Chapter 13
Generating Random Variables for Simulation

We've seen many examples of distributions. How do we generate instances of a given distribution?

Two methods:

- ❑ Inverse Transform method
- ❑ Accept-Reject method

Both assume we already have generator for $U(0,1)$

Inverse Transform Method

Requirement: To generate instances of r.v. X:

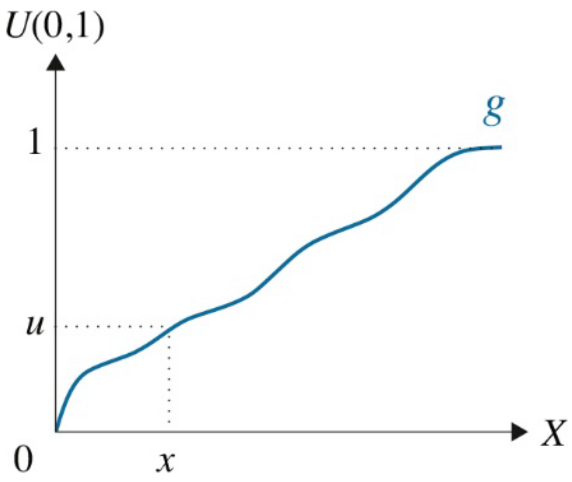
1. Need to know c.d.f. of X, that is, $F_X(x) = P\{X \leq x\}$

2. Need to be able to easily invert $F_X(x)$, i.e. get x from $F_X(x)$

Inverse Transform Method

Let u be our random instance of $U(0,1)$. Want to map u to an instance x of the r.v. X.

Let g^{-1} be the mapping that takes u's to x's:



Q: What properties should g have?

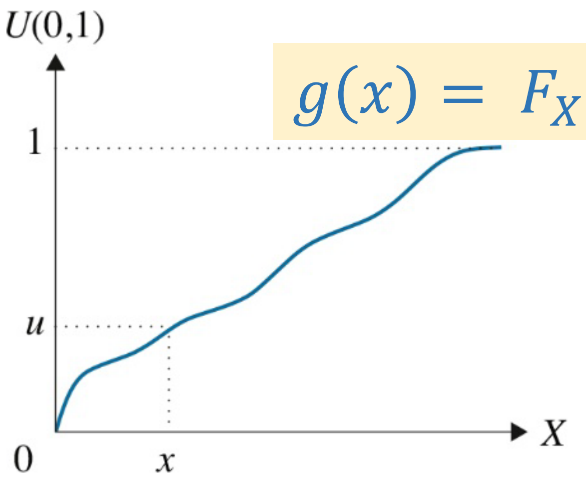
- A value in $[0, x]$ should be output with probability: __$F_X(x)$__
- A value in $[0, x]$ is output with probability: __u__

➡ $u = F_X(x)$ ➡ $x = F_X^{-1}(u)$ ➡ $g(\cdot) = F_X(\cdot)$

Inverse Transform Method

Let u be our random instance of $U(0,1)$. Want to map u to an instance x of the r.v. X.

Let g^{-1} be the mapping that takes u's to x's:



$g(x) = F_X(x)$

Inverse Transform method to generate continuous r.v. X:

1. Generate random $u \in U(0,1)$.
2. Return x such that $F_X(x) = u$.

Inverse Transform Method: Example

Q: Generate $X \sim Exp(\lambda)$, given $u \in U(0,1)$.

$$F_X(x) = u$$

$$1 - e^{-\lambda x} = u$$

$$e^{-\lambda x} = 1 - u$$

$$-\lambda x = \ln(1-u)$$

Return $x = -\frac{1}{\lambda}\ln(1-u)$

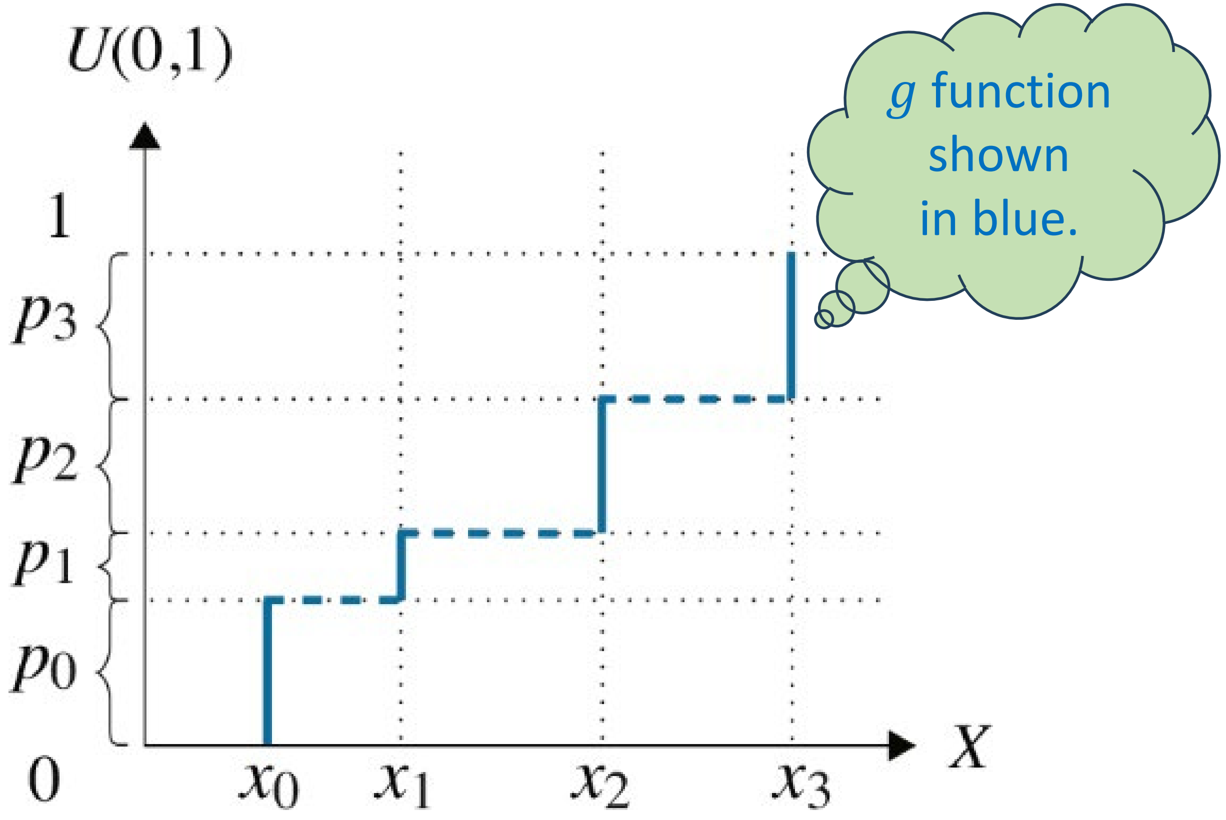
Inverse Transform Method for Discrete R.V.

Same idea applies to **discrete** r.v.s.

Let u be our random instance of $U(0,1)$. Want to map u to an instance x of the r.v. X.

$$X = \begin{cases} x_0 & \text{w.p.} \quad p_0 \\ x_1 & \text{w.p.} \quad p_1 \\ \vdots & \\ x_k & \text{w.p.} \quad p_k \end{cases}$$

Warning: Need to start by arranging values s.t. $x_0 < x_1 < x_2 < \cdots < x_k$



Inverse Transform Method for Discrete R.V.

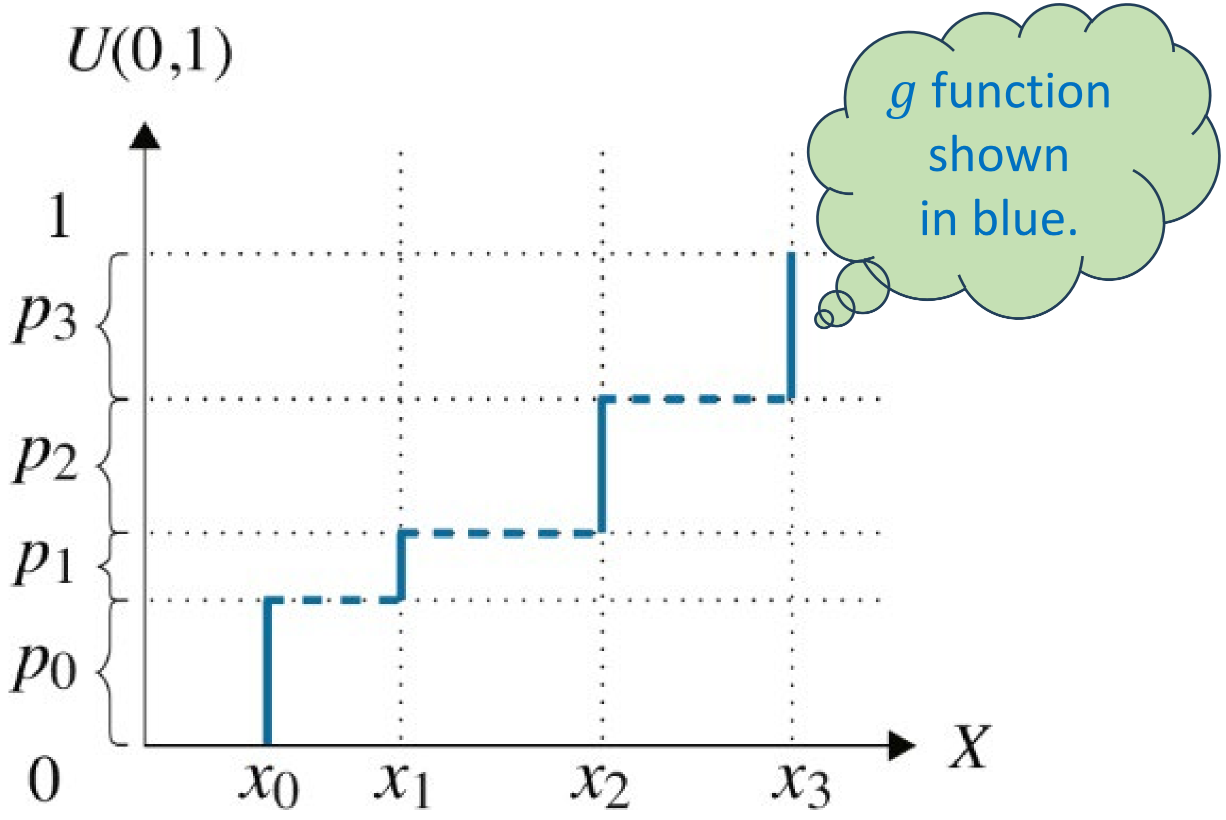
Inverse Transform method to generate discrete r.v. X:

1. Generate random $u \in U(0,1)$.

2. If $0 < u \le p_0$, output $\underline{\ x_0\ }$.
If $p_0 < u \le p_0 + p_1$, output $\underline{\ x_1\ }$.
If $p_0 + p_1 < u \le p_0 + p_1 + p_2$, output $\underline{\ x_2\ }$.

If $\sum_{i=0}^{\ell-1} p_i < u \le \sum_{i=0}^{\ell} p_i$, then output x_ℓ ,
where $0 \le \ell \le k$

To be practical, requires closed-form and invertible expression for $F_X(x)$.



Inverse Transform Method for Discrete R.V.

Example: Generate instance x of X given instance u of $U(0,1)$.

$$X = \begin{cases} 1 & \text{w.p. } 0.1 \\ 2 & \text{w.p. } 0.1 \\ \vdots & \\ 10 & \text{w.p. } 0.1 \end{cases}$$

Solution: We want to determine x as a function of u.

$$p_X(i) = 0.1$$

$$F_X(x) = \boldsymbol{P}\{X \leq x\} = \sum_{i=1}^{x} p_X(i) = x \cdot (0.1)$$

Set $u = F_X(x) = x \cdot (0.1)$

Hence $x = \lceil 10u \rceil$

Note we need ceiling.

Inverse Transform Method for Discrete R.V.

Example:

Generate instances of $X \sim Geometric(p)$

Solution:

See Exercise 13.3 in your textbook.

Interview Question

Generate instance of X where:

$$X = \begin{cases} 1 & \text{w.p.} \quad p_1 = 10^{-6} \cdot k_1 = 10^{-6} \cdot 0.33 \\ 2 & \text{w.p.} \quad p_2 = 10^{-6} \cdot k_2 = 10^{-6} \cdot 1.63 \\ 3 & \text{w.p.} \quad p_3 = 10^{-6} \cdot k_3 = 10^{-6} \cdot 0.75 \\ & \quad \cdots \\ 10^6 & \text{w.p.} \quad p_{10^6} = 10^{-6} \cdot k_{10^6} = 10^{-6} \cdot 1.02 \end{cases}$$

Given:
All of the constants,
$k_1, k_2, k_3, \ldots, k_{10^6}$
are between $(0, 2)$.

Question: Suppose you know all the p_i's. How do you generate an instance of X?

Inverse Transform:

1. Compute all the partial sums. Store in an array $A[1]$ to $A[10^6]$.
2. Given u, find *approximate* bin in array: $x \approx [u \cdot 10^6]$,
   then search around there for partial sum interval that contains u.

Interview Question

Generate instance of X where:

$$X = \begin{cases} 1 & \text{w.p.} \quad p_1 = 10^{-6} \cdot k_1 = 10^{-6} \cdot 0.33 \\ 2 & \text{w.p.} \quad p_2 = 10^{-6} \cdot k_2 = 10^{-6} \cdot 1.63 \\ 3 & \text{w.p.} \quad p_3 = 10^{-6} \cdot k_3 = 10^{-6} \cdot 0.75 \\ & \quad \cdots \\ 10^6 & \text{w.p.} \quad p_{10^6} = 10^{-6} \cdot k_{10^6} = 10^{-6} \cdot 1.02 \end{cases}$$

Given:
All of the constants, $k_1, k_2, k_3, \ldots, k_{10^6}$ are between $(0, 2)$.

Question: Now suppose you don't know the p_i's. Can look up one at a time, but takes time to look up each one, and they change over time.

Can't get partial sums ➔ Can't use Inverse Transform Method
➔ Need new approach!

Interview Question

Generate instance of X where:

$$X = \begin{cases} 1 & \text{w.p.} \quad p_1 = 10^{-6} \cdot k_1 = 10^{-6} \cdot 0.33 \\ 2 & \text{w.p.} \quad p_2 = 10^{-6} \cdot k_2 = 10^{-6} \cdot 1.63 \\ 3 & \text{w.p.} \quad p_3 = 10^{-6} \cdot k_3 = 10^{-6} \cdot 0.75 \\ & \quad \cdots \\ 10^6 & \text{w.p.} \quad p_{10^6} = 10^{-6} \cdot k_{10^6} = 10^{-6} \cdot 1.02 \end{cases}$$

Given:
All of the constants, $k_1, k_2, k_3, \ldots, k_{10^6}$ are between $(0, 2)$.

Question: Now suppose you don't know the p_i's. Can look up one at a time, but takes time to look up each one, and they change over time.

- Imagine r.v. Y, where $p_i = 10^{-6}, \ \forall i$
- Generate instance i of Y (easy to do).
- Now look up $p_X(i)$. Return i as instance of X if $p_X(i)$ is high compared to $\mathrm{p_Y(i)}$?

Intuition behind Accept/Reject method

A method with fewer requirements …

Goal: To generate instances of r.v. X

Inverse Transform Method

1. Need to know closed-form $F_X(x)$
2. Need to know be able to easily invert $F_X(x)$, i.e. get x from $F_X(x) = u$

Accept-Reject Method

1. Need the p.m.f. of X (or p.d.f., if continuous)
2. Need to know how to generate some other r.v. Y, where X and Y take on the same set of values, i.e.,

$$p_X(i) > 0 \iff p_Y(i) > 0$$

A method with fewer requirements …

Goal: To generate instances of r.v. X

<u>High-level idea</u>:
We generate an instance of Y.
Then we some probability we return that value as our instance of X (accept).
Otherwise, we reject that value and try again.

Accept-Reject Method

1. Need the p.m.f. of X (or p.d.f., if continuous)

2. Need to know how to generate some other r.v. Y, where X and Y take on the same set of values, i.e.,

$$p_X(i) > 0 \iff p_Y(i) > 0$$

Accept-Reject Method

Accept-Reject method to generate discrete r.v. X:

1. Find a discrete r.v. Y which we already know how to generate, where
$$p_X(i) > 0 \iff p_Y(i) > 0$$

2. Let $c > 1$ be the smallest constant s.t.
$$\frac{p_X(i)}{p_Y(i)} \leq c, \qquad \forall i \text{ s.t. } p_X($$

3. Generate an instance of Y. Call this instance i

4. With probability $AcceptRatio(i) = \frac{p_X(i)}{c p_Y(i)}$ accept

Else, reject i and return to step 3.

Relative likelihood of i being an instance of X versus an instance of Y.

The c ensures $AcceptRatio < 1$.

Accept-Reject Method: Correctness

Claim: $\boldsymbol{P}\{X \text{ is set to } i\} = p_X(i)$

Proof:

$$\text{Frac of time } i \text{ is generated \& accepted} = \boldsymbol{P}\{i \text{ is generated}\} \cdot \boldsymbol{P}\{i \text{ is accepted} \mid i \text{ is generated}\}$$

$$= p_Y(i) \cdot \frac{p_X(i)}{c\, p_Y(i)} = \boxed{\frac{p_X(i)}{c}}$$

$$\text{Frac of time any value is accepted} = \sum_i \text{Frac of time } i \text{ is generated \& accepted}$$

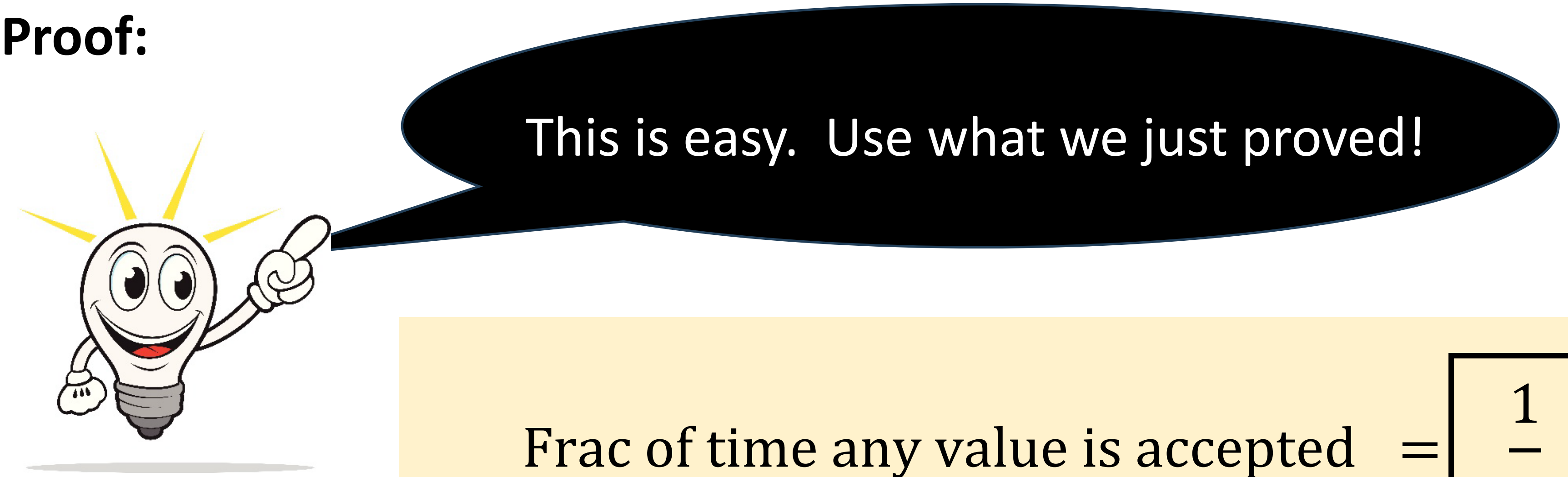
$$= \sum_i \frac{p_X(i)}{c} = \boxed{\frac{1}{c}}$$

$$\boldsymbol{P}\{X \text{ is set to } i\} = \frac{\text{Frac of time } i \text{ is generated and accepted}}{\text{Frac of time any value is accepted}} = \frac{\frac{p_X(i)}{c}}{\frac{1}{c}} = \boxed{p_X(i)}$$

Accept-Reject Method

Theorem: On average, only need to generate c values of Y before one is accepted.

Proof:



$$\text{Frac of time any value is accepted } = \boxed{\frac{1}{c}}$$

$$\Rightarrow \boldsymbol{E}[\#\text{ values generated until one is accepted}] = c$$

Back to the Interview Question

Generate instance of X where:

$$X = \begin{cases} 1 & \text{w.p.} \quad p_1 = 10^{-6} \cdot k_1 = 10^{-6} \cdot 0.33 \\ 2 & \text{w.p.} \quad p_2 = 10^{-6} \cdot k_2 = 10^{-6} \cdot 1.63 \\ 3 & \text{w.p.} \quad p_3 = 10^{-6} \cdot k_3 = 10^{-6} \cdot 0.75 \\ & \quad \cdots \\ 10^6 & \text{w.p.} \quad p_{10^6} = 10^{-6} \cdot k_{10^6} = 10^{-6} \cdot 1.02 \end{cases}$$

Given: All the constants, $k_1, k_2, k_3, \ldots, k_{10^6}$ are between $(0, 2)$.

Question: You don't know the p_i's. Can look up one at a time, but takes time to look up each one, and they change over time.

Solution: Accept-Reject!

1. Let $Y = [1, 2, 3, \ldots, 10^6]$, each with probability 10^{-6}
2. Observe $\frac{p_X(i)}{p_Y(i)} < c, \ \forall i$
3. Generate i: instance of Y. Look up $p_X(i)$ only.
4. With probability $\frac{p_X(i)}{c \cdot p_Y(i)}$: ACCEPT i, RETURN $X = i$.

   Else, Return to Step 3.

What is c?

$c = 2$

How many lookups are required in expectation?

2

Accept-Reject Method: Continuous

Accept-Reject method to generate continuous r.v. X with p.d.f. $f_X(t)$:

1. Find a discrete r.v. Y which we already know how to generate, where
$$f_X(t) > 0 \iff f_Y(t) > 0$$

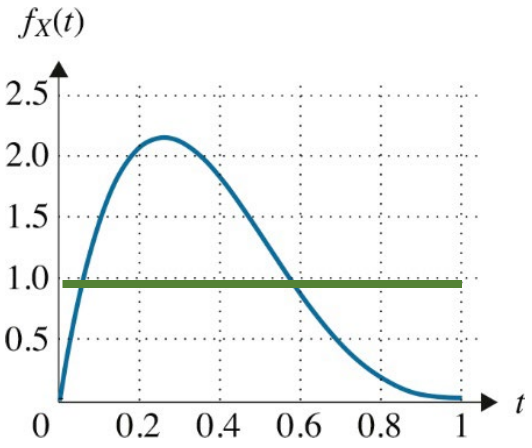
2. Let $c > 1$ be the smallest constant s.t.
$$\frac{f_X(t)}{f_Y(t)} \leq c, \qquad \forall t \text{ s.t. } f_X(t) > 0.$$

3. Generate an instance t of Y.

4. With probability $AcceptRatio(t) = \frac{f_X(t)}{cf_Y(t)}$, accept t and return $\mathrm{X} = t$.
   Else, reject t and return to step 3.

Accept-Reject Method: Example 1

GOAL: Generate r.v. X where $f_X(t) = 20t(1-t)^3, \quad 0 < t < 1$



Q: Given this image of $f_X(t)$, what's a good choice for Y ?

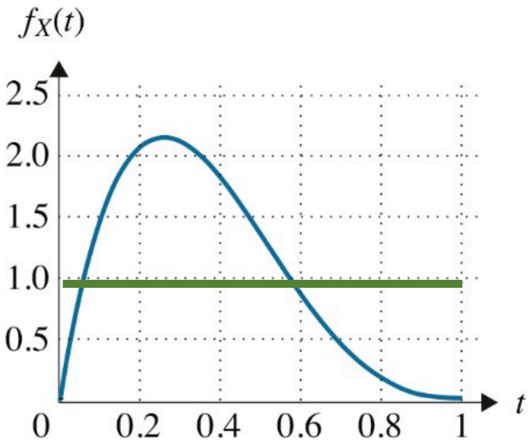
A: $f_Y(t) = 1, \quad 0 < t < 1$ has same range and is easy to generate.

Q: Looking at the picture, how high is c approximately ?

A: $c \approx 2$.
So only need 2 guesses to attempts on averge to generate X.

Accept-Reject Method: Example 1

GOAL: Generate r.v. X where $f_X(t) = 20t(1-t)^3, \quad 0 < t < 1$



More precisely:

$$c = \max_t \left\{ \frac{f_X(t)}{f_Y(t)} \right\} = \max_{\text{t}} 20\text{t}(1-\text{t})^3$$

$$\frac{d}{dt}\left(20t(1-t)^3\right) = 0 \quad \Leftrightarrow \quad t = \frac{1}{4}$$

$$c = \frac{f_X\left(\frac{1}{4}\right)}{f_Y\left(\frac{1}{4}\right)} = 20\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3 = \frac{135}{64}$$

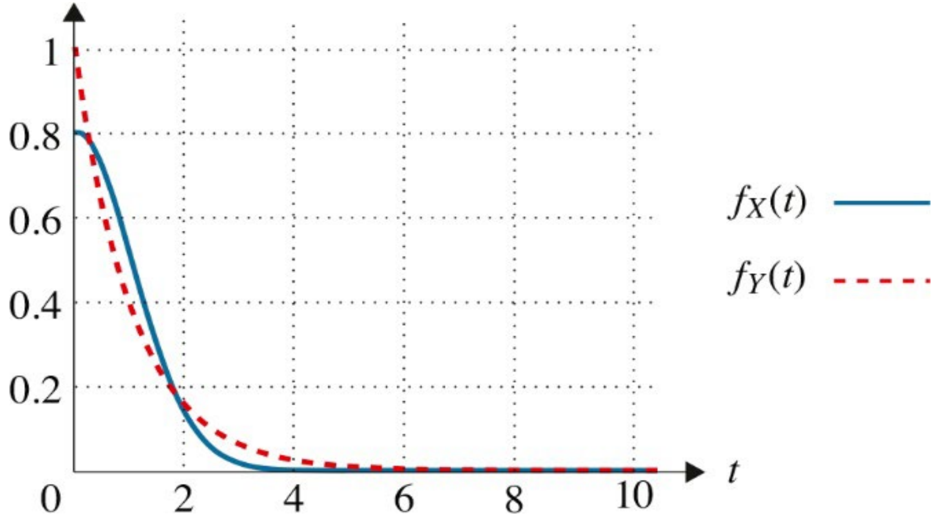
Accept-Reject Method: Example 2

GOAL: Generate r.v. N where $N \sim Normal(0,1)$

Suffices to generate $X = |N|$ and then multiply X by -1 with probability $\frac{1}{2}$

Q: What's a good choice for r.v. Y with range 0 to ∞?

A: Let $Y \sim Exp(1)$. $f_Y(t) = e^{-t}, \quad 0 < t < \infty$



Q: From the picture, how high is c approximately ?

A: $c \approx 1.3$.
So only need 1.3 guesses to attempts on averge to generate X.

Accept-Reject Method: Example 2

More precisely:

$$c = \max_t \left\{ \frac{f_X(t)}{f_Y(t)} \right\} = \max_t \sqrt{\frac{2}{\pi}}\, e^{t - \frac{t^2}{2}}$$

Suffices to maximize $t - \frac{t^2}{2}$

$$0 = \frac{d}{dt}\left(t - \frac{t^2}{2} \right) = 1 - t \quad \Leftrightarrow \quad t = 1$$

$$c = \frac{f_X(1)}{f_Y(1)} = \sqrt{\frac{2e}{\pi}} \approx 1.3$$