Sample Solution for Problem 1.a

1 Inverted Pendulum Model (IPM)

1.1 Equations of Motion and Ground Reaction Forces

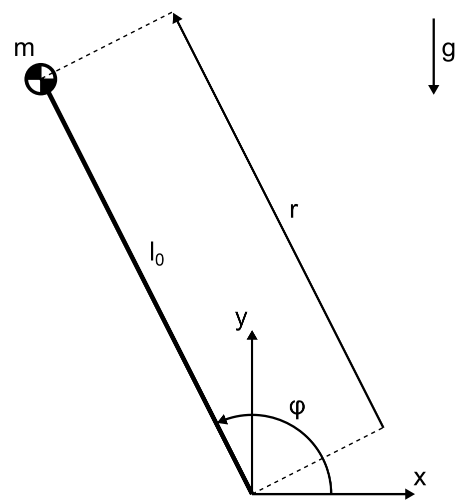

Figure 1: Scheme of the Inverted Pendulum Model (IPM).

The equations of motion of this dynamical system can be derived with Lagrange's method:

$$\begin{cases} E_k &= \frac{1}{2}m(r^2\dot{\varphi}^2 + \dot{r}^2) \\ V &= mgr\sin\varphi\ , \end{cases}$$

where E_k represents the system's kinetic energy and V its potential energy. This leads to the following equations of motion:

$$m\ddot{r} - mr\dot{\varphi}^2 + mg\sin\varphi = 0 \tag{1}$$

$$mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} + mgr\cos\varphi = 0\ , \tag{2}$$

corresponding respectively to the radial direction (free coordinate r) and to the angular direction (free coordinate φ).

From Eq. (1) it is possible to obtain the axial force on the pendulum leg:

$$F_r = m\ddot{r}$$

$$\Rightarrow \boxed{F_r = ml_0\dot{\varphi}^2 - mg\sin\varphi}$$

The inverted pendulum has a fixed pendulum length, meaning that $r = l_0$ and $\dot{r} = 0$. Substituting this constraint in Eq. (2) reduces the total degrees of freedom to one and yields the IPM equation of motion:

$$\boxed{\ddot{\varphi} = -\frac{g}{l_0}\cos\varphi}$$

The ground reaction force GRF is equal and opposite to the leg force ($GRF = -F_r$). Its horizontal and vertical components are

$$\boxed{\begin{aligned} GRF_x &= -F_r\cos\varphi \\ GRF_y &= -F_r\sin\varphi \end{aligned}}$$

1.2 Initial Conditions

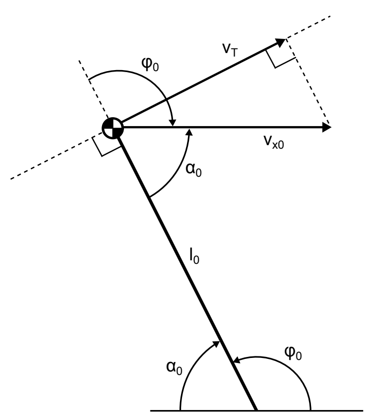

Figure 2: Scheme of the IPM initial conditions.

The initial angle of the pendulum is

$$\boxed{\varphi_0 = \pi - \alpha_0}$$

The tangential speed of the COM in the first instant is

$$v_T = v_{x0}\sin\varphi_0 = v_{xo}\sin\alpha_0$$

The initial angular speed of the pendulum is

$$\dot{\varphi} = -\frac{v_T}{l_0}$$

$$\Rightarrow \boxed{\dot{\varphi} = -\frac{v_{x0}\sin\alpha_0}{l_0}}$$

1.3 Implementation

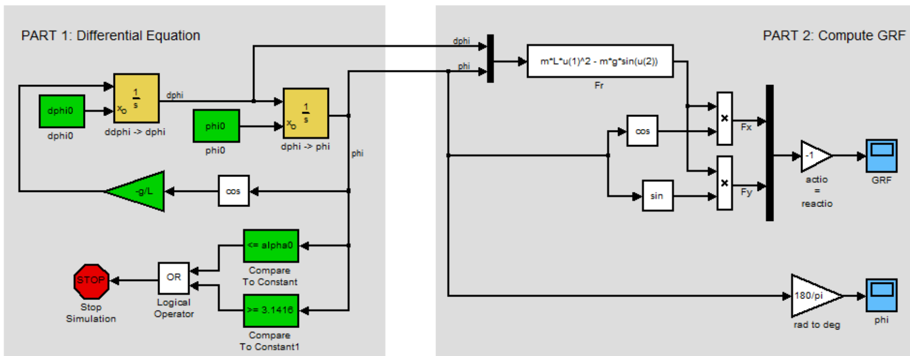

Figure 3: Implementation of the IPM and computation of the ground reaction forces.

The IPM has been implemented and integrated with Simulink. A related m-file initializes the position and velocity of the pendulum. The ground reaction forces in the case of the assigned parameters are shown in Fig.(4).

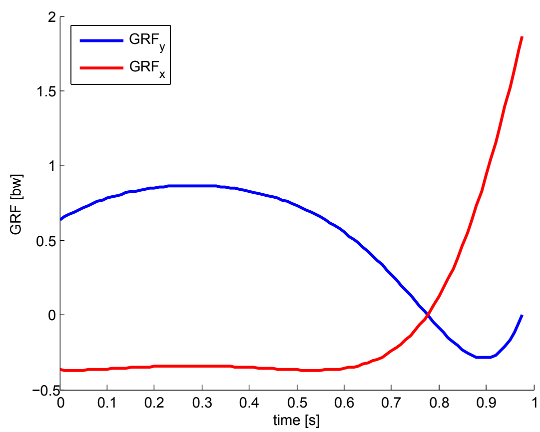

Figure 4: GRF of the IPM with the assigned parameters.

The ground reaction forces have an unexpected shape because the pendulum does not complete the stance, but falls behind before reaching the peak of the circular trajectory. This is evident considering Fig.(5), where the position of the pendulum is shown. The angle initially rises, but it stops before reaching midstance ($\varphi = 90$) and it inverts its motion, eventually falling back on the ground ($\varphi = 180$).

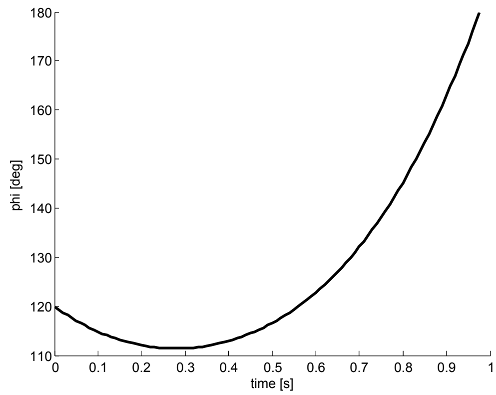

Figure 5: Inverted pendulum angular position with the assigned parameters.

2 Spring-Mass Model

2.1 Equations of Motion and Ground Reaction Forces

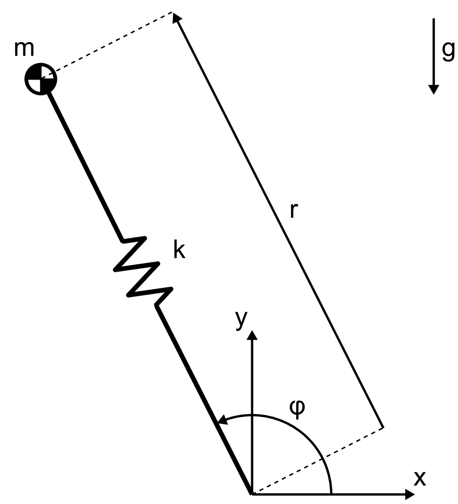

Figure 6: Scheme of the Spring-Mass Model.

The equations of motion of this dynamical system can be derived with Lagrange's method:

$$\begin{cases} E_k &= \frac{1}{2}m(r^2\dot{\varphi}^2 + \dot{r}^2) \\ V &= mgr\sin\varphi + \frac{1}{2}k(l_0 - r)^2 \end{cases},$$

where E_k represents the system's kinetic energy and V its potential energy.

This leads to the following equations of motion:

$$\boxed{\ddot{r} = r\dot{\varphi}^2 - g\sin\varphi + \frac{k}{m}(l_0 - r)}$$

$$\boxed{\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi} + g\cos\varphi}{r}},$$

corresponding respectively to the radial direction (free coordinate r) and to the angular direction (free coordinate φ).

The COM is connected to the ground through the elastic leg. This means that the force exerted by the ground on the point foot (the ground reaction force) is equal to the force exerted by the spring on the COM:

$$\boxed{GRF = F_s = k(l_0 - r)}$$

Its horizontal and vertical componets are

$$\boxed{\begin{aligned} GRF_x &= F_s\cos\varphi \\ GRF_y &= F_s\sin\varphi \end{aligned}}$$

2.2 Initial Conditions

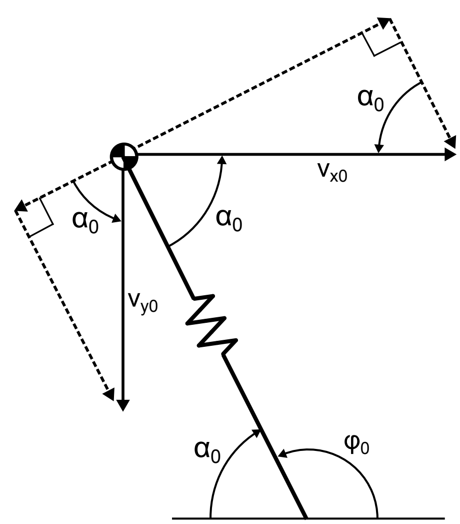

Figure 7: Scheme of the Spring-Mass Model initial conditions.

The initial position of the pendulum is determined by

$$\boxed{\begin{aligned} r_0 &= l_0 \\ \varphi_0 &= \pi - \alpha_0 \end{aligned}}$$

The initial tangential velocity v_T and radial velocity v_R of the COM are

$$\begin{aligned} v_T &= -v_{x0}\sin\alpha_0 + v_{y0}\cos\alpha_0 \\ v_R &= -v_{x0}\cos\alpha_0 - v_{y0}\sin\alpha_0 \ , \end{aligned}$$

where the signs follow the conventions shown in Fig.(7).
In polar coordinates, the initial velocity of the pendulum is

$$\begin{aligned} \dot{r}_0 &= v_R \\ \dot{\varphi}_0 &= v_T/l_0 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \dot{r}_0 &= -v_{x0}\cos\alpha_0 - v_{y0}\sin\alpha_0 \\ \dot{\varphi}_0 &= (-v_{x0}\sin\alpha_0 + v_{y0}\cos\alpha_0)/l_0 \end{aligned}}$$

2.3 Implementation

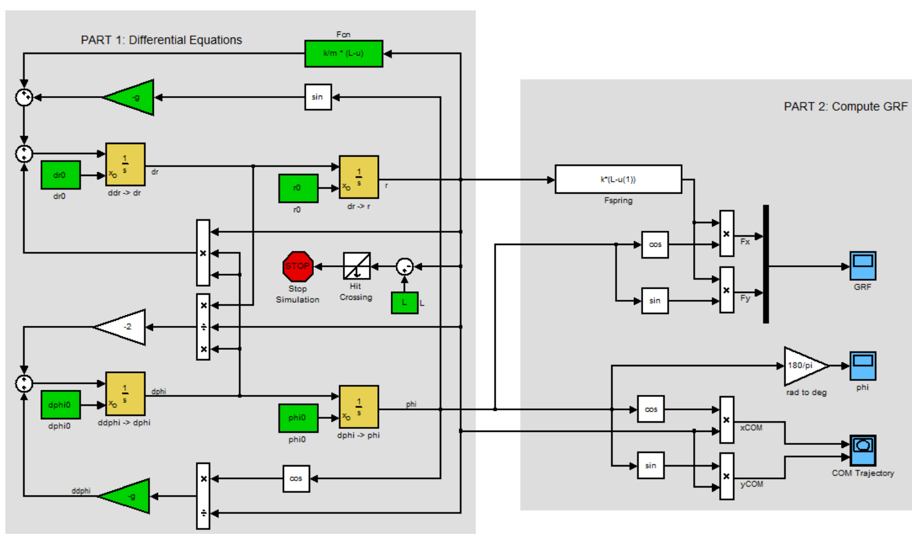

Figure 8: Implementation of the Spring-Mass Model and computation of the ground reaction forces.

The Spring-Mass Model has been implemented and integrated with Simulink. A related m-file initializes the position and velocity of the system. The ground reaction forces in the case of the assigned parameters are shown in Fig.(9).

The ground reaction forces are symmetric with respect to the time spent in stance. However, Fig.(10) shows that with the assigned parameters the stance phase starts at $\varphi = 110$ and ends at $\varphi \approx 55$. This means that the ground reaction forces are not not symmetric with respect to midstance (which is crossed by the COM shortly after touch down).

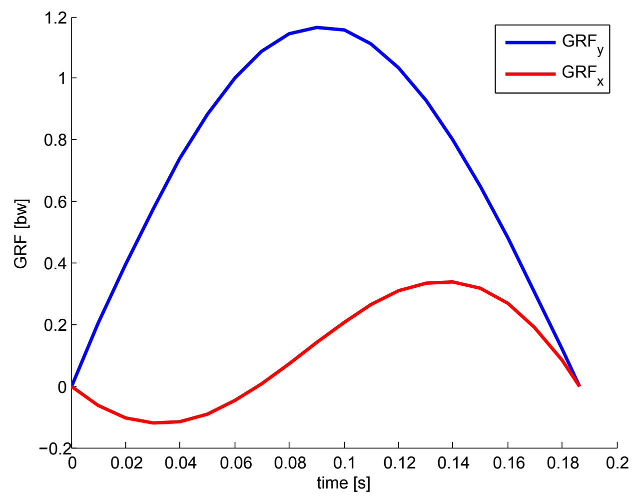

Figure 9: GRF of the Spring-Mass Model with the assigned parameters.

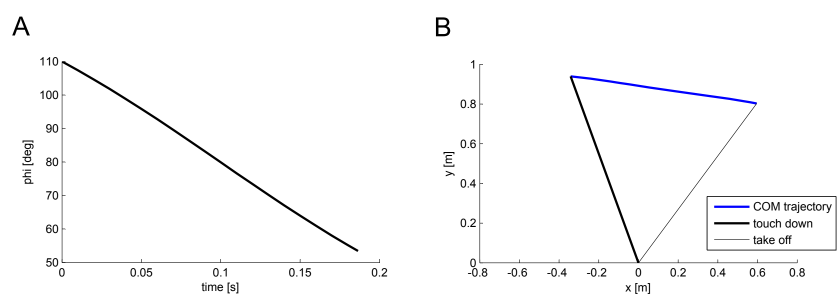

Figure 10: Spring-Mass Model angular position (A) and COM trajectory (B) with the assigned parameters.

Sample Solution for Problem 1.b

1 Inverted Pendulum Model (IPM)

There are many ways to change the parameters in order to make the IPM ground reaction forces symmetric about midstance. First of all, the inverted pendulum must successfully reach midstance (the peak of its circular trajectory) and go beyond it. This could be achieved, for example, by increasing the biped's initial horizontal speed v_{x0} or its angle of attack α_0. One possible solution consists in leaving all the assigned parameters unchanged except for the initial speed, whose value is increased from 1.3 m/s to 1.9 m/s. This yields a successful stance and ground reaction forces symmetric about midstance (Fig. 1 and 2).

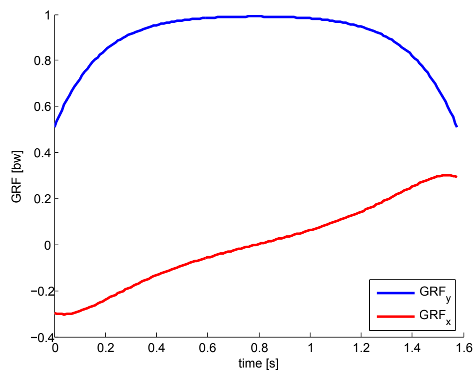

Figure 1: GRF of the IPM with modified parameters; symmetry about midstance is now achieved.

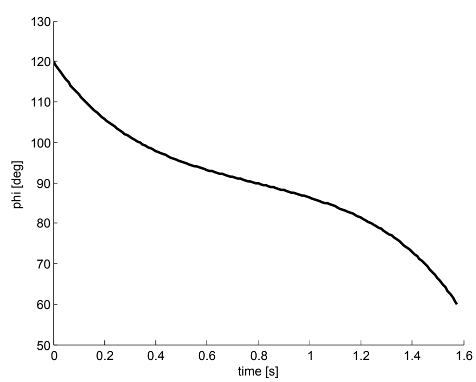

Figure 2: Inverted pendulum angular position with modified parameters; symmetry about midstance is now achieved.

2 Spring-Mass Model

As in the previous case, there are many possible combinations of parameters capable of making the ground reaction forces of the Spring-Mass Model symmetric about midstance. This time, though, the assigned parameters result in a COM trajectory which is skewed to the right. Intuitive solutions to make the stance symmeric include reducing the initial horizontal velocity v_{x0} of the system, and decreasing its angle of attack α_0. One possible solution consists in leaving all the parameters unchanged except for the angle of attack, whose value is decreased from 70° to 60°. This makes both stance and ground reaction forces symmetric about midstance (Fig. 3 and 4).

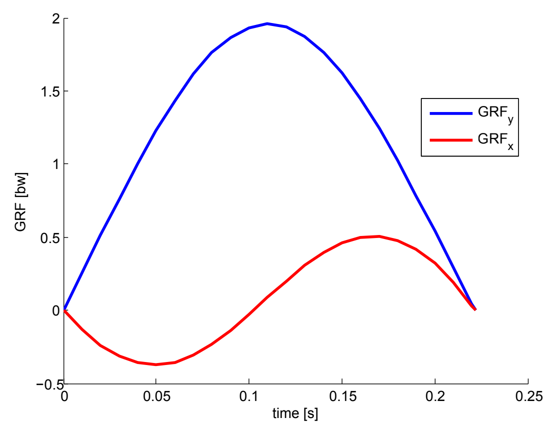

Figure 3: GRF of the Spring-Mass Model with modified parameters; symmetry about midstance is now achieved.

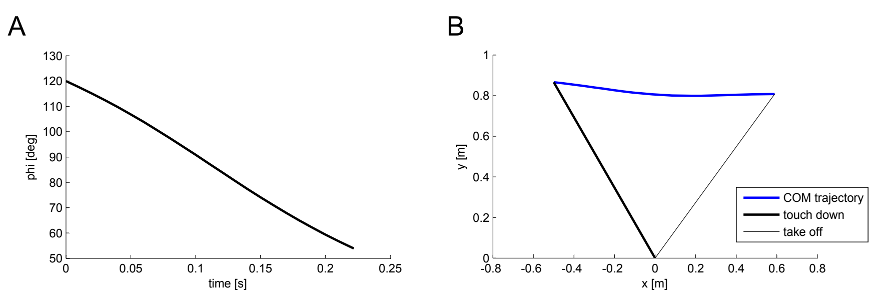

Figure 4: Spring-Mass Model angular position (A) and COM trajectory (B) with modified parameters; symmetry about midstance is now achieved.

Sample Solution for Problem 1.c

1 Inverted Pendulum Model (IPM)

The ground reaction forces computed with the IPM (Fig. 1) show some similarities with respect to the typical patterns experimentally measured in human walking (Fig. 2). However, a detailed comparison (Fig. 3) reveals that the inverted pendulum model is unable to accurately reproduce the dynamics of human walking.

The vertical ground reaction force (GRF_y) recorded for walking humans and animals presents a particular M shape. This pattern is determined by the dynamic load sharing that characterizes walking gaits. The first peak indicates the progressive loading of the front leg after touch down, while the second peak reflects the passage of the body weight onto the other leg. The central decrease corresponds to the upward motion of the center of mass at midstance (after it has been pushed by the combined stiffness of both legs).

The IPM does not model double support, and this makes it unable to capture the load sharing dynamics. As a consequence, instead of the M shape (with both peaks above 1 bw) appears a convex pattern (which never reaches 1 bw). This pattern also introduces unrealistic finite discontinuities in GRF_y, both at touch down and at take off.

The experimental horizontal ground reaction force (GRF_x) of walking animals presents a modulated sinusoidal shape. In this case the IPM predictions are more accurate, although they still introduce discontinuities at the beginning and at the end of stance.

The shape of ground reaction forces in human walking is discussed in [1]. A bipedal spring-mass model is able to overcome the limits of the inverted pendulum model, capturing the essential dynamics of both walking and running gaits [2].

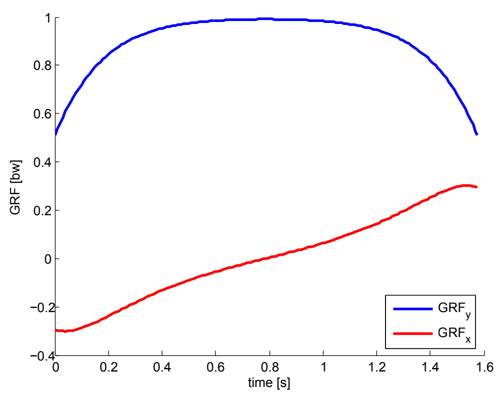

Figure 1: Ground reaction forces of the Inverted Pendulum Model (IPM).

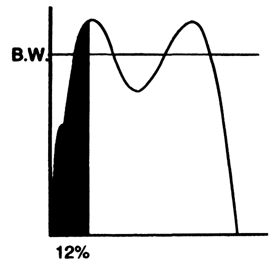

Figure 2: Experimental ground reaction forces typical of human walking. The full loading of the front leg after touch down (loading response, colored in black) takes about 12% of the total stance length [1].

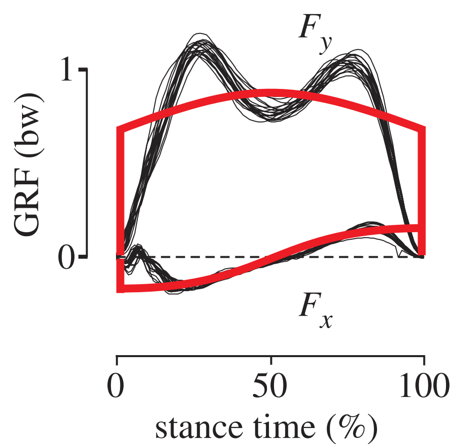

Figure 3: Comparison between the ground reaction forces of the Inverted Pendulum Model (red) and the ones typical of human walking (black) [2].

2 Spring-Mass Model

Experimental measurements of the ground reaction forces which characterize human and animal running identify typical patterns both for GRF_y and GRF_x [3]. Vertical forces present a bell-shaped behavior which peaks at about mid-stance, while horizontal forces show a sinusoidal behavior (simpler than the one observed during walking). Both patterns start and end at zero (the swing leg bears no load).

The spring mass model is able to describe the dynamic behavior of running or hopping animals [4] [2]. The overall shape of both components of the ground reaction force is correctly rendered. No discontinuities are introduced at touch down or take off.

The only particular in running dynamics that the spring-mass model does not include is the small hump in GRF_y that immediately follows touch down. This local variation is caused by the impact of the swing leg with the ground, and by the consequent wobbling of muscle tissue. A similar phenomenon occurs, with a lower degree of intensity, during walking [1].

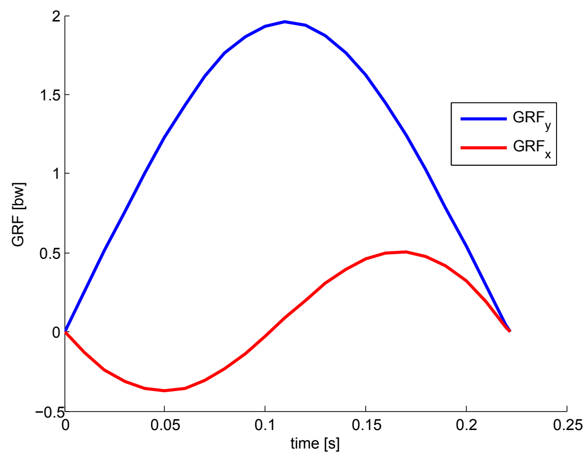

Figure 4: Ground reaction forces of the Spring-Mass Model.

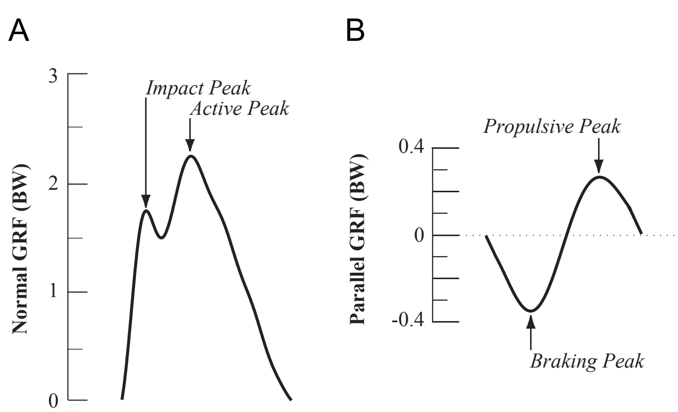

Figure 5: Experimental ground reaction forces typical of human running. Horizontal component (A) and vertical component (B) [3].

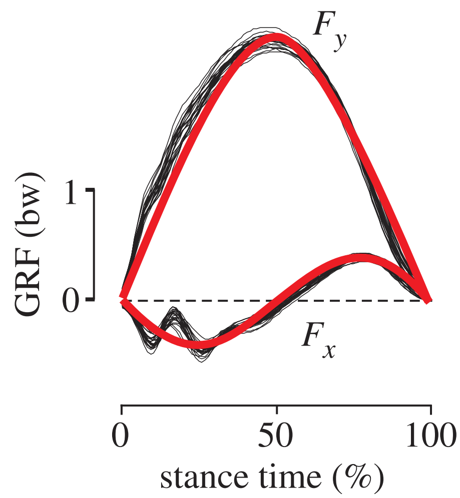

Figure 6: Comparison between the ground reaction forces of the Spring-Mass Model (red) and the ones typical of human running (black) [2].

References

[1] J. Perry, *Gait Analysis: Normal and Pathological Function*, 1st ed. Delmar Learning, January 1992.

[2] H. Geyer, A. Seyfarth, and R. Blickhan, "Compliant leg behaviour explains basic dynamics of walking and running." *Proceedings. Biological sciences / The Royal Society*, vol. 273, no. 1603, pp. 2861–2867, November 2006.

[3] J. S. Gottschall and R. Kram, "Ground reaction forces during downhill and uphill running," *Journal of Biomechanics*, vol. 38, no. 3, pp. 445 – 452, 2005.

[4] R. Blickhan, "The spring-mass model for running and hopping," *Journal of Biomechanics*, vol. 22, no. 11-12, pp. 1217 – 1227, 1989.