance from x to $S \cap SP$ is at least $e/2$. Suppose x and S are such a point and 3-simplex. Consider the following integral similar to the one in the proof of Lemma 3.6:

$$\int_{y \in S \cap SP} d(x,z)d(x,y)\theta(x,z)(dy)^2 \geq b \tag{17}$$

where b is the surface area of S exterior to SP. By our hypothesis we know the $d(x,y) \geq e/2$. Further, $d(x,z) \leq dia(S)$. It is not too hard to see that $\theta(x,z)/b \leq dia(S)/|S|$. Therefore, $|S \cap SP| \geq e \cdot |S|/Dia(S)^2$. The case where S does not exist is slightly messier and will be handled in the full paper.

3.2 Picking a Linear Approximation of the Sphere

In this subsection we define our linear approximation of the sphere. We first pick a collection of half-cells, $H-C$. From the half-simplexes we shall obtain the 2-dimensional subcomplex that will be our separator.

For each cell C that intersects SP we pick the half-cell of C that is not sufficient, if it exists. Otherwise, we pick the half-cell with the fewer vertices, if it exists. Otherwise, we arbitrarily pick one of the half-cells. Let $H-C$ be this set of nontrivial half-cells. Thus, our linear approximation will simply be the "pull back" in the direction of the chosen half-simplexes.

The separator is the following subcomplex. Let K' consist of all simplexes in K that are contained in the union of half-3-simplexs of H−C. The vertex separator of K is the vertices V' of K'. To prove that V' is a small separator we need to examine K' more closely.

Claim: Every interior 2-simplex of K that intersects SP has two half-simplexes in H − C, either two copies of the same half-simplex or a half-simplex and its complement. Since exterior simplexes are common to only one 3-simplex of K, they have exactly one half-2-simplex in H − C.

Lemma 3.8 ***Every edge E that intersects SP has at least one vertex in V'.***

Proof: The 1-simplex E is contained in a 3-simplex S that intersects SP. The vertices of E belongs to different half-3-simplexes of S. Since one of these half-3-simplexes is in H $-C$ one of the vertices of E belong to H $-$ C and thus to K'. □

We next bound the number of vertices in K' as a function of the number of faces in K'. The vertices of K' fall into three types. Each **Type 2** vertex belongs to some face of K'. While each **type 1** vertex does not belong to any face of K' but it does belong to some edge of K'. Finally, each **type 0** vertex is isolated; it belongs to no face or edge of K'.

Observe that if there is a type 0 vertex v then it is either on the boundary of K or else K' consists of just v. If v is of type 1 then it must belong to an exposed edge. Thus, the number of type 1 vertices is at most twice the number of exposed edges in K'. Similarly the number of type **2** edges is at most three times the number of faces of K'. This discussion proves the following lemma.

Lemma 3.9 ***The number of vertices in K' is at most $2e_1+3f+\bar{v}$, where e_1 is the number of exposed edges in K', f is the number of faces in K', and $\bar{v}$ is the number of exterior vertices of h'.***

3.3 Bounding the Number of Vertices in the Separator

Let K, K' be as in Section 3.2. We define a function $f(x)$ on the underlying space of K as follows: If x is a point on the interior of a 3-simplex S of K, then set $f(x) = (1/|S|)^{1/3}$. If x is not in the underlying space of K, set $f(x) = 0$. Otherwise, set $f(x)$ equal to some arbitrarily value.

We define the cost of a 3-simplex S, with respect to f, to be

$$\int_{x\in S} f^3(x)(dx)^3. \qquad (18)$$

Thus, the cost of any 3-simplex in K is 1 and the total cost of K is the number of 3-simplexes.

Note that the opposite of each half-3-simplex in H $-$ C is a sufficient half-3-simplex. But the half-simplex in H $-$ C need not be sufficient. We next show that, if a sphere divides a 3-simplex with bounded aspect-ratio into two sufficient half-simplexes, then the boundary between the two half-simplexes is bounded from below.

Lemma 3.10 ***Suppose that S is a cell with aspect-ratio a and S_1 and S_2 is a division of S into two suficient half-cells then the cost in the intersection of S_1 and S_2 is at least τ/α.***

We show that Lemma 3.10 follows from Lemma 3.5.

Proof: Let A be as in Lemma 3.5, and d = $Dia(S)$ and $v = |S|$. Let f be the density in S, i.e., $vf^3 = 1$. We must bound Af^2. First we bound df. Using the fact that S has an aspect-ratio at most a, $d \leq \alpha\sqrt[3]{v}$. Thus,

$$df \leq \alpha\sqrt[3]{vf^3} \leq \alpha \qquad (19)$$

Using the inequality $(\tau v)/d \leq$ A from Lemma 3.5 we get:

$$Af^2 \geq \frac{\tau v f^2}{d} = \frac{\tau v f^3}{df} = \frac{\tau}{df} \geq \frac{\tau}{a}. \qquad (20)$$

□

Lemma 3.11 ***Let F be a 2-simplex in K' where S_1 and S_2 are the two 3-simplexes in K containing F, then either S_1 or S_2 is split into two sufficient half-8-simplexes by SP.***

Proof: The sphere SP splits F into two half-2-simplexes F_1 and F_2. Since both F_1 and F_2 belong to $H-C$, they must each belong to sufficient half-3-simplex, say, F_1 belongs to one in S_1 and F_2 belongs to one in S_2. On the other hand, either F_1 or F_2 is a major half-2-simplex. Suppose F_1 is major. Thus the half-simplex of S_2 containing F_1 is also sufficient. Therefore both half-3-simplexes of S_2 are sufficient. □

Lemma 3.12 *If E is an exposed edge in K', then there exists a cell C containing E such that the cost of $C \cap SP$ is at least $\rho/(\alpha)^4$.*

Proof: The proof is very similar to that of Lemma 3.10, except here we use Lemma 3.7.

Theorem 3.13 *The number of vertices in K' is at most $(\frac{2\alpha^4}{\rho} + \frac{12\alpha}{\tau})m + \bar{v}$ where m is the cost of SP.*

Proof: By Lemma 3.9 the number of vertices in K' is at most $2e_1 + 3f + \bar{v}$ where e_1 is the number of exposed edges in K', f is the number of faces in K', and $\bar{v}$ is the number of exterior vertices of K. By Lemma 3.12 $e_1 \leq \frac{m\alpha^4}{\rho}$. By Lemma 3.11 each face in K' is contained in a cell whose intersection with SP is at least τ/α, Lemma 3.5. But, each cell in K can be common to at most 4 faces in K'. Therefore $f \leq \frac{4m\alpha}{\tau}$. Substituting these inequalities into $2e_1 + 3f + \bar{v}$ proves the theorem. □

Theorem 1.5 now follows from Theorems 2.3 and 3.13.

4 Finding a Family of Separators

In this section we show how to find a family of separators. We must show that small separators exist in the subcomplexes that have been created by the removal of prior separators. The problem is that the separator constructed in Theorem 1.5 grows as a function of the boundary. We show that the algorithm works unchanged. But we modify the analysis. We assume that K is a complex in $\Re^3$ and K has at most $O(c^{2/3})$ external vertices. We include in the root separator all of the external vertices of K. Let K' be a subcomplex of K which we want to separate. Let $\bar{K}'$ be the subcomplex of K consisting of simplexes in K' plus all 3-simplexes in K that contain a vertex in K'. We also include in $\bar{K}'$ all subsimplexes so that $\bar{K}'$ is a simplicial complex. We bound the number of 3-simplexes in $\bar{K}'$ in terms of the number of vertices in K'.

Lemma 4.1 *If K is an α-stable complex in $\Re^3$ then the number of 3-simplexes common to a vertex is at most $\sigma\alpha^3$ where σ is some fixed constant.*

We now apply Theorem 1.5 to separate the vertices of K' but use the 3-simplexes of $\bar{K}'$ to determine the cost function. Observe that Theorem 1.5 can be strengthened to separate any subset of the vertices, i.e., it will separate a weighted combination of the vertices.

5 Finding a Centerpoint

Finding centerpoints is the only nontrivial computational step in the construction of the separator given in Theorem 1.5. Theorem 1.5 finds a centerpoint in $\Re^4$ where all the points lay on the unit 3-sphere. We know of no randomized or deterministic algorithm which finds these centerpoints in less than $O(n^4)$ time. On the other hand, there are randomized algorithms which find points p such that any hyperplane not containing p contains at most $(4/5 + \epsilon)n$ points. We call these ecenterpoints. ϵ-centerpoints in $\Re^4$ can be found in constant time where the probability of error is at most 1/2, [VC71, HW87, MT90]. RNC algorithms exist that always return an ϵ-centerpoint, [MT90].

6 A 3-complex with only Large Separators

We exhibit a complex in $\Re^3$ with $O(n \log n)$ simplexes such that any separator has size $\Omega(n)$. The idea is to embed the Cube-Connected-Cycles graph in the 1-skeleton. The vertices will be integer points in $\Re^3$ and $n = 2^k$. The vertices are triples (x, y, z) such that $0 \leq z \leq k$, $0 \leq y \leq 2^z$, and $0 \leq x \leq 2^{k-z}$. We view vertices with the same z value as on a level. We connect the vertices on a level together to form a mesh. Between consecutive levels we connect as follows: vertex $(2p, q, z)$ is connected to $(p, 2q, z+1)$ and vertex $(2p+1, q, z)$ is connected to $(p, 2q+1, z+1)$. Thus, the four corners of two adjacent squares at level z are connected to four corners of two adjacent squares at level $z+1$. As it is now defined this is not the 1-skeleton of a 3-complex. But, we can refine it until we have a 3-complex. Recall that the CCC has as its vertex set all pairs $(a_0 \cdots a_{n-1}, i)$ where a_1 to a_n are zero or one and $0 \leq i < n$. We connect $(a_0 \cdots a_{n-1}, i)$ to $(a_0 \cdots a_{n-1}, i \pm 1)$ and to $(a_0 \cdots \bar{a}_i \cdots a_{n-1}, i)$ where $\bar{a}$ the complement of a. We view the pair of integers (x, y) as the binary string $x_{k-z} \cdots x_1 y_1 \cdots y_z$ where $x_1 \cdots x_{k-z}$ and $y_1 \cdots y_z$ binary representation of x and y respectively. Thus, changing level corresponds to changing the value of i in the CCC. Therefore any separator of the complex will be a separator of the CCC graph. But it is known that any separator of the CCC has size $\Omega(n)$, [Tho80b].

7 Solving Linear Systems use Parallel Nested Dissection

In this section we show that the fill-in when we apply Nested Dissection is not too large. One could possibly use Theorem 6 from [LRT79]. We can analyze our separators directly. The main advantages of a direct analysis will be: (1) the constants will be smaller, (2) we can apply our results to finite element graphs and there generalization to $\Re^3$, and (3) we get faster parallel algorithms.

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