Chapter 11

Laplace Transforms

There are different types of transforms

Back in Chapter 6 we covered a type of generating function called the **z-transform**.

The z-transform is particularly well suited to discrete, integer-valued random variables.

In this chapter we introduce a new generating function called the **Laplace transform**, which is well suited to common continuous random variables.

The structure of this chapter will closely mimic that of Chapter 6.

Motivation

Let $X \sim Exp(\lambda)$

What is $\boldsymbol{E}[X^3]$?

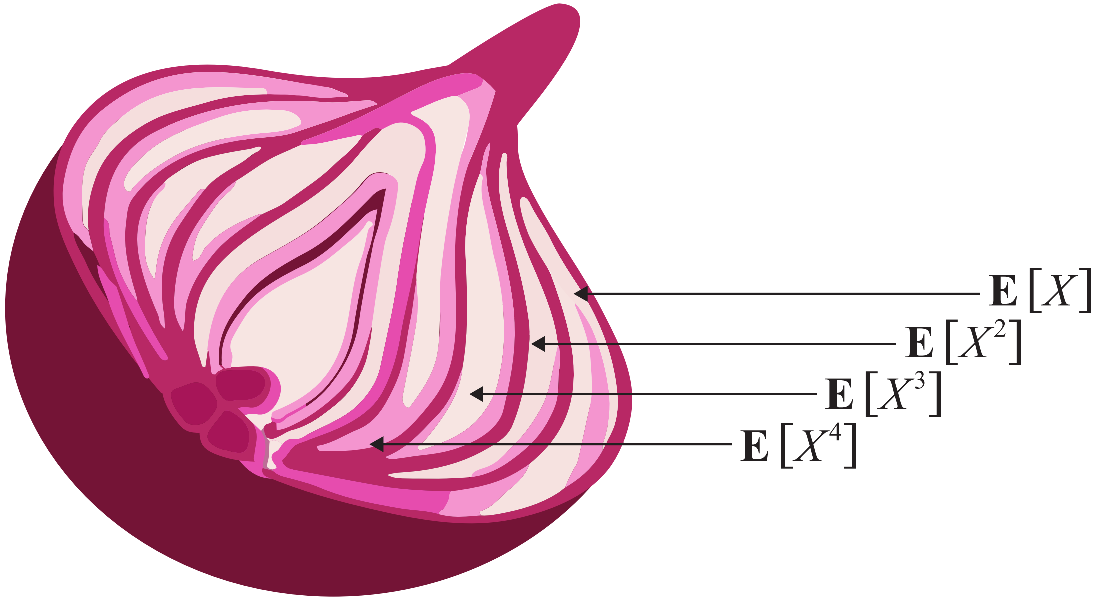
$$\boldsymbol{E}[X^3] = \int_0^\infty t^3 \cdot \lambda e^{-\lambda t} dt$$

Seems complicated to evaluate!

The Laplace transform will make this very easy!

The Laplace transform as an onion

Onion represents Laplace transform of r.v. X



Lower moments are in the outer layers → less effort/tears
Higher moments are deeper inside → more effort/tears

Laplace transform of continuous r.v.

Defn: Let X be a non−negative continuous r.v. with p.d.f. $f_X(t)$.
Then the **Laplace transform** of X is

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}] = \int_0^\infty e^{-st} f_X(t)dt$$

Assume s is a constant where $s \geq 0$.

Note: The Laplace transform can be defined for any r.v., or even for just a function $f(t)$, where $t \geq 0$. However convergence is only guaranteed when X is a non-negative r.v. and $s \geq 0$.

Pop Quiz

<u>Defn</u>: Let X be a non−negative continuous r.v. with p.d.f. $f_X(t)$.
Then the **Laplace transform** of X is

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}] = \int_0^\infty e^{-st} f_X(t)dt$$

Assume s is a constant where $s \geq 0$.

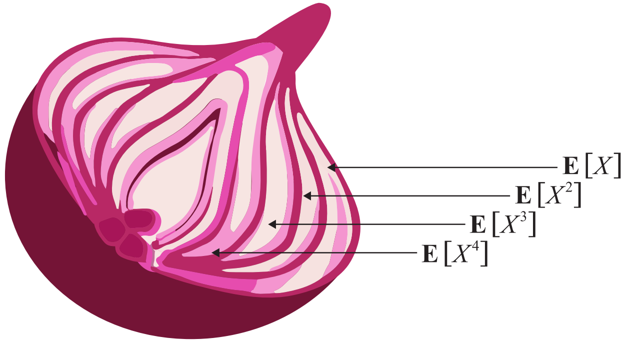
Q: What is $\widetilde{X}(0)$?

A: $\widetilde{X}(0) = \boldsymbol{E}[e^{-0 \cdot X}] = 1$

Example of Onion Building

$X \sim Exp(\lambda)$

Create the onion!



$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}]$$

$$= \int_0^\infty e^{-st} f_X(t)dt$$

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}] = \int_0^\infty e^{-st}\lambda e^{-\lambda t}dt$$

$$= \lambda \int_0^\infty e^{-(s+\lambda)t}dt$$

$$= \frac{\lambda}{s+\lambda}$$

Example of Onion Building

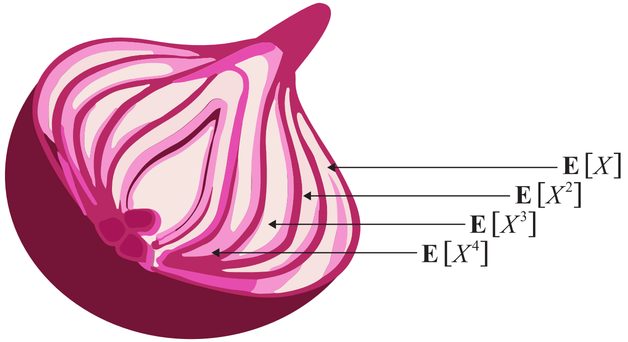
$X = 3$

Create the onion!

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}]$$

$$= \boldsymbol{E}[e^{-3s}]$$

$$= e^{-3s}$$



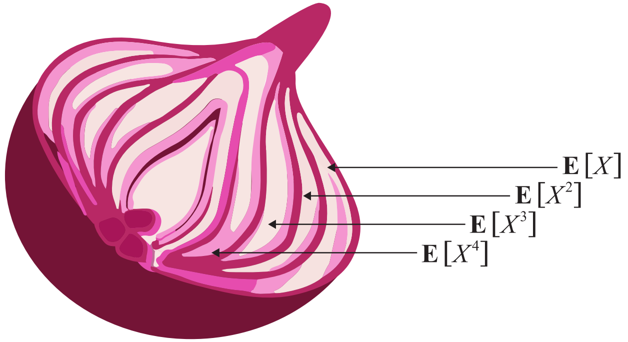
$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}]$$

$$= \int_0^\infty e^{-st} f_X(t) dt$$

Example of Onion Building

$X \sim Uniform(a, b)$, where $a, b \geq 0$

Create the onion!



$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}] = \int_a^b e^{-st} \cdot \frac{1}{b-a} dt$$

$$= \frac{1}{b-a} \cdot \frac{1}{s} \cdot \left(e^{-sa} - e^{-sb}\right)$$

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}]$$

$$= \int_0^\infty e^{-st} f_X(t) dt$$

Convergence of Laplace transform

Theorem 11.7: $\tilde{X}(s)$ is bounded for any non−negative continuous r.v. X, assuming $s \geq 0$.

Proof:

$$e^{-t} \leq 1, \qquad \forall t \geq 0$$

$$\Rightarrow (e^{-t})^{s} \leq 1, \qquad \forall s \geq 0$$

$$\Rightarrow e^{-st} \leq 1, \qquad \forall t, s \geq 0$$

$$\Rightarrow \tilde{X}(s) = \int_0^\infty e^{-st} \, f_X(t)dt \leq \int_0^\infty 1 \cdot f_X(t)dt = 1$$

Getting moments: Onion peeling

Theorem 11.8: (Onion Peeling) Let X be a non−negative, continuous r.v. with p.d.f. $f_X(t),\ t \geq 0$. Then,

$$\tilde{X}'(s)\Big|_{s=0} = -\mathbf{E}[X]$$

$$\tilde{X}''(s)\Big|_{s=0} = \mathbf{E}[X^2]$$

$$\tilde{X}'''(s)\Big|_{s=0} = -\mathbf{E}[X^3]$$

$$\tilde{X}''''(s)\Big|_{s=0} = \mathbf{E}[X^4]$$

If can't evaluate at $s = 0$, instead consider limit as $s \to 0$ (use L'Hospital's Rule).

Proof of onion peeling theorem

$$e^{-st} = 1 - (st) + \frac{(st)^2}{2!} - \frac{(st)^3}{3!} + \frac{(st)^4}{4!} - \cdots$$

(Taylor Series Expansion)

$$e^{-st} f(t) = f(t) - (st) f(t) + \frac{(st)^2}{2!} f(t) - \frac{(st)^3}{3!} f(t) + \frac{(st)^4}{4!} f(t) - \cdots$$

$$\int_0^\infty e^{-st} f(t) dt = \int_0^\infty f(t) dt - \int_0^\infty (st) f(t) dt + \int_0^\infty \frac{(st)^2}{2!} f(t) dt - \int_0^\infty \frac{(st)^3}{3!} f(t) dt + \cdots$$

$$\tilde{X}(s) = 1 - s\,\boldsymbol{E}[X] + \frac{s^2}{2!}\boldsymbol{E}[X^2] - \frac{s^3}{3!}\boldsymbol{E}[X^3] + \frac{s^4}{4!}\boldsymbol{E}[X^4] - \frac{s^5}{5!}\boldsymbol{E}[X^5] + \cdots$$

Proof of onion peeling theorem

$$\tilde{X}(s) = 1 - s\,\mathbf{E}[X] + \frac{s^2}{2!}\mathbf{E}[X^2] - \frac{s^3}{3!}\mathbf{E}[X^3] + \frac{s^4}{4!}\mathbf{E}[X^4] - \frac{s^5}{5!}\mathbf{E}[X^5] + \frac{s^6}{6!}\mathbf{E}[X^6] \cdots$$

$$\tilde{X}'(s) = -\mathbf{E}[X] + s\mathbf{E}[X^2] - \frac{s^2}{2!}\mathbf{E}[X^3] + \frac{s^3}{3!}\mathbf{E}[X^4] - \frac{s^4}{4!}\mathbf{E}[X^5] + \frac{s^5}{5!}\mathbf{E}[X^6] \cdots$$

$$\tilde{X}'(0) = -\mathbf{E}[X]$$ ✓

$$\tilde{X}''(s) = \mathbf{E}[X^2] - s\mathbf{E}[X^3] + \frac{s^2}{2!}\mathbf{E}[X^4] - \frac{s^3}{3!}\mathbf{E}[X^5] + \frac{s^4}{4!}\mathbf{E}[X^6] \cdots$$

$$\tilde{X}''(0) = \mathbf{E}[X^2]$$ ✓

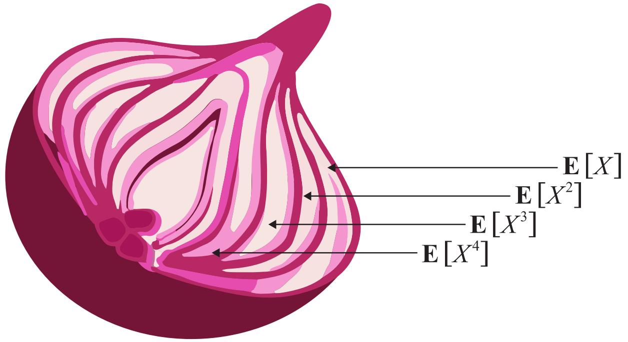
$$\tilde{X}'''(s) = -\mathbf{E}[X^3] + s\mathbf{E}[X^4] - \frac{s^2}{2!}\mathbf{E}[X^5] + \frac{s^3}{3!}\mathbf{E}[X^6] \cdots$$

$$\tilde{X}'''(0) = -\mathbf{E}[X^3]$$ ✓

Example of onion peeling

$X \sim Exp(\lambda)$

$$\tilde{X}(s) = \frac{\lambda}{\lambda + s} = \lambda(\lambda + s)^{-1}$$



Q: Peel the onion to get $\boldsymbol{E}[X], \boldsymbol{E}[X^2], \boldsymbol{E}[X^3], \boldsymbol{E}[X^4]$,…

$$\tilde{X}'(s) = -\lambda(\lambda + s)^{-2} \quad \Longrightarrow \boldsymbol{E}[X] = \frac{1}{\lambda}$$

$$\tilde{X}''(s) = 2\lambda(\lambda + s)^{-3} \quad \Longrightarrow \boldsymbol{E}[X^2] = \frac{2}{\lambda^2}$$

$$\tilde{X}'''(s) = -3!\,\lambda(\lambda + s)^{-4} \quad \Longrightarrow \boldsymbol{E}[X^3] = \frac{3!}{\lambda^3}$$

$$\Longrightarrow \boldsymbol{E}[X^k] = \frac{k!}{\lambda^k}$$

Linearity of Transforms

Theorem 11.10: (Linearity) Let X and Y be independent continuous r.v.s.
Let

$$Z = X + Y$$

Then the Laplace transform of Z is:

$$\tilde{Z}(s) = \tilde{X}(s) \cdot \tilde{Y}(s)$$

Proof:

$$\begin{aligned}\tilde{Z}(s) = \boldsymbol{E}[e^{-sZ}] &= \boldsymbol{E}[e^{-s(X+Y)}] \\ &= \boldsymbol{E}[e^{-sX} \cdot e^{-sY}] \\ &= \boldsymbol{E}[e^{-sX}] \cdot \boldsymbol{E}[e^{-sY}] \\ &= \tilde{X}(s) \cdot \tilde{Y}(s)\end{aligned}$$

Conditioning with Transforms

Theorem 11.11: Let X, A, and B be continuous r.v.s. where

$$X = \begin{cases} A & \text{w.p.} \quad p \\ B & \text{w.p.} \quad 1-p \end{cases}$$

Then,

$$\tilde{X}(s) = p \cdot \tilde{A}(s) + (1-p) \cdot \tilde{B}(s)$$

Proof:

$$\tilde{X}(s) = \boldsymbol{E}[e^{-sX}]$$

$$= \boldsymbol{E}[e^{-sX} | X = A] \cdot p + \boldsymbol{E}[e^{-sX} | X = B] \cdot (1-p)$$

$$= \boldsymbol{E}[e^{-sA}] \cdot p + \boldsymbol{E}[e^{-sB}] \cdot (1-p)$$

$$= p \cdot \tilde{A}(s) + (1-p) \cdot \tilde{B}(s)$$

Conditioning

Theorem 11.12:
Let Y be a continuous r.v. and let X_Y be a continuous r.v. that dependes on Y.
Let $f_Y(y)$ denote the p.d.f. of Y.
Then:

$$\widetilde{X_Y}(s) = \int_{y=0}^{\infty} \widetilde{X_y}(s) \cdot f_Y(y) dy$$

Proof:

$$\widetilde{X_Y}(s) = \boldsymbol{E}[e^{-sX_Y}] = \int_{y=0}^{y=\infty} \boldsymbol{E}[e^{-sX_Y} | Y = y] \cdot f_Y(y) dy$$

$$= \int_{y=0}^{y=\infty} \boldsymbol{E}\left[e^{-sX_y}\right] \cdot f_Y(y) dy$$

$$= \int_{y=0}^{\infty} \widetilde{X_y}(s) \cdot f_Y(y) dy$$