# Chapter 12
# The Poisson Process

# Where we're heading …

Goal of the next 3 chapters:

**How to model and simulate computer systems**

**Chapter 12:** The Poisson Process – how to model an arrival process (jobs arriving to a data center, requests arriving to a web server, etc.)

**Chapter 13:** Generating r.v.s for simulation – methods for generating instances of different distributions

**Chapter 14:** Event-driven simulation – simulating systems with queues and networks of queues

# Chapter 12 outline

The Poisson process is intimately related to the Exponential distribution.
Before discussing the Poisson process, we must revisit the Exponential.

**STEP 1:** Revisiting the Exponential distribution

- ❑ Relating the Exponential distribution to the Geometric
- ❑ More properties of the Exponential distribution

**STEP 2:** The celebrated Poisson Process

- ❑ First definition vs. Second definition
- ❑ Properties of the Poisson Process

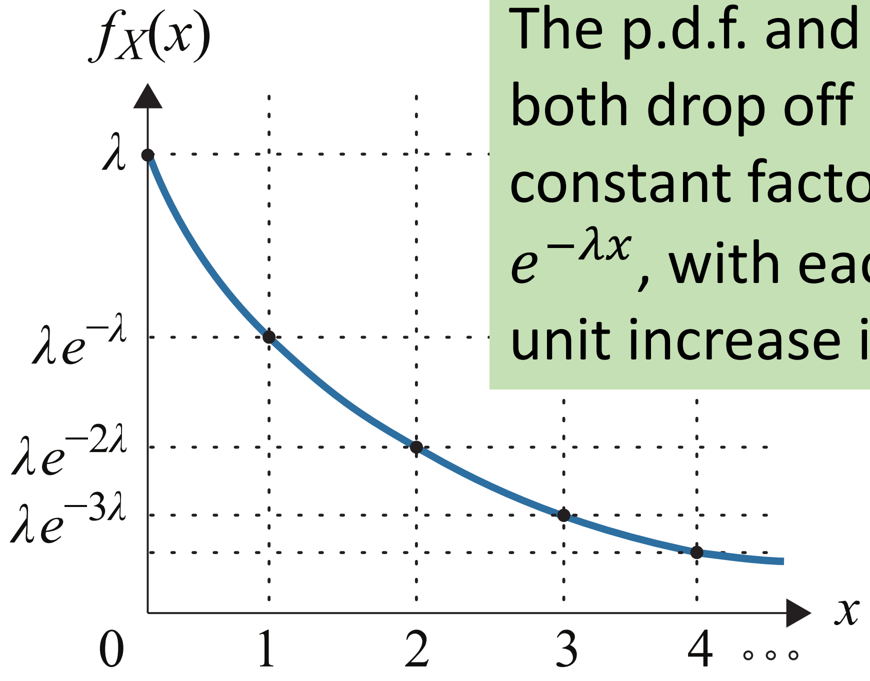
# Review of the Exponential distribution

<u>Defn</u>: A r.v. is **Exponentially distributed** with rate $\lambda$, written $\boldsymbol{X \sim Exp(\lambda)}$, if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = 1 - e^{-\lambda x} \quad \text{if } x \geq 0$$

$$\overline{F_X}(x) = e^{-\lambda x} \quad \text{if } x \geq 0$$




The p.d.f. and tail both drop off by a constant factor, $e^{-\lambda x}$, with each unit increase in $x$.

# Review of the Exponential distribution

$\underline{X \sim Exp(\lambda)}$

$$\boldsymbol{E}[X] = \frac{1}{\lambda} \qquad \boldsymbol{Var}(X) = \frac{1}{\lambda^2} \qquad C_X^2 = \frac{\boldsymbol{Var}(X)}{\boldsymbol{E}[X]^2} = 1$$

- Memoryless Property

$$\mathbf{P}\{X > s + t \mid X > s\} = \boldsymbol{P}\{X > t\}$$

- Constant Failure Rate

$$\mathbf{P}\{X \in (t, t+\delta) \mid X > t\} = \frac{f_X(t) \cdot \delta}{\overline{F_X}(t)} = \lambda\delta$$

Failure rate

Constant indpt of $t$

# Relating Exponential distribution to Geometric

The Exponential and Geometric are the only memoryless distributions. How are they related?

Geometric($p$) is the **number** of flips until see first head.


How to relate these?

Exp($\lambda$) is the **time** until success.

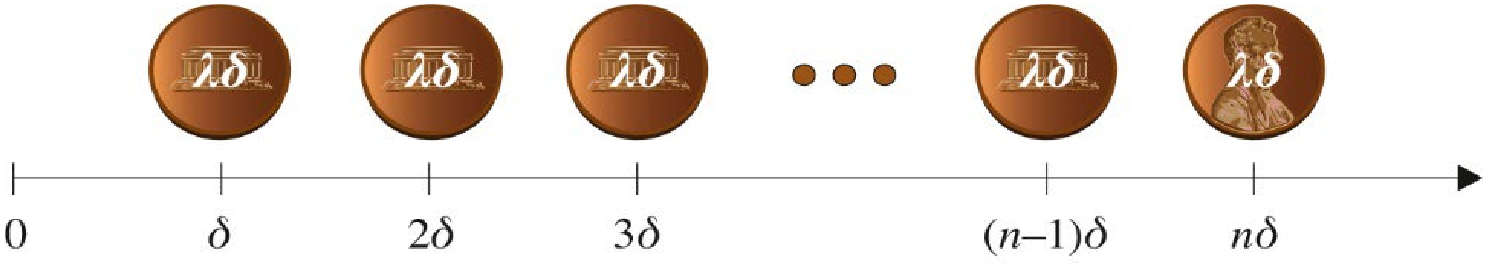
Range is 0 to ∞.

discrete

continuous

# Relating Exponential distribution to Geometric

- Flip coin every $\delta$–step, where $\delta \to 0$.
- Coin has very small probability, $\lambda\delta$, of heads.



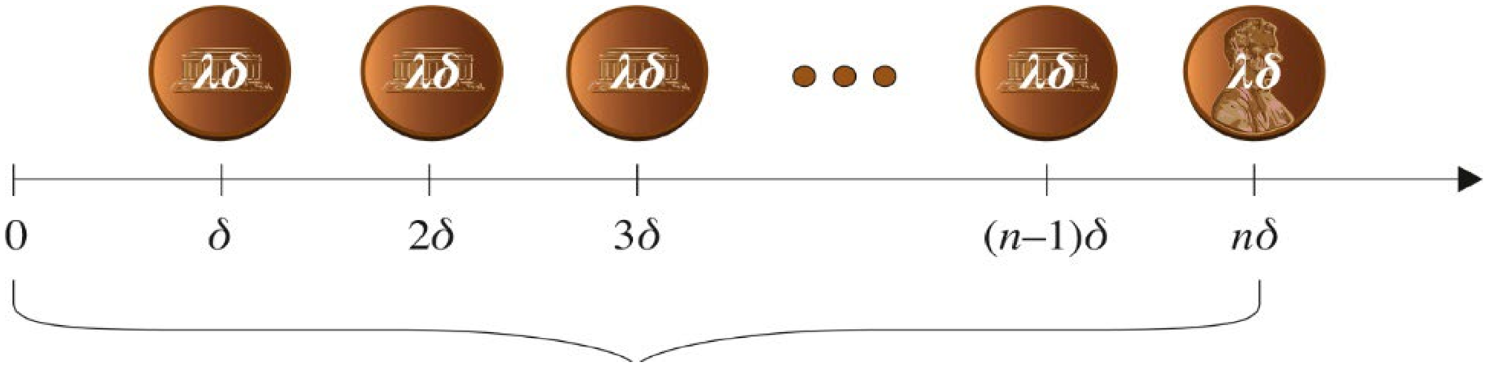

$Y \sim Geom(p = \lambda\delta)$
where flip every $\delta$–step

↑ discrete

$Y^*$ = time associated with $Y$.

↑ continuous

# Relating Exponential distribution to Geometric




Time is $Y^*$

**Claim:** $Y^* \sim Exp(\lambda)$

$Y \sim Geom(p = \lambda\delta)$
where flip every $\delta$–step

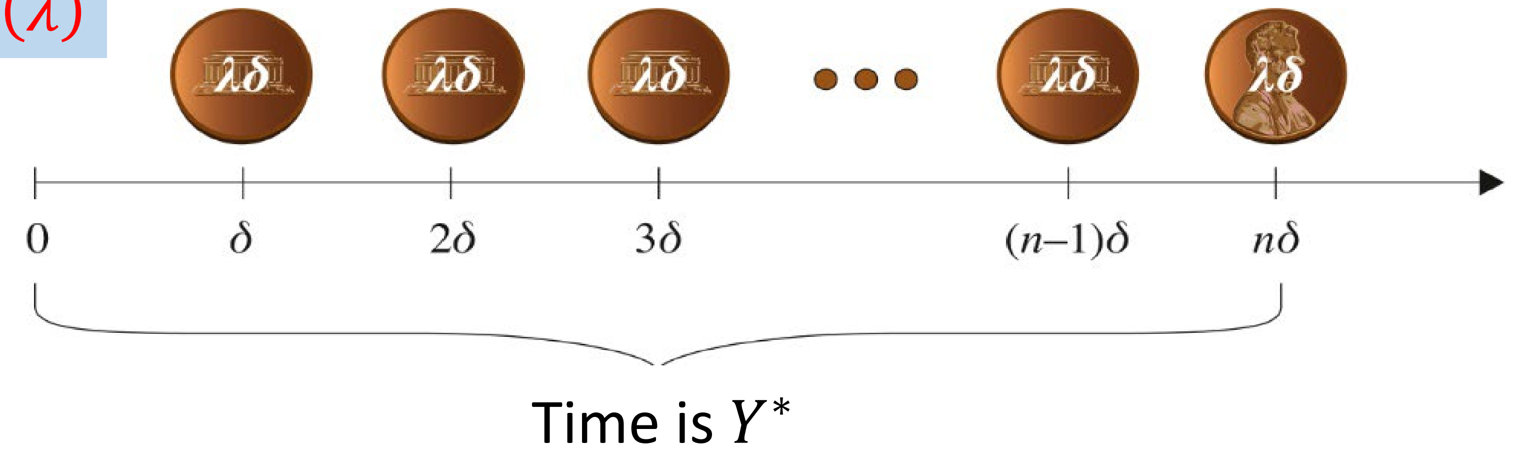
↑ discrete

$Y^*$ = time associated with $Y$.

↑ continuous

# Relating Exponential distribution to Geometric

**Claim:** $Y^* \sim Exp(\lambda)$




**Q:** Does $\boldsymbol{E}[Y^*]$ check out?

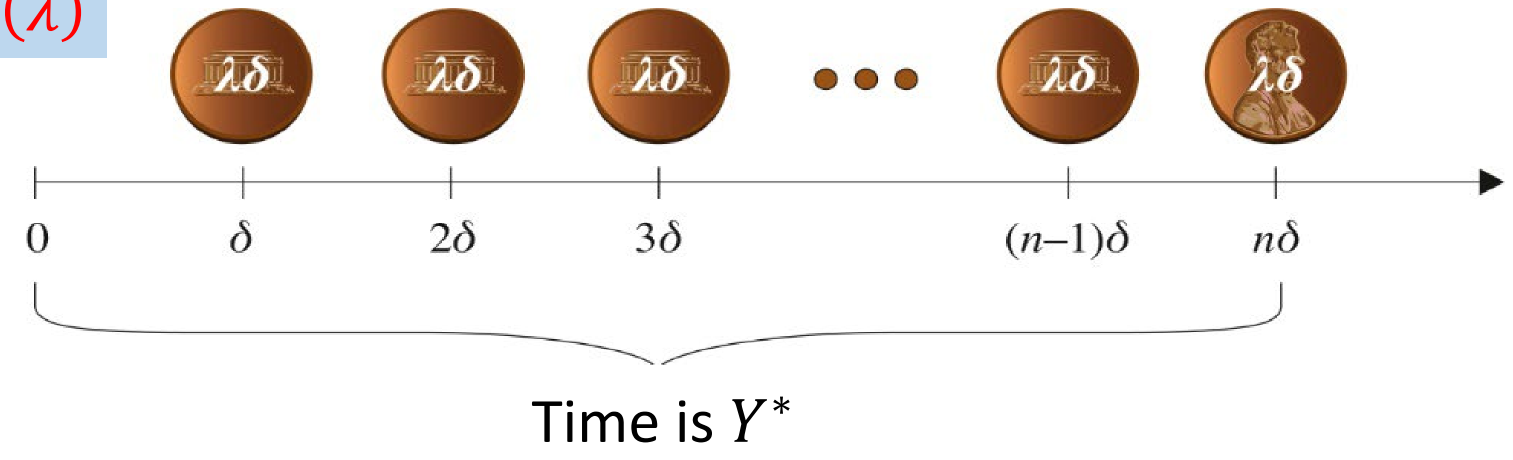
$$\boldsymbol{E}[Y^*] = (avg.\#\ flips\ until\ H) \cdot (time\ per\ flip)$$

$$= \frac{1}{\lambda\delta} \cdot (\delta)$$

$$= \frac{1}{\lambda}$$

# Relating Exponential distribution to Geometric

**Claim:** $Y^* \sim Exp(\lambda)$



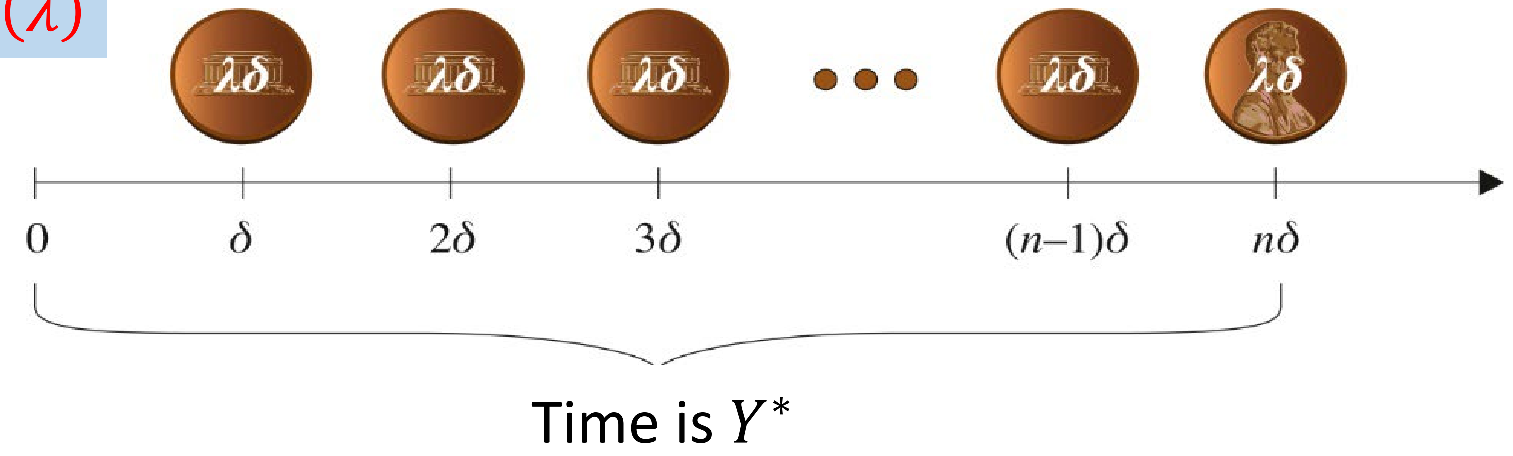

**Q:** What is $\boldsymbol{P}\{Y^* > t\}$ ?

$$\boldsymbol{P}\{Y^* > t\} = \boldsymbol{P}\left\{> \frac{t}{\delta}\ failures\right\} = (1-\lambda\delta)^{\frac{t}{\delta}} = \left(1 - \frac{1}{\frac{1}{\lambda\delta}}\right)^{\frac{t}{\delta}}$$

$$= \left(1 - \frac{1}{\frac{1}{\lambda\delta}}\right)^{\frac{1}{\lambda\delta}\cdot\lambda\cdot t} = (e^{-1})^{\lambda t} = e^{-\lambda t}$$

# Relating Exponential distribution to Geometric

**Claim:** $Y^* \sim Exp(\lambda)$




So $\boldsymbol{P}\{Y^* > t\} = e^{-\lambda t}$ ➡ $Y^* \sim Exp(\lambda)$

**Theorem 12.1:** Let $X \sim Exp(\lambda)$. Then $X$ represents the time to get a H, given we flip every $\delta$-step, and a flip is successful with probability $\lambda\delta$, where $\delta \rightarrow 0$ .

# A new view of asymptotic notation

<u>Defn</u>:

$$f = o(\delta) \quad \text{if} \quad \lim_{\delta \to 0} \frac{f}{\delta} = 0$$

Rather than
$n \to \infty$
we have
$\delta \to 0$

**Q:** Which of these are $o(\delta)$?

a. $f = \delta^3$
b. $f = \delta^2$
c. $f = \delta\sqrt{\delta}$
d. $f = \sqrt{\delta}$

**A:** (a), (b), (c)

# Properties of Exponential

<u>Theorem 12.3</u>: Given $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2,$

$$\boldsymbol{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

**Pf: (traditional algebraic proof)**

$$\boldsymbol{P}\{X_1 < X_2\} = \int_0^\infty \boldsymbol{P}\{X_1 < X_2 \mid X_2 = x\} \cdot f_{X_2}(x)dx$$

$$= \int_0^\infty \boldsymbol{P}\{X_1 < X_2 \mid X_2 = x\} \cdot \lambda_2 e^{-\lambda_2 x} dx$$

$X_1 \perp X_2$

after integration

$$= \int_0^\infty \boldsymbol{P}\{X_1 < x\} \cdot \lambda_2 e^{-\lambda_2 x} dx = \int_0^\infty (1 - e^{-\lambda_1 x}) \cdot \lambda_2 e^{-\lambda_2 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

# Properties of Exponential

Theorem 12.3: Given $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), X_1 \perp X_2$,

$$\boldsymbol{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

**Pf: (δ-step proof)**

- ❑ Success of type 1 occurs with probability $\lambda_1 \delta$ on each $\delta$-step.
- ❑ Success of type 2 occurs with probability $\lambda_2 \delta$ on each $\delta$-step.

**Q:** What is $\boldsymbol{P}\{X_1 < X_2\}$ saying?

**A:** Given that a success of type 1 or type 2 has occurred,
what is the probability that it is a success of type 1?

# Properties of Exponential

Theorem 12.3: Given $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), X_1 \perp X_2,$

$$\boldsymbol{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

**Pf: (δ-step proof)**

$$\boldsymbol{P}\{X_1 < X_2\} = \mathbf{P}\{\text{type 1} \mid \text{type 1 or type 2}\}$$

$$= \frac{\mathbf{P}\{\text{type 1}\}}{\mathbf{P}\{\text{type 1 or 2}\}}$$

$$= \frac{\lambda_1\delta}{\lambda_1\delta + \lambda_2\delta - (\lambda_1\delta)(\lambda_2\delta)}$$

What do we call this product?

$$= \frac{\lambda_1\delta}{\lambda_1\delta + \lambda_2\delta - \mathrm{o}(\delta)}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 - \frac{\mathrm{o}(\delta)}{\delta}}$$

What happens to this as $\delta \to 0$

$$\to \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad \text{as } \delta \to 0$$

# Properties of Exponential

<u>Theorem 12.5</u>: Given $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2$.

Let $X = \min(X_1, X_2)$. Then $X \sim Exp(\lambda_1 + \lambda_2)$.

**Pf: (traditional algebraic proof)**

$$\boldsymbol{P}\{X > t\} = \boldsymbol{P}\{\min(X_1, X_2) > t\}$$

$$= \boldsymbol{P}\{X_1 > t \ \text{ and } \ X_2 > t\}$$

$X_1 \perp X_2$

$$= \boldsymbol{P}\{X_1 > t\} \cdot \boldsymbol{P}\{X_2 > t\}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)} \Longrightarrow X \sim Exp(\lambda_1 + \lambda_2).$$

# Properties of Exponential

<u>Theorem 12.5</u>: Given $X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$, $X_1 \perp X_2$.

Let $X = \min(X_1, X_2)$. Then $X \sim Exp(\lambda_1 + \lambda_2)$.

**Pf: (δ-step proof)**

- A trial occurs every $\delta$–step
- The trial is "successful of type 1" with probability $\lambda_1 \delta$
- The trial is "successful of type 2" with probability $\lambda_2 \delta$
- We are looking for the time until there is a success of either type
- A trial is "successful" (either type) with probability

$$\lambda_1\delta + \lambda_2\delta - (\lambda_1\delta)(\lambda_2\delta) = \delta\left(\lambda_1 + \lambda_2 - \frac{o(\delta)}{\delta}\right)$$

- So time until success $\sim Exp\left(\lambda_1 + \lambda_2 - \frac{o(\delta)}{\delta}\right)$ → $Exp\,(\lambda_1 + \lambda_2)$ as $\delta \to 0$

# Example

Our system has 2 potential points of failure (assume these are independent):

Time to failure is Exponentially-distributed with mean 500

Time to failure is Exponentially-distributed with mean 1000

**Q:** What is the time until there is a system failure of either type?

$$Time \sim Exp\left(\frac{1}{500} + \frac{1}{1000}\right)$$

**Q:** What is $\boldsymbol{P}\{\text{system failure caused by power supply}\}$?

$$\frac{\frac{1}{500}}{\frac{1}{500} + \frac{1}{1000}}$$

# The Poisson Process (P.P.)

The Poisson process is the most widely used model for the arrivals into a system


- P.P. is analytically tractable

- P.P. occurs in nature whenever look at aggregate stream from a large number of independent users (as is typical for mail server, web server, data center, etc.)

# Before we start … recall the Poisson distribution

$$X \sim Poisson(\lambda)$$

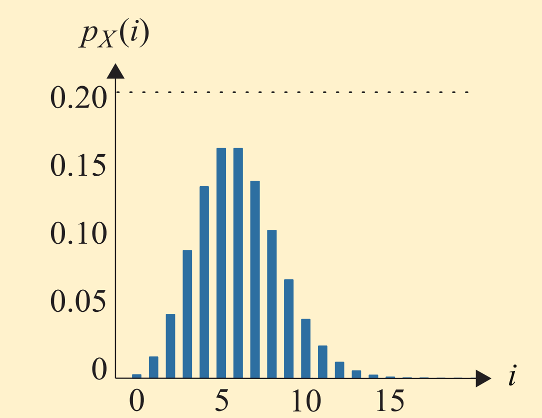
**Q:** What is $p_X(i)$?

$$p_X(i) = \frac{e^{-\lambda} \cdot \lambda^i}{i!} \quad \text{where } i = 0, 1, 2, 3, \ldots$$

**Q:** What is $\boldsymbol{E}[X]$? What is $\boldsymbol{Var}(X)$?

$$\boldsymbol{E}[X] = \boldsymbol{Var}(X) = \lambda$$

**Q:** What is the shape of the Poisson?




Like a Normal,
but starts at 0.

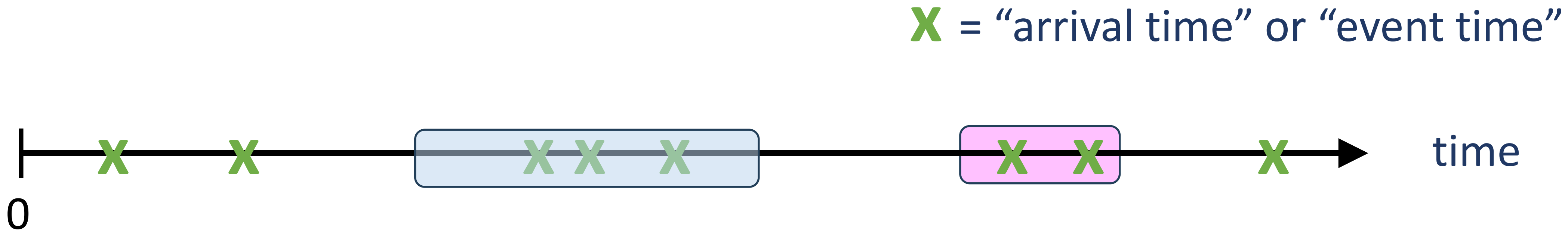
# 2 Properties of the Poisson process

1. Independent Increments

2. Stationary Increments

These two properties define the Poisson Process

# Independent Increments




$N_{blue} = \#\ events\ in\ blue\ interval$

$N_{pink} = \#\ events\ in\ pink\ interval$

If blue and pink intervals are non-overlapping $\Rightarrow\ N_{blue} \perp N_{pink}$

# Independent Increments

X = "arrival time" or "event time"


$N(t) = \#\ events\ occurring\ by\ time\ t$

Defn 12.7: An event sequence has **independent increments** if the numbers of events that occur in disjoint time intervals are independent. Specifically, for all $t_0 < t_1 < t_2 < \cdots < t_n$:

$$N(t_1) - N(t_0) \perp N(t_2) - N(t_1) \perp \ldots \perp N(t_n) - N(t_{n-1})$$

# Independent Increments

**X** = “arrival time” or “event time”

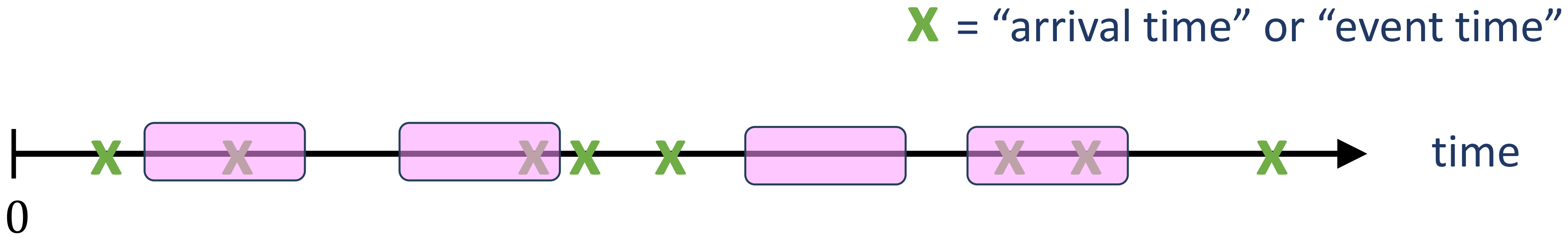

<u>For discussion</u>: Which of these event sequences have independent increments?

1. Births of children 

2. People entering a store 

3. Goals scored by a particular soccer player 

# Stationary Increments




Defn 12.9: An event sequence has **stationary increments** if the number of events during a time period depends only on the length of the time period and not on its starting point. That is:

$$N(t+s) - N(s)$$

has the same distribution for all $s$.

# Poisson Process: First definition

The **rate** of an event sequence refers to the average number of events per unit time.

**X** = "arrival time" or "event time"


<u>First definition of the Poisson Process</u>:

A **Poisson Process with rate $\boldsymbol{\lambda}$** is a sequence of events s.t.

1. $N(0) = 0$
2. The process has independent increments
3. The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$.

$$\forall s, t \geq 0, \qquad \mathbf{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

**Q:** Why only independent increments?

**A:** Third item implies stationary increments.


First definition of the Poisson Process:

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$$\forall s, t \geq 0, \qquad \boldsymbol{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

**Q:** What do we know about $N(t)$?

**A:**

$$N(t) \sim Poisson(\lambda t)$$


First definition of the Poisson Process:

A **Poisson Process with rate $\lambda$** is a sequence of events s.t.

1. $N(0) = 0$
2. The process has independent increments
3. The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$.

$$\forall s, t \geq 0, \qquad \mathbf{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

**Q:** Why is $\lambda$ called the "rate" of the process?

**A:**

$$Avg.rate\ of\ events = \frac{E[N(t)]}{t} = \frac{\lambda t}{t} = \lambda.$$


<u>First definition of the Poisson Process</u>:

A **Poisson Process with rate $\boldsymbol{\lambda}$** is a sequence of events s.t.

1. $N(0) = 0$
2. The process has independent increments
3. The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$.

$$\forall s, t \geq 0, \qquad \boldsymbol{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

**Q:** What does an event process with *both* stationary and independent increments look like?

**A:** At any point in time, the process probabilistically restarts itself.

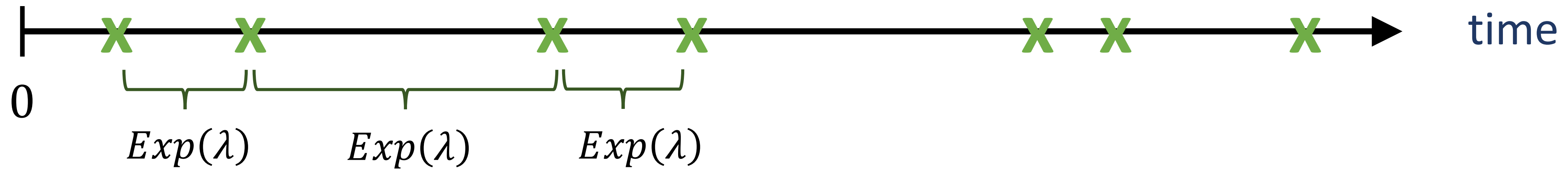
- Independent Increments → From any point on, the process is independent of all that occurred previously.
- Stationary Increments → When the process restarts itself, it has the same distribution as the original process

So the process has no memory!

# Poisson Process: Second definition

The **rate** of an event sequence refers to the average number of events per unit time.

**X** = "arrival time" or "event time"

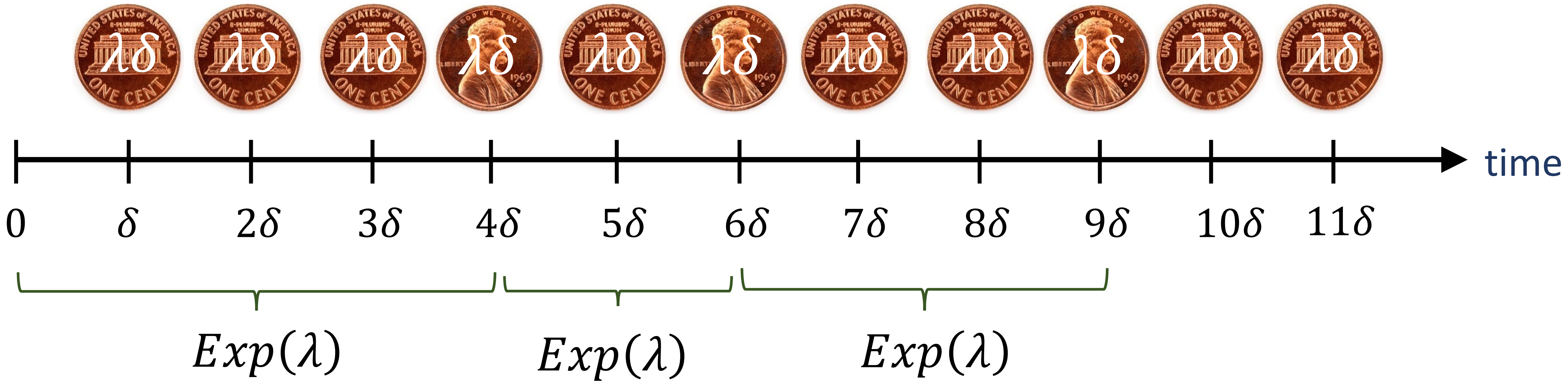



Second definition of the Poisson Process:

A **Poisson Process with rate $\lambda$** is a sequence of events s.t.

1. $N(0) = 0$
2. The sequence of inter-event times are i.i.d. $Exp(\lambda)$ random variables.

# Poisson Process: Second definition




Every $\delta$-step, flip a coin with probability $\lambda\delta$ of heads.
The sequence of heads forms a Poisson process!

# Definition 1 ➔ Definition 2

Let $\tau_1, \tau_2, \tau_3, \ldots$ be the inter-event times of a sequence of events.

**WTS:** $\tau_i \sim Exp(\lambda)$

**PF:**

$$\boldsymbol{P}\{\tau_1 > t\} = \boldsymbol{P}\{N(t) = 0\} = \frac{e^{-\lambda t} \cdot (\lambda t)^0}{0!} = \mathrm{e}^{-\lambda t}$$

$$\boldsymbol{P}\{\tau_{n+1} > t \mid \textstyle\sum_{i=1}^{n} \tau_i = s\} = \boldsymbol{P}\{0\ events\ in\ (s, s+t) \mid \textstyle\sum_{i=1}^{n} \tau_i = s\}$$

$$= \boldsymbol{P}\{0\ events\ in\ (s, s+t)\}$$

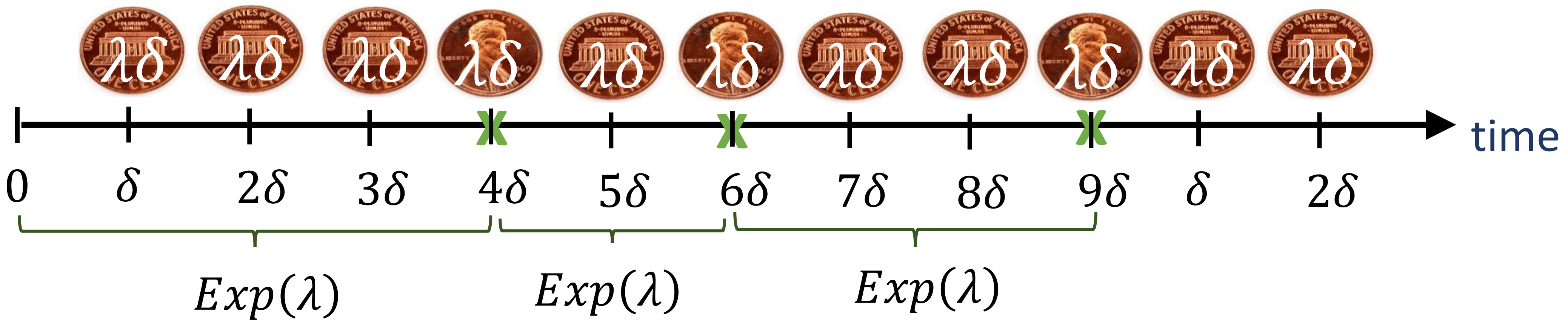
indpt incr.

$$= e^{-\lambda \mathrm{t}}$$

stat. incr.

# Definition 2 ➔ Definition 1

By Defn 2:




**WTS:** N(t) ~ Poisson(λt), and also stationary increments

$N(t) =$ Number of successes by time t

$\sim Binomial$ (#flips , probability of success of each flip)

$\sim Binomial\left(\frac{t}{\delta}, \lambda\delta\right)$

$\rightarrow Poisson\left(\frac{t}{\delta} \cdot \lambda\delta\right)$ as $\delta \rightarrow 0$

$= Poisson(\lambda t)$

**Q:** What do we know about $Binomial(n, p)$ for large $n$ and tiny $p$?

**A:** By Exercise 3.8, $Binomial(n, p) \rightarrow Poisson(np)$

# Number Poisson Arrivals during a Random Time

Jobs arrive to a system according to a Poisson process with rate $\lambda$.

Let $A_S$ = # arrivals occur during random time $S$ (assume $S$ is independent of the arrival process)

**Q:** What is $\boldsymbol{E}[A_S]$?

**Q:** How is $A_t$ distributed?

Let $A_t$ = # arrivals occur during $t$

$$A_t \sim Poisson(\lambda t) \quad \Rightarrow \quad \boldsymbol{E}[A_t] = \lambda t$$

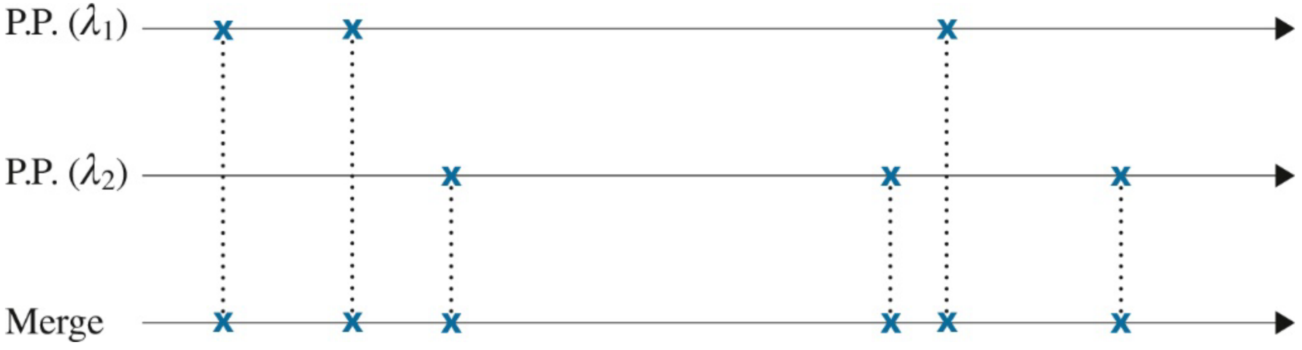
$$\boldsymbol{E}[A_S] = \int_{t=0}^{\infty} \boldsymbol{E}[A_S \mid S = t] \cdot f_S(t) dt$$

$$= \int_{t=0}^{\infty} \boldsymbol{E}[A_t] \cdot f_S(t) dt = \int_{t=0}^{\infty} \lambda t \cdot f_S(t) dt = \lambda \, \boldsymbol{E}[S]$$

# Merging Independent Poisson processes

**Theorem 12.13 (Poisson Merging):** Given two independent Poisson processes, where process 1 has rate $\lambda_1$ and process 2 has rate $\lambda_2$, the merge of these is a single Poisson process with rate $\lambda_1 + \lambda_2$.

2 proofs at least!




**Proof:**

Time to next event of merged process

$$= \min\big(Exp(\lambda_1), Exp(\lambda_2)\big) \sim Exp(\lambda_1 + \lambda_2)$$

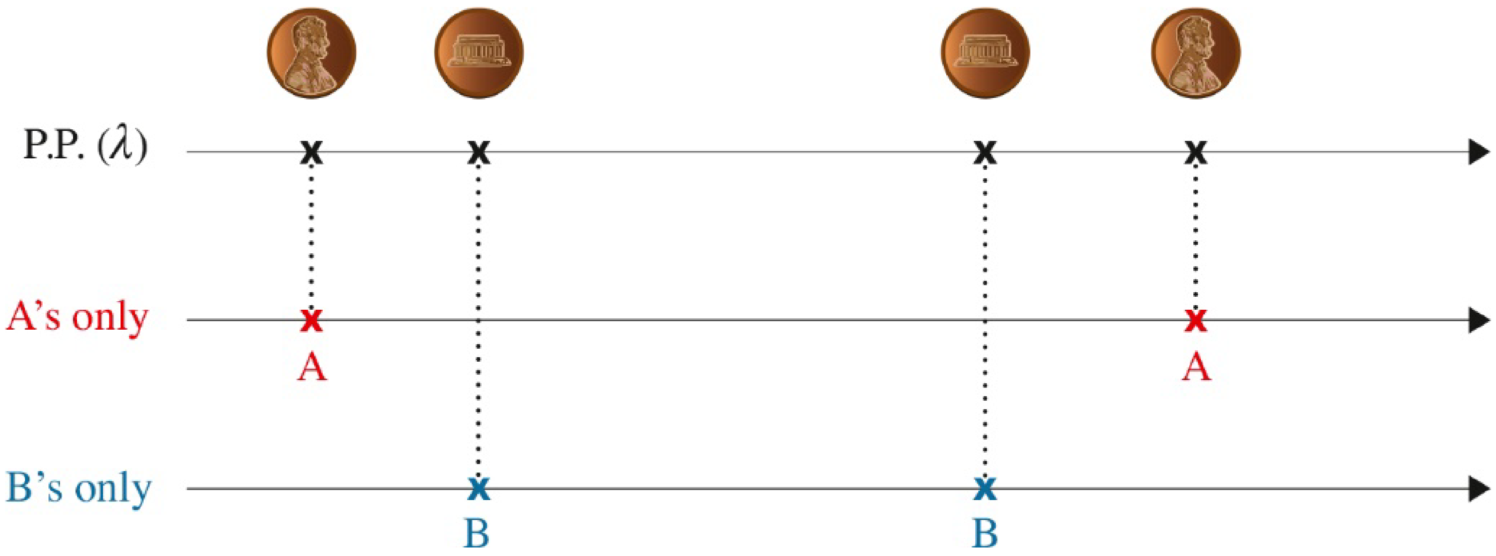
**Alternate Proof:**

$N_1(t) \sim Poisson(\lambda_1 t)$ $\qquad N_2(t) \sim Poisson(\lambda_2 t)$

$$N_{merge}(t) = N_1(t) + N_2(t) \sim Poisson(\lambda_1 t + \lambda_2 t)$$

# Poisson Splitting

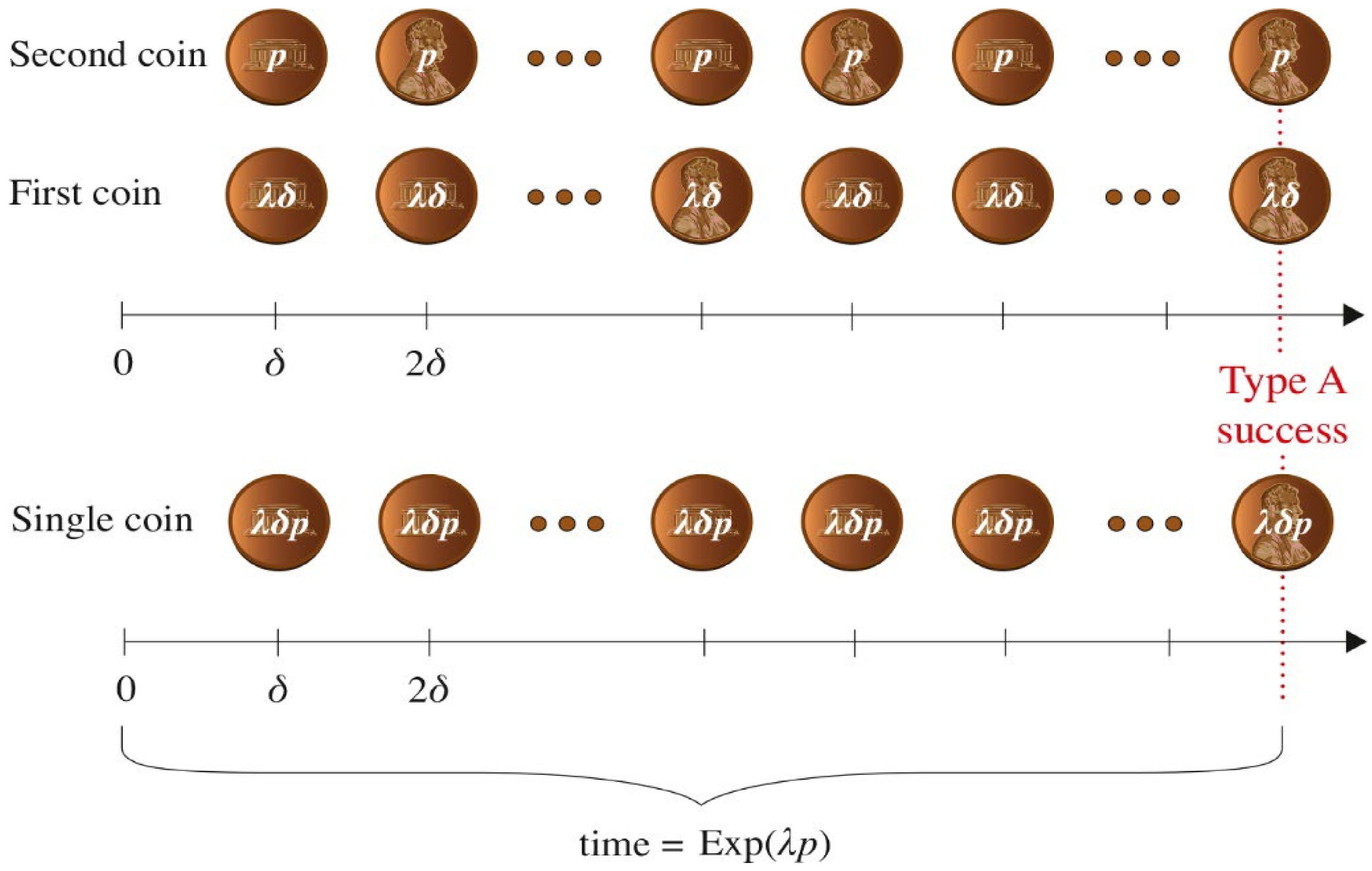
**Theorem 12.14 (Poisson Splitting):** Given a Poisson process with rate $\lambda$, suppose each event is independently classified "type A" with probability $p$ and "type B" with probability $1 - p$. Then the A's form a Poisson process with rate $\lambda p$ and the B's form a Poisson process with rate $\lambda(1 - p)$, and these processes are independent




Hard to believe because time between A's doesn't look Exponentially distributed!

# Poisson Splitting

**Intuition:**




Type A success happens only when **both** coins are heads!

Time to next type A is time until Heads for coin with probability $(\lambda p)\delta$.

$\therefore \; Time \sim Exp(\lambda p)$

See your textbook for full proof.



"Introduction to Probability for Computing", Harchol-Balter '24

# Uniformity

**Theorem 12.15 (Uniformity):** Given that one event of a Poisson process has occurred by time $t$, that event is equally likely to have occurred anywhere in $[0, t]$.

**Proof:** Let $T_1$ be the time of that one event:

$$\boldsymbol{P}\{T_1 < s \mid N(t) = 1\} = \frac{\boldsymbol{P}\{T_1 < s \ \& \ N(t) = 1\}}{\boldsymbol{P}\{N(t) = 1\}} = \frac{\boldsymbol{P}\{1 \text{ event in } [0, s] \ \& \ 0 \text{ events in } [s, t]\}}{\frac{e^{-\lambda t}(\lambda t)^1}{1!}}$$

$$= \frac{\boldsymbol{P}\{1 \text{ event in } [0, s]\} \cdot \boldsymbol{P}\{0 \text{ events in } [s, t]\}}{e^{-\lambda t} \cdot (\lambda t)}$$

$$= \frac{e^{-\lambda s} \cdot \lambda s \cdot e^{-\lambda(t-s)} \cdot \left(\lambda(t-s)\right)^0}{e^{-\lambda t} \cdot \lambda t} = \frac{s}{t}$$

**Theorem 12.15 (Uniformity -- Generalization):** If $k$ events of a Poisson process occur by time $t$, the $k$ events are distributed independently and uniformly in $[0, t]$.