10-301/601: Introduction to Machine Learning Lecture 20 - Naïve Bayes

Henry Chai

7/12/23

Front Matter

- Announcements:
  - PA4 released 6/15, due 7/13 (tomorrow) at 11:59 PM
  - PA5 released 7/13 (tomorrow), due 7/20 at 11:59 PM
  - Quiz 7: Unsupervised Learning & Naïve Bayes on 7/18
    - Based on your responses to the midsemester feedback survey, we have decided to drop everyone's lowest quiz grade!
- Recommended Readings:
  - Mitchell, draft chapter on Naïve Bayes & logistic regression
  - Murphy, Chapter 3.5

What is ~~Machine Learning~~ 10-301/601?

- Supervised Models
  - Decision Trees
  - KNN
  - Naïve Bayes
  - Perceptron
  - Logistic Regression
  - SVMs
  - Linear Regression
  - Neural Networks
- Unsupervised Models
  - K-means
  - GMMs
  - PCA
- Graphical Models
  - Bayesian Networks
  - HMMs
- Learning Theory
- Reinforcement Learning
- Important Concepts
  - Feature Engineering and Kernels
  - Regularization and Overfitting
  - Experimental Design
  - Ensemble Methods

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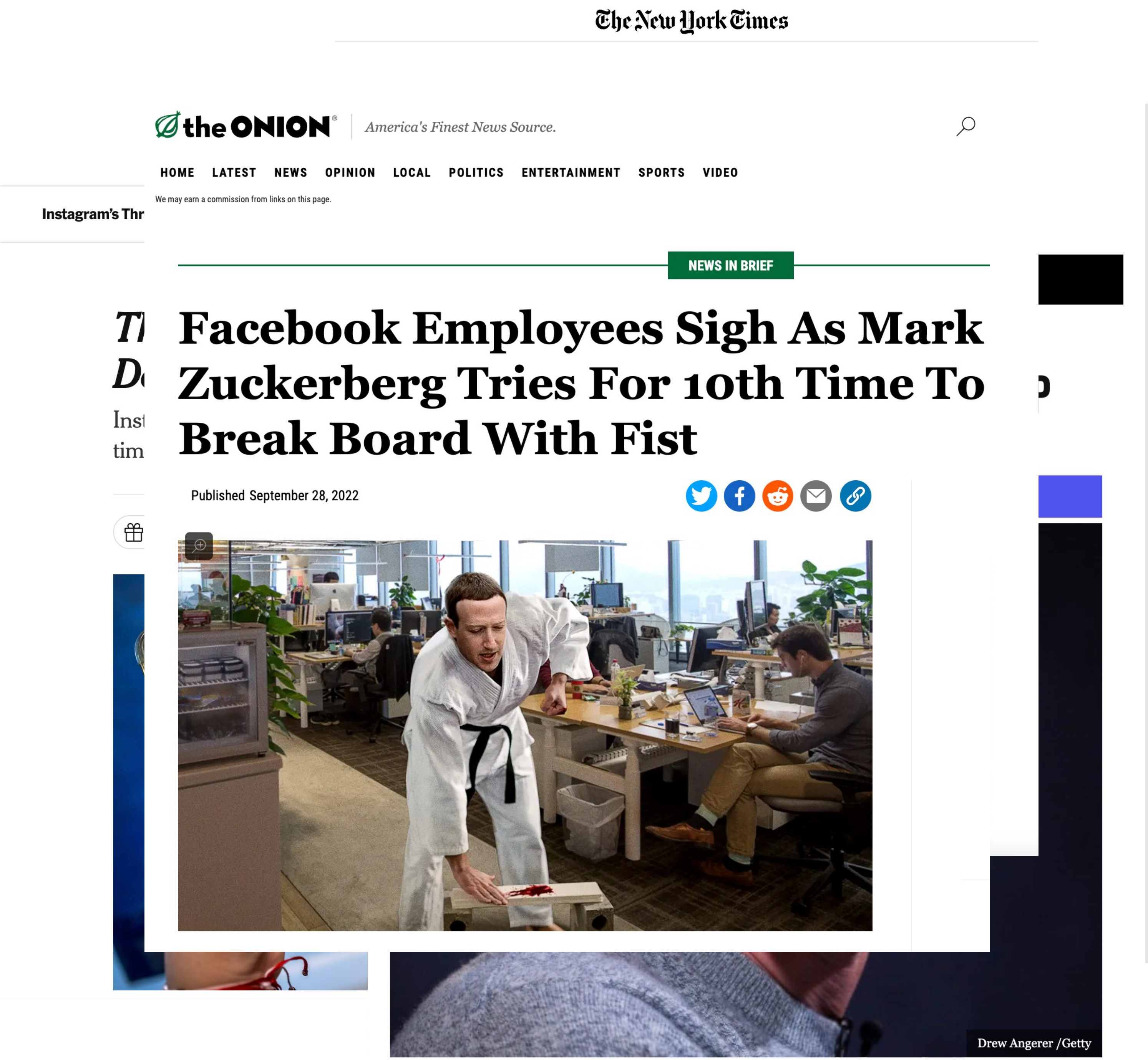
Text Data

- https://www.nytimes.com/2023/07/06/technology/threads-downloads-twitter.html?searchResultPosition=1
- https://www.breitbart.com/tech/2023/07/06/sanely-run-mark-zuckerbergs-twitter-clone-censors-donald-trump-jr-on-day-one/
- https://www.nytimes.com/2023/07/01/technology/elon-musk-mark-zuckerberg-cage-match.html?searchResultPosition=2
- https://www.theonion.com/facebook-employees-sigh-as-mark-zuckerberg-tries-for-10-1849518797



Text Data

The TOOTH Book
Dr. Seuss
POCKET ROCKS
THE CAT IN THE HAT
By Dr. Seuss
HOP ON POP
By Dr. Seuss
The SIMPLEST SEUSS for YOUNGEST USE
Touch, Feel
Lois Ehlert
My Mother?
by P.D. Eastman
Go, Dog. Go!
by P.D.Eastman
BLUE FISH
Chicka Chicka Boom Boom
by Bill Martin Jr and John Archambault
illustrated by Lois Ehlert
The CRAYONS' Book of FEELINGS
幼儿启蒙大全
YOU ER QI MENG DA QUAN
Big Dog... Little Dog
by P.D. Eastman
FRIENDS COLLECTION
Pooh & Eeyore

Bag-of-Words Model

x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	y (Dr. Seuss)

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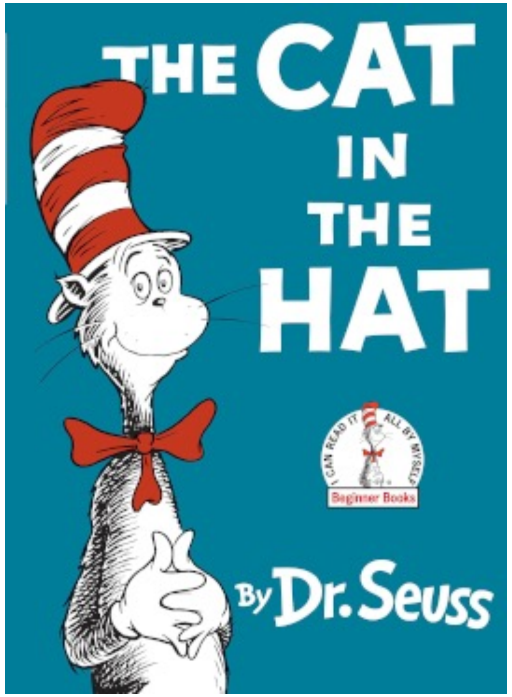
x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	y (Dr. Seuss)
1	1	0	0	0	0	1

The Cat in the Hat
(by Dr. Seuss)

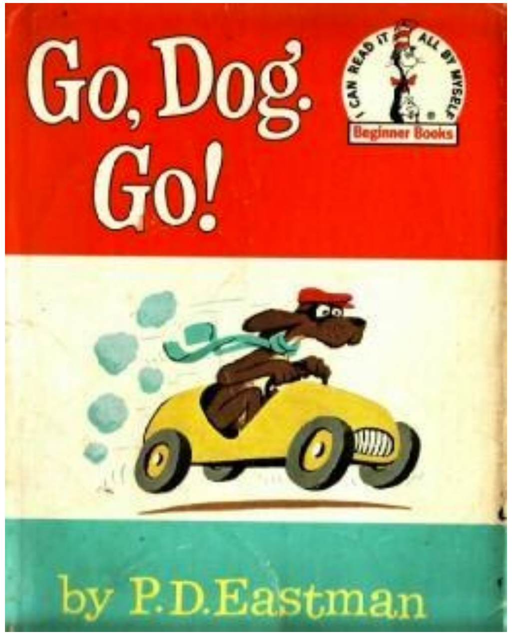


Source: https://en.wikipedia.org/wiki/The_Cat_in_the_Hat#/media/File:The_Cat_in_the_Hat.png

Bag-of-Words Model

x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	y (Dr. Seuss)
1	1	0	0	0	0	1
0	0	1	0	0	0	0

Go, Dog. Go!
(by P. D. Eastman)



Source: https://en.wikipedia.org/wiki/Go,_Dog._Go!#/media/File:Go_Dog_Go.jpg

Bag-of-Words Model

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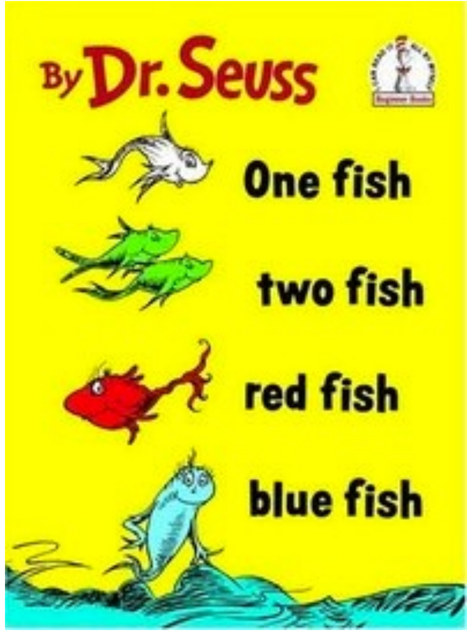
1	1	0	0	0	0	1
0	0	1	0	0	0	0
0	0	0	1	0	0	1

One Fish, Two Fish, Red Fish, Blue Fish (by Dr. Seuss)



Source: https://en.wikipedia.org/wiki/One_Fish,_Two_Fish,_Red_Fish,_Blue_Fish#/media/File:One_Fish_Two_Fish_Red_Fish_Blue_Fish_(cover_art).jpg

Bag-of-Words Model

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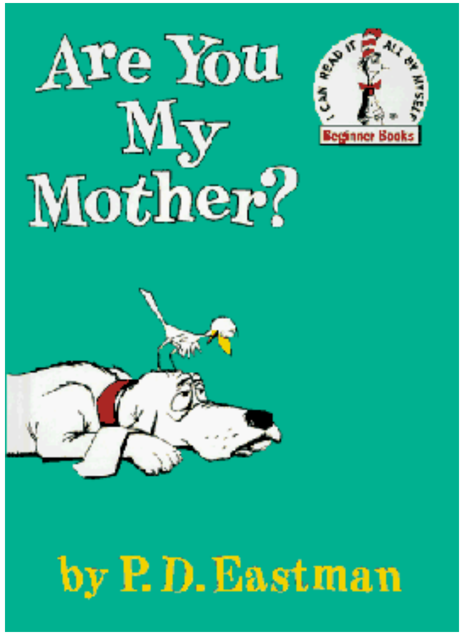
1	1	0	0	0	0	1
0	0	1	0	0	0	0
0	0	0	1	0	0	1
0	0	0	0	1	0	0

Are You My Mother?
(by P. D. Eastman)



Source: https://en.wikipedia.org/wiki/Are_You_My_Mother%3F#/media/File:Areyoumymother.gif

Recall: Building a Probabilistic Classifier

- Define a decision rule
  - Given a test data point $\boldsymbol{x}'$, predict its label $\hat{y}$ using the posterior distribution $P(Y = y | X = \boldsymbol{x}')$
  - Common choice: $\hat{y} = \underset{y}{\operatorname{argmax}} P(Y = y | X = \boldsymbol{x}')$
- Model the posterior distribution
  - Option 1 - Model $P(Y|X)$ directly as some function of X (Logistic Regression)
  - Option 2 - Use Bayes' rule (today!):

$$P(Y|X) = \frac{P(X|Y)\,P(Y)}{P(X)} \propto P(X|Y)\,P(Y)$$

How hard is modelling $P(X|Y)$?

- Define a decision rule
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How hard is modelling $P(X|Y)$?

x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	$P(X|Y = 1)$
0	0	0	0	0	0	θ_1
1	0	0	0	0	0	θ_2
1	1	0	0	0	0	θ_3
1	0	1	0	0	0	θ_4

When survey is active, respond at **pollev.com/301601polls**

Lecture 20 Polls

0 done

0 underway

Start the presentation to see live content. For screen share software, share the entire screen. Get help at **pollev.com/app**

Given 6 binary features $\mathbf{x} = [x_1, \ldots, x_6]^T$ and a binary label y, how many parameters are needed to fully specify the distribution $P(\mathbf{x}|Y = y)$?

- $2^6 = 64$
- $2^6 - 1 = 63$
- $2(2^6) = 128$
- $2(2^6 - 1) = 126$

How hard is modelling $P(X|Y)$?

x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	$P(X\|Y = 1)$	$P(X\|Y = 0)$
0	0	0	0	0	0	θ_1	θ_{64}
1	0	0	0	0	0	θ_2	θ_{65}
1	1	0	0	0	0	θ_3	θ_{66}
1	0	1	0	0	0	θ_4	θ_{67}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	1	1	1	1	$1 - \sum_{i=1}^{63} \theta_i$	$1 - \sum_{i=64}^{126} \theta_i$

Given 6 binary features $\mathbf{x} = [x_1, \ldots, x_6]^T$ and a binary label y, how many parameters are needed to fully specify the distribution $P(\mathbf{x}|Y = y)$ with the naïve Bayes assumption?

- 6
- $6 - 1 = 5$
- $2(6) = 12$
- $2(6 - 1) = 10$

Naïve Bayes Assumption

- ***Assume*** features are conditionally independent given the label:

$$P(X|Y) = \prod_{d=1}^{D} P(X_d|Y)$$

- Pros:
  - Significantly reduces computational complexity
  - Also reduces model complexity, combats overfitting
- Cons:
  - Is a strong, often illogical assumption
    - We'll see a relaxed version of this later in the semester when we discuss Bayesian networks

General Recipe for Machine Learning

- Define a model and model parameters
- Write down an objective function
- Optimize the objective w.r.t. the model parameters

Recipe for Naïve Bayes

- Define a model and model parameters
  - Make the Naïve Bayes assumption
  - Assume independent, identically distributed (iid) data
  - Parameters: $\pi = P(Y = 1), \theta_{d,y} = P(X_d = 1 | Y = y)$
- Write down an objective function
  - Maximize the log-likelihood
- Optimize the objective w.r.t. the model parameters
  - Solve in *closed form*: take partial derivatives, set to 0 and solve

Setting the Parameters via MLE

$$\ell_{\mathcal{D}}(\pi, \boldsymbol{\theta}) = \log P\left(\mathcal{D} = \left\{\boldsymbol{x}^{(1)}, y^{(1)}, \dots, \boldsymbol{x}^{(N)}, y^{(N)}\right\} \middle| \pi, \boldsymbol{\theta}\right)$$

$$= \log \prod_{n=1}^{N} P\left(\boldsymbol{x}^{(n)}, y^{(n)} \middle| \pi, \boldsymbol{\theta}\right) = \log \prod_{n=1}^{N} P\left(\boldsymbol{x}^{(n)} \middle| y^{(n)}, \boldsymbol{\theta}\right) P\left(y^{(n)} \middle| \pi\right)$$

$$= \log \prod_{n=1}^{N} \left(\prod_{d=1}^{D} P\left(x_d^{(n)} \middle| y^{(n)}, \theta_{d,1}, \theta_{d,0}\right) \right) P\left(y^{(n)} \middle| \pi\right)$$

$$= \sum_{n=1}^{N} \left(\sum_{d=1}^{D} \log P\left(x_d^{(n)} \middle| y^{(n)}, \theta_{d,1}, \theta_{d,0}\right) \right) + \log P\left(y^{(n)} \middle| \pi\right)$$

$$= \sum_{n: y^{(n)}=1} \left(\sum_{d=1}^{D} \log P\left(x_d^{(n)} \middle| \theta_{d,1}\right) \right)$$

$$+ \sum_{n: y^{(n)}=0} \left(\sum_{d=1}^{D} \log P\left(x_d^{(n)} \middle| \theta_{d,0}\right) \right) + \sum_{n=1}^{N} \log P\left(y^{(n)} \middle| \pi\right)$$

Setting the Parameters via MLE

- Binary label
  - $Y \sim \text{Bernoulli}(\pi)$
  - $\hat{\pi} = {}^{N_{Y=1}}/_{N}$
    - N = # of data points
    - $N_{Y=1}$ = # of data points with label 1
- Binary features
  - $X_d | Y = y \sim \text{Bernoulli}(\theta_{d,y})$
  - $\hat{\theta}_{d,y} = {}^{N_{Y=y,\, X_d=1}}/_{N_{Y=y}}$
    - $N_{Y=y}$ = # of data points with label y
    - $N_{Y=y,\, X_d=1}$ = # of data points with label y and feature $X_d = 1$

Bernoulli Naïve Bayes

- Binary label
  - $Y \sim \text{Bernoulli}(\pi)$
  - $\hat{\pi} = {N_{Y=1}}/{N}$
    - N = # of data points
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- Binary features
  - $X_d | Y = y \sim \text{Bernoulli}(\theta_{d,y})$
  - $\hat{\theta}_{d,y} = {N_{Y=y,\, X_d=1}}/{N_{Y=y}}$
    - $N_{Y=y}$ = # of data points with label y
    - $N_{Y=y,\, X_d=1}$ = # of data points with label y and feature $X_d = 1$

Multinomial Naïve Bayes

- Binary label
  - $Y \sim \text{Bernoulli}(\pi)$
  - $\hat{\pi} = {}^{N_{Y=1}}/_{N}$
    - N = # of data points
    - $N_{Y=1}$ = # of data points with label 1
- Discrete features (X_d can take on one of K possible values)
  - $X_d | Y = y \sim \text{Categorical}\left(\theta_{d,1,y}, \ldots, \theta_{d,K-1,y}\right)$
  - $\hat{\theta}_{d,k,y} = {}^{N_{Y=y,\, X_d=k}}/_{N_{Y=y}}$
    - $N_{Y=y}$ = # of data points with label y
    - $N_{Y=y,\, X_d=k}$ = # of data points with label y and feature $X_d = k$

Gaussian Naïve Bayes

- Binary label
  - $Y \sim \text{Bernoulli}(\pi)$
  - $\hat{\pi} = {N_{Y=1}}/{N}$
    - N = # of data points
    - $N_{Y=1}$ = # of data points with label 1
- Real-valued features
  - $X_d | Y = y \sim \text{Gaussian}\left(\mu_{d,y}, \sigma_{d,y}^2\right)$
  - $\hat{\mu}_{d,y} = \frac{1}{N_{Y=y}} \sum_{n: y^{(n)} = y} x_d^{(n)}$
  - $\hat{\sigma}_{d,y}^2 = \frac{1}{N_{Y=y}} \sum_{n: y^{(n)} = y} \left(x_d^{(n)} - \hat{\mu}_{d,y}\right)^2$
    - $N_{Y=y}$ = # of data points with label y

Visualizing Gaussian Naïve Bayes

- Fisher (1936) used 150 measurements of flowers from 3 different species: Iris setosa (0), Iris virginica (1), Iris versicolor (2) collected by Anderson (1936)

Species	Sepal Length	Sepal Width
0	4.3	3.0
0	4.9	3.6
0	5.3	3.7
1	4.9	2.4
1	5.7	2.8
1	6.3	3.3
1	6.7	3.0

Deleted two of the four features, so that input space is 2D

Visualizing Gaussian Naïve Bayes (2 classes)

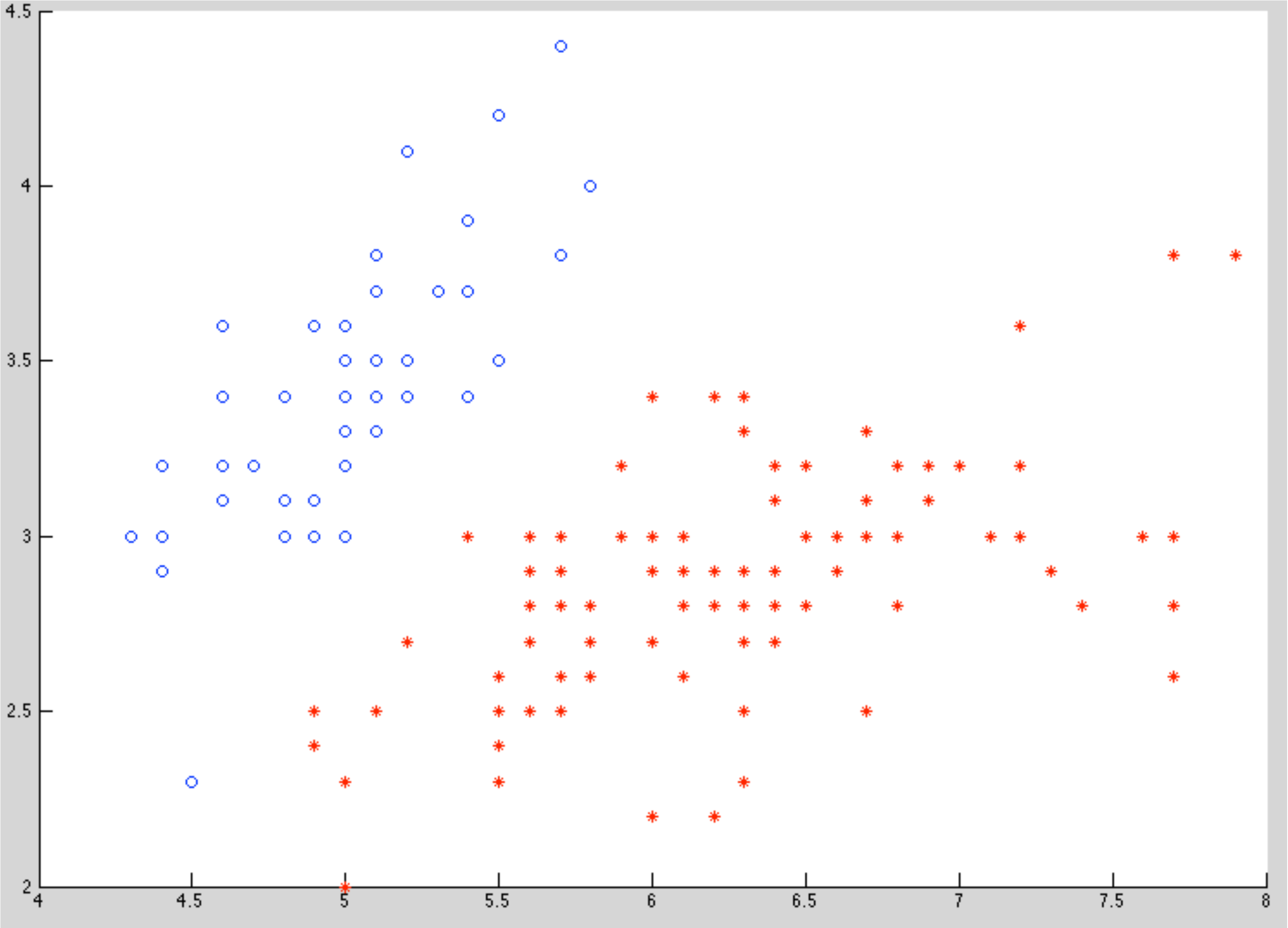

Figure courtesy of William Cohen

Visualizing Gaussian Naïve Bayes (2 classes)

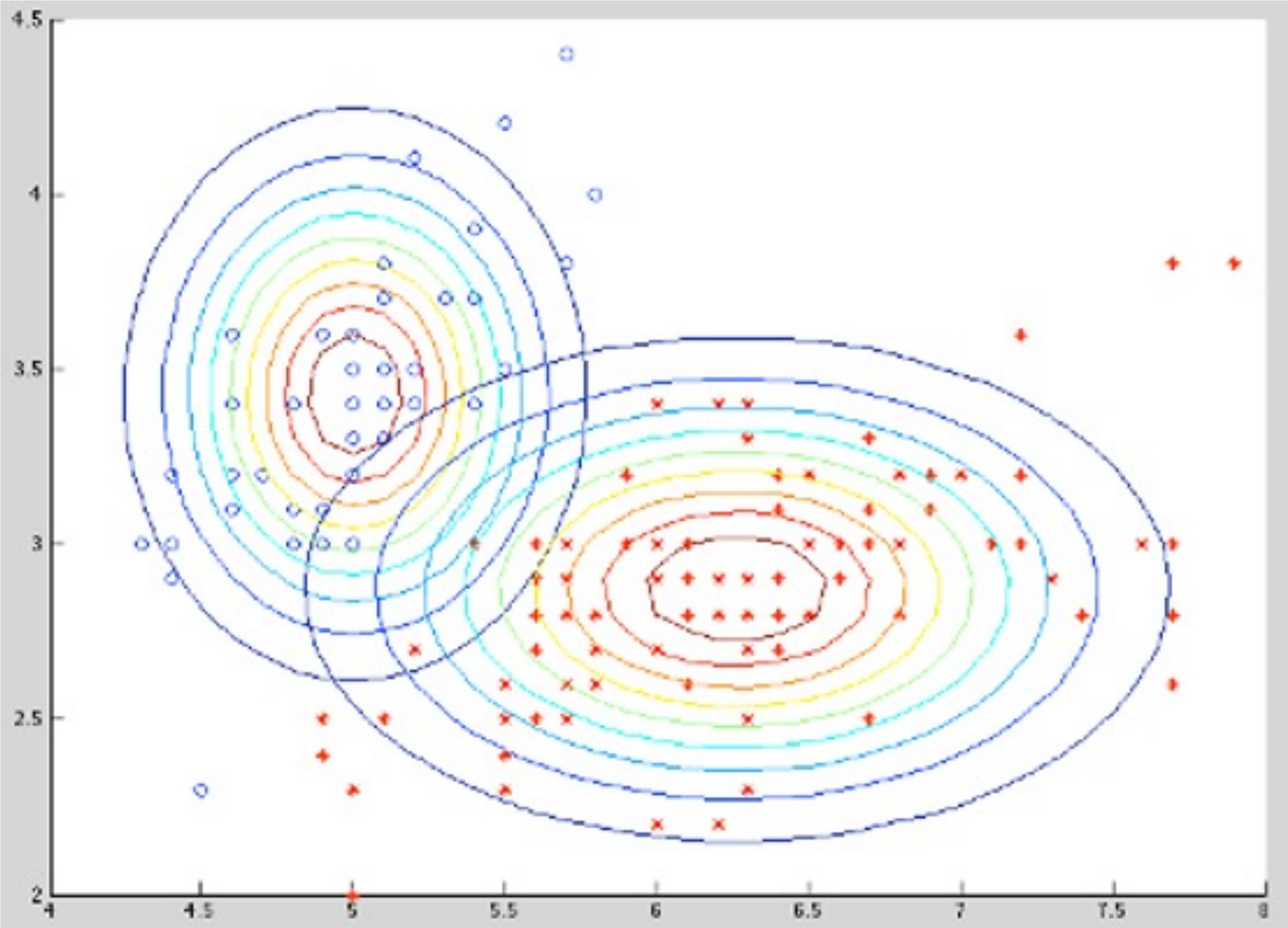

Figure courtesy of William Cohen

Visualizing Gaussian Naïve Bayes (2 classes, equal variances)

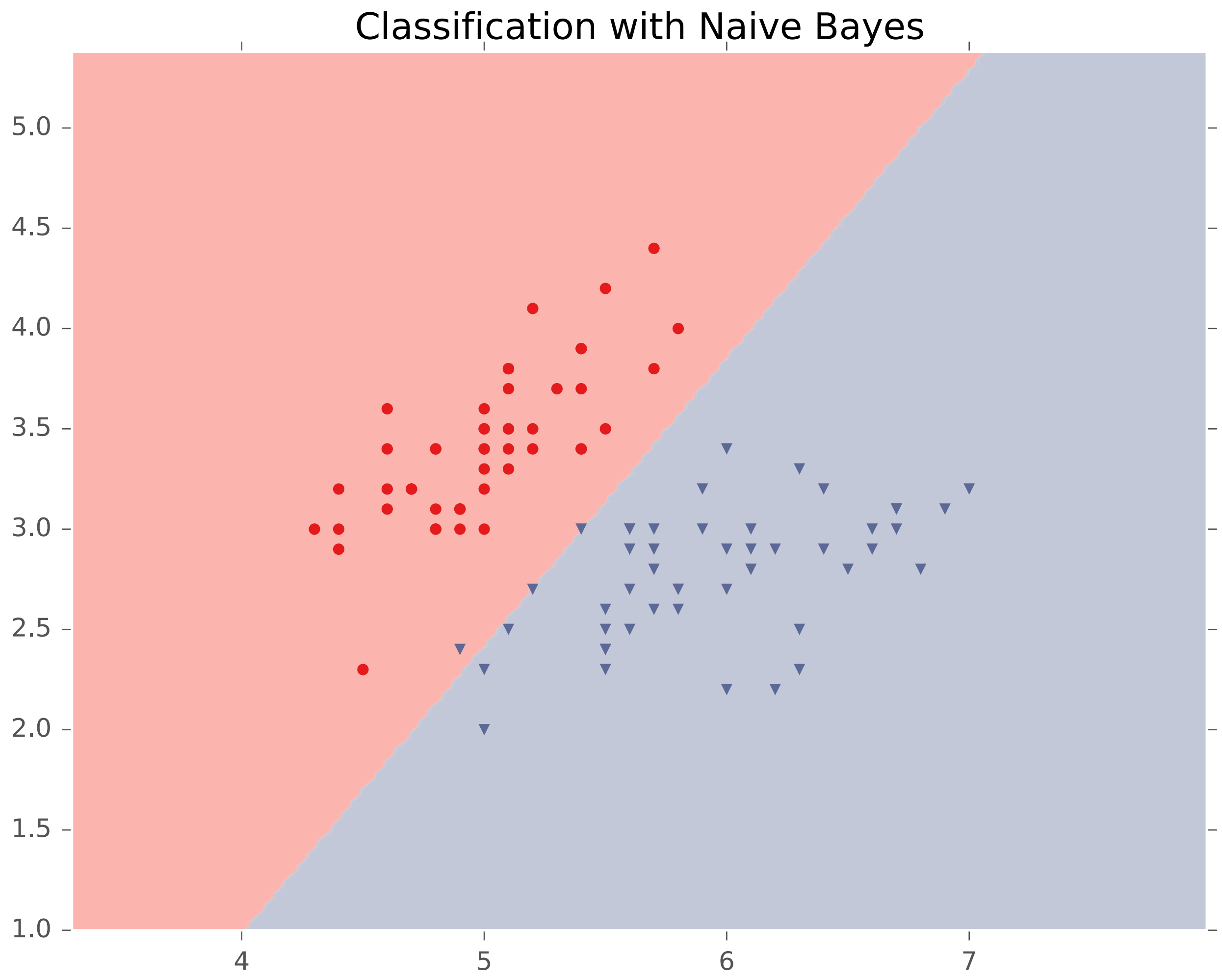

Figure courtesy of Matt Gormley

Visualizing Gaussian Naïve Bayes (2 classes, learned variances)

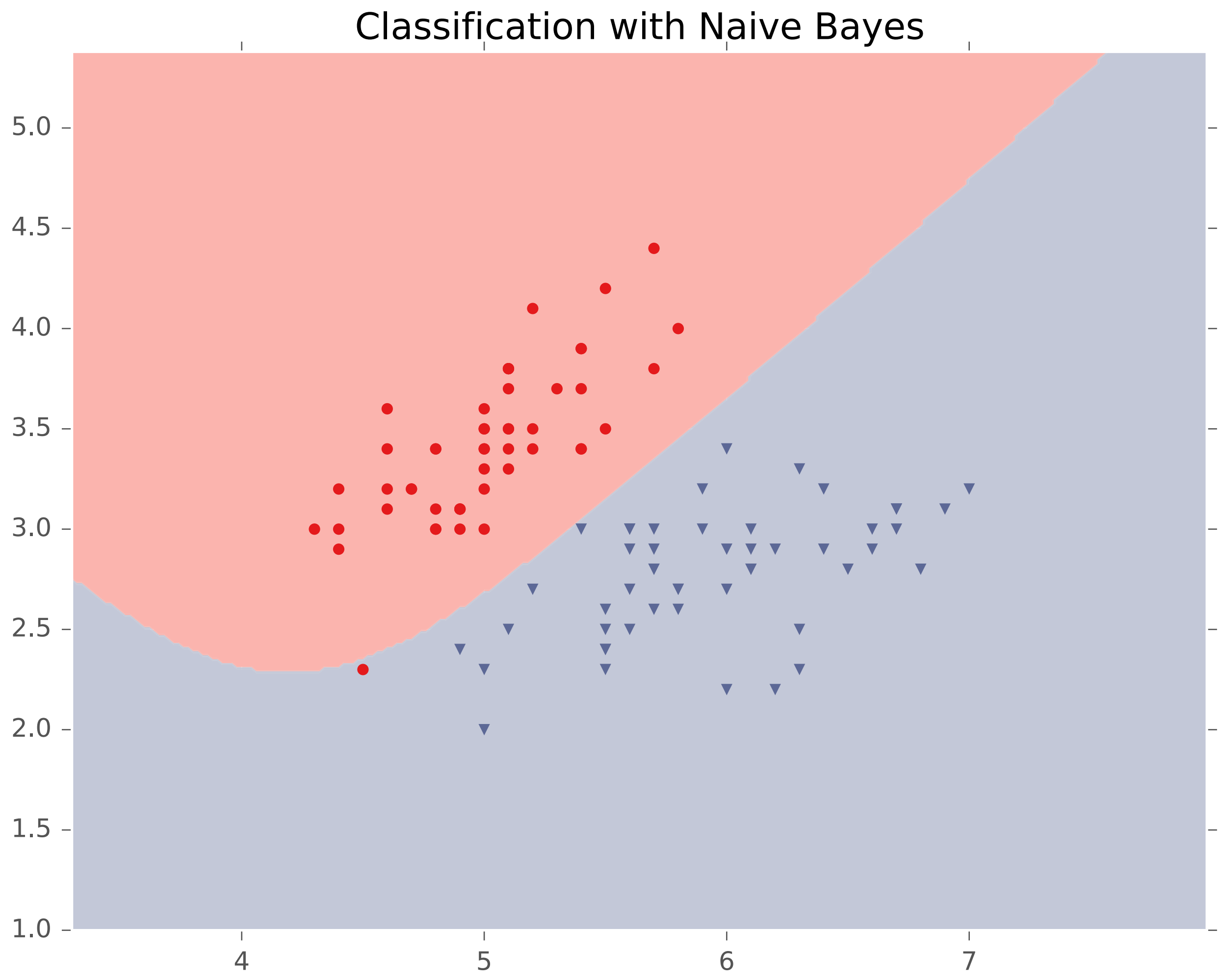

Figure courtesy of Matt Gormley

Bernoulli Naïve Bayes: Making Predictions

- Given a test data point $\boldsymbol{x}' = [x'_1, \ldots, x'_D]^T$

$$P(Y = 1|\boldsymbol{x}') \propto P(Y = 1)P(\boldsymbol{x}'|Y = 1)$$
$$= \hat{\pi} \prod_{d=1}^{D} \hat{\theta}_{d,1}^{x'_d} \left(1 - \hat{\theta}_{d,1}\right)^{1-x'_d}$$

$$P(Y = 0|\boldsymbol{x}') \propto (1 - \hat{\pi}) \prod_{d=1}^{D} \hat{\theta}_{d,0}^{x'_d} \left(1 - \hat{\theta}_{d,0}\right)^{1-x'_d}$$

$$\hat{y} = \begin{cases} 1 \text{ if } \hat{\pi} \prod_{d=1}^{D} \hat{\theta}_{d,1}^{x'_d} \left(1 - \hat{\theta}_{d,1}\right)^{1-x'_d} > \\ \quad (1 - \hat{\pi}) \prod_{d=1}^{D} \hat{\theta}_{d,0}^{x'_d} \left(1 - \hat{\theta}_{d,0}\right)^{1-x'_d} \\ 0 \text{ otherwise} \end{cases}$$

What if some Word-Label pair never appears in our training data?

- Given a test data point $\boldsymbol{x}' = [x'_1, \dots, x'_D]^T$

$$P(Y = 1|\boldsymbol{x}') \propto P(Y = 1)P(\boldsymbol{x}'|Y = 1)$$
$$= \hat{\pi} \prod_{d=1}^{D} \hat{\theta}_{d,1}^{x'_d} \left(1 - \hat{\theta}_{d,1}\right)^{1-x'_d}$$

$$P(Y = 0|\boldsymbol{x}') \propto (1 - \hat{\pi}) \prod_{d=1}^{D} \hat{\theta}_{d,0}^{x'_d} \left(1 - \hat{\theta}_{d,0}\right)^{1-x'_d}$$

$$\hat{y} = \begin{cases} 1 \text{ if } \hat{\pi} \prod_{d=1}^{D} \hat{\theta}_{d,1}^{x'_d} \left(1 - \hat{\theta}_{d,1}\right)^{1-x'_d} > \\ \quad (1 - \hat{\pi}) \prod_{d=1}^{D} \hat{\theta}_{d,0}^{x'_d} \left(1 - \hat{\theta}_{d,0}\right)^{1-x'_d} \\ 0 \text{ otherwise} \end{cases}$$

What if some Word-Label pair never appears in our training data?

x_1 ("hat")	x_2 ("cat")	x_3 ("dog")	x_4 ("fish")	x_5 ("mom")	x_6 ("dad")	y (Dr. Seuss)
1	1	0	0	0	0	1
0	0	1	0	0	0	0
0	0	0	1	0	0	1
0	0	0	0	1	0	0

The Cat in the Hat gets a Dog (by ???)

- If some $\hat{\theta}_{d,y} = 0$ and that word appears in our test data $\boldsymbol{x}'$, then $P(Y = y|\boldsymbol{x}') = 0$ even if all the other features in $\boldsymbol{x}'$ point to the label being y!
- The model has been overfit to the training data…
- We can address this with a prior over the parameters!

Setting the Parameters via MAP

- Binary label
  - $Y \sim \text{Bernoulli}(\pi)$
  - $\hat{\pi} = {}^{N_{Y=1}}/_{N}$
    - N = # of data points
    - $N_{Y=1}$ = # of data points with label 1
- Binary features
  - $X_d | Y = y \sim \text{Bernoulli}(\theta_{d,y})$ and $\theta_{d,y} \sim \text{Beta}(\alpha, \beta)$
  - $\hat{\theta}_{d,y} = {}^{N_{Y=y,\, X_d=1} + (\alpha - 1)}/_{N_{Y=y} + (\alpha-1) + (\beta-1)}$
    - $N_{Y=y}$ = # of data points with label y
    - $N_{Y=y,\, X_d=1}$ = # of data points with label y and feature $X_d = 1$
    - α and β are "pseudocounts" of imagined data points that help avoid zero-probability predictions.
    - Common choice: $\alpha = \beta = 2$

What can we do when this is a bad/incorrect assumption, e.g., when our features are words in a sentence?

- Define a model and model parameters
  - **Make the Naïve Bayes assumption**
  - Assume independent, identically distributed (iid) data
  - Parameters: $\pi = P(Y = 1), \theta_{d,y} = P(X_d = 1|Y = y)$
- Write down an objective function
  - Maximize the log-likelihood
- Optimize the objective w.r.t. the model parameters
  - Solve in *closed form*: take partial derivatives, set to 0 and solve

Key Takeaways

- Text data
  - Bag-of-words feature representation
- Naïve Bayes
  - Conditional independence assumption
    - Pros and cons
  - Different Naïve Bayes models based on type of features
  - MLE vs. MAP for Bernoulli Naïve Bayes