Chapter 6
z-Transforms

Motivation

Let $X \sim Binomial(n, p)$

What is $\boldsymbol{E}[X^3]$?

$$\boldsymbol{E}[X^3] = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \cdot i^3$$

Seems complicated to evaluate!

Motivation

Let $X \sim Poisson(\lambda)$

What is $\boldsymbol{E}[X^5]$?

$$\boldsymbol{E}[X^5] = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\,\lambda^i}{i!} \cdot i^5$$

Seems complicated to evaluate!

The wonderful world of transforms

Laplace transforms

Fourier transforms

z-transforms

characteristic functions

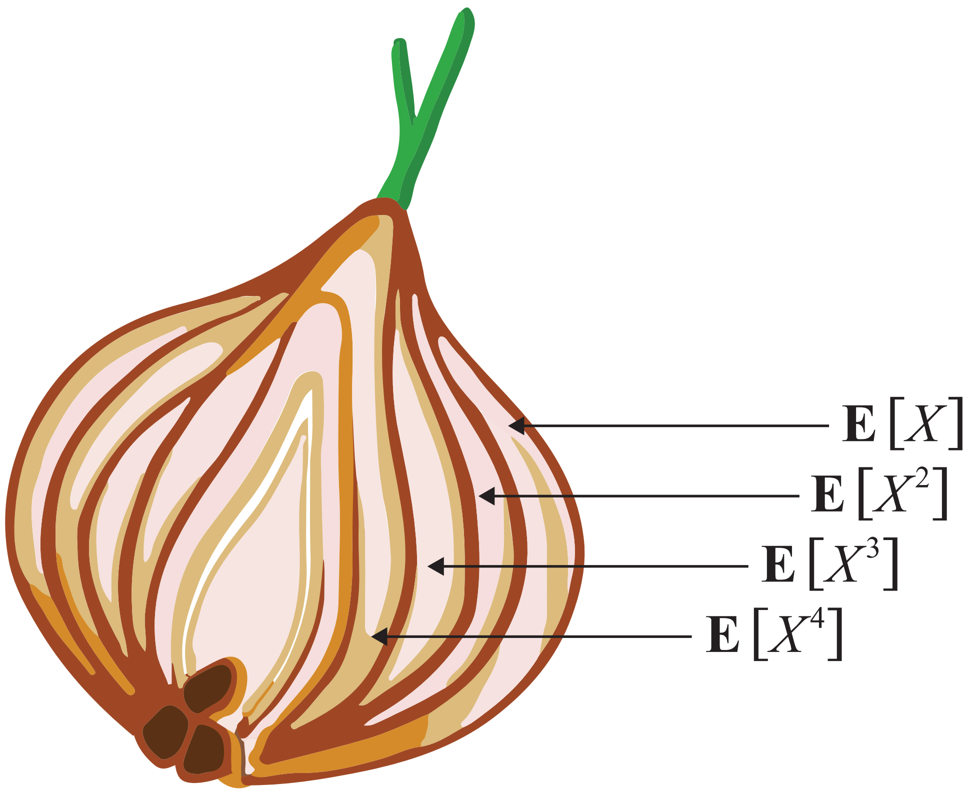
moment-generating functions

Two common uses:

1. Computing higher moments of random variables
2. Solving recurrence relations

The z-transform as an onion

Onion represents z-transform of r.v. X



Lower moments are in the outer layers → less effort/tears
Higher moments are deeper inside → more effort/tears

z-transform of discrete r.v.

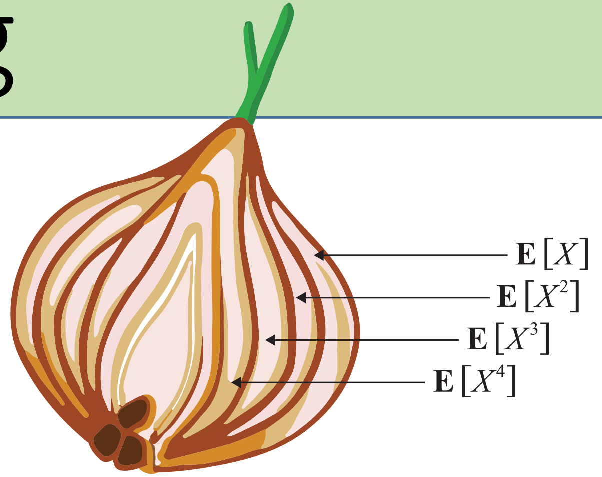
<u>Defn</u>: Let X be a non−negative discrete r.v. with p.m.f. $p_X(i)$, where $i = 0, 1, 2, \ldots$
Then the **z-transform** of X is

$$\hat{X}(z) = \boldsymbol{E}[z^X] = \sum_{i=0}^{\infty} p_X(i) \cdot z^i$$

Assume z is a constant and $|z| \leq 1$.

Note: The z−transform can be defined for any r.v., or even for just a sequence of $p(i)'s$.
However convergence is only <u>guaranteed</u> when X is a non-negative r.v. and $|z| \leq 1$.

Example of Onion Building



$X \sim Binomial(n, p)$

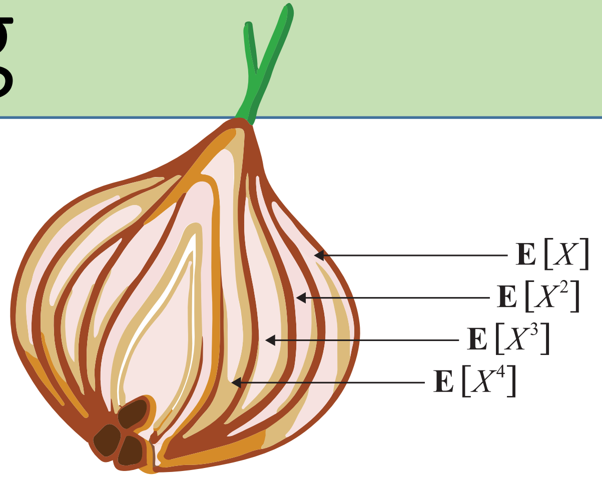
Create the onion!

$$\hat{X}(z) = \boldsymbol{E}[z^X] = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i$$

$$= \sum_{i=0}^{n} \binom{n}{i} (zp)^i (1-p)^{n-i}$$

$$= \big(zp + (1-p)\big)^n$$

Example of Onion Building



$X \sim Geometric(p)$

Create the onion!

$$\hat{X}(z) = \boldsymbol{E}[z^X] = \sum_{i=1}^{\infty} (1-p)^{i-1} p\, z^i$$

$$= zp \sum_{i=1}^{\infty} (z(1-p))^{i-1}$$

$$= \frac{zp}{1 - z(1-p)}$$

Where did we use the fact that $|z| \leq 1$?

Convergence of z-transform

Theorem: $\hat{X}(z)$ is bounded for any non−negative discrete r.v. X, assuming $|z| \leq 1$.

Proof:

$$-1 \leq z \leq 1$$

$$-1 \leq z^i \leq 1$$

$$-p_X(i) \leq p_X(i)z^i \leq p_X(i)$$

$$-\sum_i p_X(i) \leq \sum_i p_X(i)z^i \leq \sum_i p_X(i)$$

$$-1 \leq \hat{X}(z) \leq 1$$

Getting moments: Onion peeling

Theorem: (Onion Peeling) Let X be a discrete, integer−valued, non−negative r.v. with p.m.f. $p_X(i)$, $\mathrm{i} = 0, 1, 2, \ldots$ Then,

$$\widehat{X}\,'(z)\Big|_{\mathrm{z}=1} = \mathbf{E}[\mathrm{X}]$$

$$\widehat{X}\,''(z)\Big|_{\mathrm{z}=1} = \mathbf{E}[\mathrm{X}(\mathrm{X}-1)]$$

$$\widehat{X}\,'''(z)\Big|_{\mathrm{z}=1} = \mathbf{E}[\mathrm{X}(\mathrm{X}-1)(\mathrm{X}-2)]$$

$$\widehat{X}\,''''(z)\Big|_{\mathrm{z}=1} = \mathbf{E}[\mathrm{X}(\mathrm{X}-1)(\mathrm{X}-2)(\mathrm{X}-3)]$$

If can't evaluate at $z = 1$, instead consider limit as $z \to 1$ (use L'Hospital's Rule).

Proof of onion peeling theorem

$$\widehat{X}(z) = p_X(0)z^0 + p_X(1)z^1 + p_X(2)z^2 + p_X(3)z^3 + p_X(4)z^4 + p_X(5)z^5 + \cdots$$

$$\widehat{X}'(z) = p_X(1) + 2p_X(2)z^1 + 3p_X(3)z^2 + 4p_X(4)z^3 + 5p_X(5)z^4 + \cdots$$

$$\widehat{X}'(z)\Big|_{z=1} = 1 \cdot p_X(1) + 2p_X(2) + 3\,p_X(3) + 4p_X(4) + 5\,p_X(5) + \cdots \;= \boldsymbol{E}[X]$$

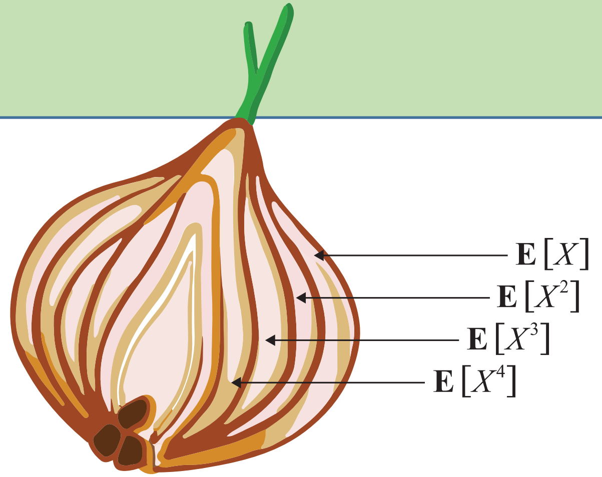
$$\widehat{X}''(z) = 2p_X(2) + 3 \cdot 2\,p_X(3)z + 4 \cdot 3\,p_X(4)z^2 + 5 \cdot 4\,p_X(5)z^3 + \cdots$$

$$\widehat{X}''(z)\Big|_{z=1} = 2 \cdot 1 p_X(2) + 3 \cdot 2 \cdot p_X(3) + 4 \cdot 3 \cdot p_X(4) + 5 \cdot 4 p_X(4) + \cdots = \boldsymbol{E}[X(X-1)]$$

$$\widehat{X}'''(z) = 3 \cdot 2\,p_X(3) + 4 \cdot 3 \cdot 2\,p_X(4)z + 5 \cdot 4 \cdot 3\,p_X(5)z^2 + \cdots$$

$$\widehat{X}'''(z)\Big|_{z=1} = 3 \cdot 2 \cdot 1\,p_X(3) + 4 \cdot 3 \cdot 2\,p_X(4) + 5 \cdot 4 \cdot 3\,p_X(5) + \cdots = \boldsymbol{E}[X(X-1)(X-2)]$$

Example of onion peeling



$X \sim Geometric(p)$ $\qquad \hat{X}(z) = \dfrac{zp}{1 - z(1-p)}$

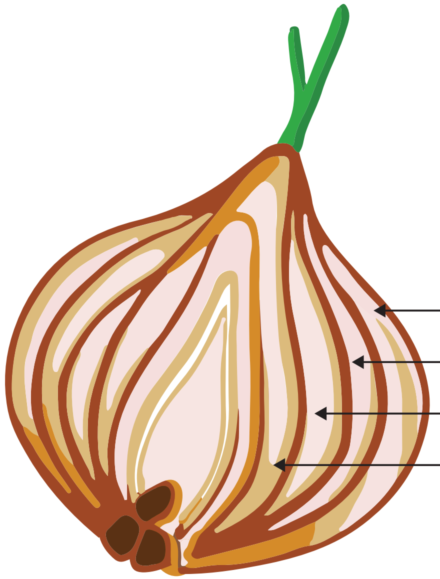
Q: Peel the onion to get $\boldsymbol{E}[X]$ and $\boldsymbol{E}[X^2]$

$$\mathbf{E}[\mathrm{X}] = \left(\hat{X}'(z)\right)\Big|_{z=1} = \frac{d}{dz}\left(\frac{zp}{1 - z(1-p)}\right)\Bigg|_{z=1} = \left(\frac{p}{\left(1 - z(1-p)\right)^2}\right)\Bigg|_{z=1} = \frac{1}{\mathrm{p}}$$

$$\mathbf{E}[\mathrm{X}^2] = \left(\hat{X}''(z)\right)\Big|_{z=1} + \mathbf{E}[\mathrm{X}] = \left(\frac{2p(1-p)}{\left(1 - z(1-p)\right)^3}\right)\Bigg|_{z=1} + \frac{1}{p} = \frac{2-p}{p^2}$$

Onion to distribution

The z−transform of X is an onion that contains all moments of X.



But does it also contain the distribution of X?

The answer is YES! The distribution of X can be extracted from its z−transform.
See Exercise 6.14 in your book.

Linearity of Transforms

Theorem 6.9: (Linearity) Let X and Y be **independent** discrete r.v.s.
Let

$$W = X + Y$$

Then the z−transform of W is:

$$\widehat{W}(z) = \hat{X}(z) \cdot \hat{Y}(z)$$

Proof:

$$\begin{aligned}\widehat{W}(z) = \boldsymbol{E}[z^W] &= \boldsymbol{E}[z^{X+Y}] \\ &= \boldsymbol{E}[z^X \cdot z^Y] \\ &= \boldsymbol{E}[z^X] \cdot \boldsymbol{E}[z^Y] \\ &= \hat{X}(z) \cdot \hat{Y}(z)\end{aligned}$$

Example: From Bernoulli to Binomial

$X \sim Bernoulli(p)$ $Y \sim Binomial(n, p)$

Q: (a) What is $\hat{X}(z)$?

(b) How can we use $\hat{X}(z)$ to get $\hat{Y}(z)$?

A:

$$\hat{X}(z) = (1-p) \cdot z^0 + p \cdot z^1 = 1 - p + pz$$

$$Y = \sum_{i=1}^{n} X_i \quad \text{where } X_i\text{'s i.i.d.} \sim X$$

$$\hat{Y}(z) = \left(\hat{X}(z)\right)^n = (1 - p + pz)^n$$

Example: Sum of Binomials

$X \sim Binomial(n, p) \qquad Y \sim Binomial(m, p) \qquad X \perp Y$

Q: What is the distribution of $Z = X + Y$?

A:

$$\hat{Z}(z) = \hat{X}(z) \cdot \hat{Y}(z)$$

$$= \left(zp + (1-p)\right)^n \cdot \left(zp + (1-p)\right)^m$$

$$= \left(zp + (1-p)\right)^{n+m}$$

$$\Rightarrow Z \sim Binomial(n+m, p)$$

Conditioning with Transforms

Theorem 6.12: Let X, A, and B be discrete r.v.s. where

$$X = \begin{cases} A & \text{w.p.} \quad p \\ B & \text{w.p.} \quad 1-p \end{cases}$$

Then,

$$\hat{X}(z) = p \cdot \hat{A}(z) + (1-p) \cdot \hat{B}(z)$$

Proof:

$$\hat{X}(z) = \boldsymbol{E}[z^X]$$

$$= \boldsymbol{E}[z^X | X = A] \cdot p + \boldsymbol{E}[z^X | X = B] \cdot (1-p)$$

$$= \boldsymbol{E}[z^A] \cdot p + \boldsymbol{E}[z^B] \cdot (1-p)$$

$$= p\hat{A}(z) + (1-p)\hat{B}(z)$$

Sum of random number of random variables

Theorem 6.13: Let $X_1, X_2, X_3, \ldots$ be i.i.d. r.v.s, where $X_i \sim X$.

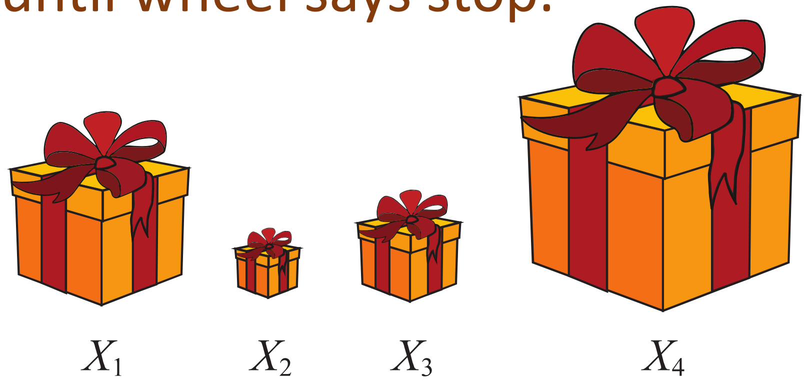
Let $S = \sum_{i=1}^{N} X_i$, where $N \perp \{X_1, X_2, X_3, \ldots\}$

Then $\hat{S}(z) = \hat{N}\left(\hat{X}(z)\right)$

N is the number of spins of the wheel.

Proof: See Exercise 6.10.

Get new prize every day, until wheel says stop.



Solving recurrence relations

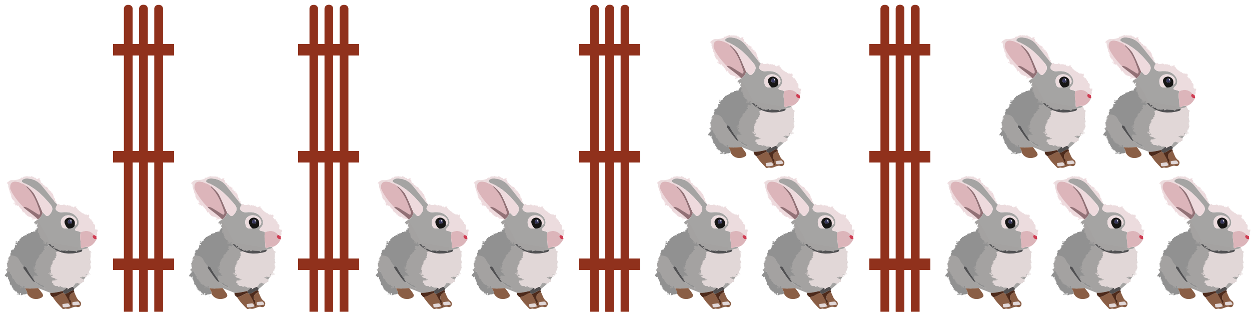
<u>Defn</u>: A linear homogeneous recurrence relation takes the form:

$$f_{i+n} = a_1 f_{i+n-1} + a_2 f_{i+n-2} + \cdots + a_n f_i$$

Such recurrences come up in many fields

<u>Example</u>:

$$f_{i+2} = f_{i+1} + f_i, \qquad f_0 = 0, \qquad f_1 = 1$$



<u>Goal</u>: Closed-form solution for f_i

Solving recurrence relations: general approach

Defn 6.14:
Given a sequence of values: $f_0, f_1, f_2, \ldots$, the **z-transform of the sequence** is

$$F(z) = \sum_{i=0}^{\infty} f_i z^i$$

z-transform approach is the best approach for solving recurrences.

$$f_{i+2} = f_{i+1} + f_i$$

$$f_{i+2}z^{i+2} = f_{i+1}z^{i+2} + f_i z^{i+2}$$

$$\sum_{i=0}^{\infty} f_{i+2}z^{i+2} = \sum_{i=0}^{\infty} f_{i+1}z^{i+2} + \sum_{i=0}^{\infty} f_i z^{i+2}$$

$$F(z) - f_1 z - f_0 = z(F(z) - f_0) + z^2 F(z)$$

1. Solve for $F(z)$
2. Express $F(z)$ as a series expansion of the form $F(z) = \sum_{i=0}^{\infty} c_i z^i$, where c_i denotes some expression not involving z.
3. Obtain f_i by setting $f_i = c_i$