A General Modal Framework for the Event Calculus and its Skeptical and Credulous Variants

Iliano Cervesato[1] Luca Chittaro[2] Angelo Montanari[2]

[1] Department of Computer Science
Carnegie Mellon University
5000 Forbes Avenue – Pittsburgh, PA 15213-3891
Phone: (412)-268-1413 – Fax: (412)-681-5739
E-mail: *iliano@cs.cmu.edu*

[2] Dipartimento di Matematica e Informatica
Università di Udine
Via delle Scienze, 206 – 33100 Udine, Italy
Phone: +39-432-5584 50|77 – Fax: +39-432-558499
E-mail: {*chittaro* | *montana*}*@dimi.uniud.it*

Abstract

We propose a general and uniform modal framework for the Event Calculus (EC) and its skeptical and credulous variants. The resulting temporal formalism, called the Generalized Modal Event Calculus (GMEC), extends considerably the expressive power of EC when information about the ordering of events is incomplete. It provides means of inquiring about the evolution of the maximal validity intervals of properties relatively to all possible refinements of the ordering data by allowing free mixing of propositional connectives and modal operators. We first give a semantic definition of GMEC; then, we propose a declarative encoding of GMEC in the language of hereditary Harrop formulae and prove the soundness and completeness of the resulting logic programs[1].

Keywords: Temporal Reasoning, Non-monotonic Reasoning, Modal Logic, Hereditary Harrop Formulae.

[1] A short version of this paper has been published in the Proceedings of the 12^{th} European Conference on Artificial Intelligence [17].

Appeared as Reasearch Report 37/96-RR, Dipartimento di Matematica e Informatica, Università di Udine, Italy, July 1996.

Contents

List of Figures

This web of times — the strands of which approach to one another, bifurcate, intersect or ignore each other through the centuries — embraces every possibility. We do not exist in most of them. In some you exist and not I, while in others I do, and you do not, and in yet others both of us exist. In this one, in which chance has favored me, you have come to my gate. In another, you, crossing the garden, have found me dead. In yet another, I say these very same words, but am an error, a phantom.

— *The Garden of the Forking Paths*, Jorge Luis Borges

1 Introduction

This paper proposes a general and uniform modal framework for Kowalski and Sergot's Event Calculus (EC) [12] and its skeptical and credulous variants [1, 2, 3]. Given a set of event occurrences, EC allows one to derive maximal validity intervals (MVIs hereafter) over which properties initiated or terminated by those events hold. As new events or additional ordering information about known events are recorded, EC updates accordingly the set of MVIs. Most approaches based on EC assume the occurrence time of each event to be known; here, we explore the case of partially ordered events devoid of an explicit occurrence time. In this case, EC is neither able to derive all admissible MVIs nor to distinguish which of the derived intervals are defeasible and which are not.

The problem of computing which facts must be or may possibly be true over certain time intervals in presence of partially ordered events has been already addressed in the literature, e.g. [1, 2, 3, 5, 6, 16, 19], and case studies in the domains of diagnosis and planning have been analyzed in [3] and [16], respectively. In [5], Dean and Boddy showed that this computation is intractable in the general case and proposed polynomial approximations that compute either a subset of necessary facts or a superset of possible ones.

In [2] and [3], we defined two variants of EC, called Skeptical EC (SKEC) and Credulous EC (CREC), which respectively compute the necessarily true MVIs and the possibly true MVIs in the restricted setting where the occurrence of events is not subject to preconditions[2]. SKEC and CREC can be given a polynomial implementation, that can be further enhanced by exploiting *transitive reduction* graph processing techniques [4]. In [1], we defined a uniform modal interpretation for EC, SKEC and CREC, called the Modal Event Calculus (MEC). MEC deals with atomic formulae (MVIs computed EC) as well as simply modalized atomic formulae, i.e. atomic formulae prefixed by only one modality (MVIs computed by SKEC and CREC). It is provided with a sound and complete axiomatic formulation in a logic programming framework.

In this paper, we define a Generalized Modal Event Calculus (GMEC) that extends MEC by allowing a free mixing of propositional connectives and modal operators. Such a capability is useful to deal with real-world applications, as pointed out in [3]. We initially capture the intuitions underlying GMEC by giving a *semantic* formulation of EC and extending it to a modal interpretation that takes into account all possible refinements of the ordering data. Then, we provide GMEC with a sound and complete *axiomatization* in the language of hereditary Harrop formulae and rely on a *proof-theoretic* approach for proving the faithfulness of our implementations with respect to the behavior of GMEC, as expressed by the semantics.

We believe that our approach contributes to the conceptual understanding of EC, an important but not yet fully understood formalism for reasoning about events and their effects. Moreover, the proposed method can be exploited to increase the confidence in alternative axiomatizations of EC by proving

[2] It is worth noting that as soon as we abandon this restricted setting and allow one to use preconditions or boolean connectives, (the generalized versions of) SKEC and CREC respectively compute a subset of the necessarily true MVIs and a superset of the possibly true MVIs. Consider the following example. Take two properties p and q which are respectively initiated by events e_1 and e_2, provided that precondition r holds. Moreover, suppose that both events terminate r. Consider a scenario where (i) both e_1 and e_2 occurred, but their ordering is unknown, and (ii) an event e_0, occurred before e_1 and e_2, initiated r. A precise temporal reasoner should conclude that *either* p *or* q must hold. On the contrary, CREC concludes that *both* p *and* q may hold, while SKEC concludes that *neither* p *nor* q necessarily hold.

them sound and complete with respect to the corresponding semantics on a syntactic (proof-theoretic) ground. Finally, it seems suited to act as a general framework for studying significant extensions of EC (e.g. GMEC). We expect this approach to be applicable to related formalisms as well (e.g. the situation calculus).

The paper is organized as follows. In Section 2, we first recall some basic definitions about orderings and tailor them to the needs of the subsequent discussion; then, we formally define GMEC and presents its fundamental properties. In Section 3, we summarize the definition and operational semantics of hereditary Harrop formulae and use this language to give two sound and complete encodings of GMEC. The conclusions provide an assessment of the work done and discuss future developments. For the sake of readability, we have collected the proofs of the results presented in Section 3 in the Appendix.

2 The Generalized Modal Event Calculus

In this section, we formally define the Generalized Modal Event Calculus (GMEC). We consider the case in which the set of event occurrences has been fixed once and for all and the input process consists in the addition of information about the relative ordering of event pairs. Furthermore, we assume that events do not happen simultaneously and that the ordering information is always consistent.

The section is organized as follows. We first recall some notions about ordering relations. Then, we provide EC with a semantic interpretation that validates, in the current knowledge state, precisely the MVIs computed by EC. By considering all possible knowledge states with the associated reachability relation, this model is naturally lifted to a modal interpretation. The corresponding extension of EC with propositional connectives and modalities substantially augments the expressive power of EC. Next, we formally state a number of properties of the proposed formalization that will be later exploited to increase the efficiency of GMEC implementations.

2.1 Ordering Relations

In the following, we will rely upon different notions of ordering and ordered set. The ordering information, as usually *represented* in EC, constitutes a quasi-order, i.e. an ordering relation missing some transitive links; however, this information is *used* in EC as a strict order. Moreover, the structure representing the possible evolutions of the ordering data constitutes a non-strict order.

Definition 2.1 (*Quasi-orders, strict orders, non-strict orders*)

Let E be a set and R a binary relation on E. R is called a quasi-order *if it is acyclic; a* strict order *if it is irreflexive, asymmetric and transitive; a* non-strict order *if it is reflexive, antisymmetric and transitive. The structure (E, R) is respectively called a* quasi-ordered set, *a* strictly ordered set *and a* non-strictly ordered set. □

We denote the sets of all quasi-orders and of all strict orders on E as O_E and W_E, respectively. It is easy to show that, for any set E, $W_E \subseteq O_E$ (actually, $W_E \subset O_E$ if E has at least three elements). We will use the letters o and w possibly subscripted to denote quasi-orders and strict orders, respectively.

We indicate the transitive closure of a relation R as R^+. Clearly, if (E, o) is a quasi-ordered set, then (E, o^+) is a strictly ordered set. Two quasi-orders $o_1, o_2 \in O_E$ are *equally informative* if $o_1^+ = o_2^+$. This induces an equivalence relation $\sim$ on O_E. It is easy to prove that, for any set E, $O_{E/\sim}$ and W_E are isomorphic. In the following, we will often identify a quasi-order o with the corresponding element o^+ of W_E.

The set $\mathbf{2}^{E\times E}$ of all binary relations on E naturally becomes a non-strictly ordered set when considered together with the usual subset relation $\subseteq$. Moreover, $(\mathbf{2}^{E\times E}, \cup, \cap, \overline{\ }, E \times E, \emptyset)$ is a boolean lattice. Since W_E is a subset of $\mathbf{2}^{E\times E}$, the restriction of $\subseteq$ to this set still forms a non-strict order. Indeed, we have that, for any set E, $(W_E, \subseteq)$ is a non-strictly ordered set. It can be easily proved

that $(W_E, \cap, \emptyset)$ forms a lower semi-lattice. Moreover, for any $w_1, w_2 \in W_E$, the relation $w_1 \uparrow w_2 = (w_1 \cup w_2)^+$ is the least upper bound (*lub*) of w_1 and w_2 whenever this element belongs to W_E. Note that $w_1 \uparrow w_2 \notin W_E$ if w_1 and w_2 contain symmetric pairs.

Given w in W_E, any $w' \in W_E$ such that $w \subseteq w'$ is called an *extension* of w. We denote the set of all extensions of w as $Ext(w)$. We have that for any $w \in W_E$, if $(e_1, e_2) \in w$, then for all $w' \in Ext(w)$, $(e_1, e_2) \in w'$. For any $w \in W_E$, $Ext(w)$ enjoys the same properties of W_E. More precisely, $(Ext(w), \subseteq)$ is a non-strictly ordered set, $(Ext(w), \cap, w)$ is a lower semi-lattice, and $\uparrow$ characterizes the partial operation of *lub* over this semi-lattice. Notice in particular that $Ext(\emptyset) = W_E$.

Whenever E is a finite set, also W_E is finite since it is a subset of $2^{E \times E}$. Moreover, all $Ext(w)$ for $w \in W_E$ are finite as well. This property allows us to prove statements by induction on the cardinality of $Ext(w)$ for $w \in W_E$ and E finite. We will need this fact in the proofs of the results of Section 3.

We conclude the treatment of orderings by giving some definitions related to the notion of interval. Let E be a set and $w \in W_E$. A pair $(e_1, e_2) \in w$ is called an *interval* of w. Given two *distinct* intervals (e_1, e_2) and (e'_1, e'_2) over w, we say that (e_1, e_2) is a *subinterval* of (e'_1, e'_2) (or (e'_1, e'_2) is a *superinterval* of (e_1, e_2)) with respect to w if either $e_1 = e'_1$ or $(e'_1, e_1) \in w$ and dually $e_2 = e'_2$ or $(e_2, e'_2) \in w$. We write in this case $(e_1, e_2) \sqsubset_w (e'_1, e'_2)$. We have that, for any ordering $w \in W_E$, $(w, \sqsubset_w)$ is a strictly ordered set.

2.2 Formalization of GMEC

EC proposes a general approach to representing and reasoning about events and their effects in a logic programming framework. It takes the notions of event, property, time-point and time-interval as primitives and defines a model of change in which *events* happen at *time-points* and initiate and/or terminate *time-intervals* over which some *property* holds. EC also embodies a notion of default persistence according to which properties are assumed to persist until an event that interrupts them occurs (an event e interrupts the validity of a property p if it initiates/terminates p itself or a property q which is incompatible with p). This formalism was originally designed to compute the maximal validity intervals (MVIs) over which properties hold uninterruptedly.

In order to formalize these entities, we define the notion of EC-structure that records the time-independent (factual) parameters of an EC problem, i.e. the sets of relevant events and properties, the relations that associate events to the properties they initiate and to the properties they terminate, and the pairs of mutually incompatible properties.

Definition 2.2 (*EC-structure*)

A structure *for the* Event Calculus *(*EC-structure*) is a quintuple* $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *such that:*

- $E = \{e_1, \ldots, e_n\}$ *and* $P = \{p_1, \ldots, p_m\}$ *are finite sets of* events *and* properties*, respectively.*
- $[\cdot\rangle : P \rightarrow \mathbf{2}^E$ *and* $\langle\cdot] : P \rightarrow \mathbf{2}^E$ *are respectively the* initiating *and* terminating map *of* $\mathcal{H}$*. For every property* $p \in P$*,* $[p\rangle$ *and* $\langle p]$ *represent the set of events that initiate and terminate* p*, respectively.*
- $]\cdot,\cdot[\subseteq P \times P$ *is an irreflexive and symmetric relation, called the* exclusivity relation*, that models exclusivity among properties.* □

Since we consider situations where events are ordered relatively to one another, we will represent an MVI for a property p as $p(e_i, e_t)$, where e_i and e_t are the events that initiate and terminate p, respectively. MVIs are thus intervals labeled by properties. Let us adopt the set of all property-labeled intervals as the language of EC. The task performed by EC reduces to deciding which formulae are MVIs and which are not. GMEC extends this language by allowing combinations of property-labeled intervals by means of propositional connectives and modal operators.

Any *EC-structure* is also a structure for the *Generalized Modal Event Calculus* (hereafter *GMEC-structure*). The language for GMEC is defined as follows.

Definition 2.3 (*GMEC-language*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure. The* base language *of* $\mathcal{H}$ *(*EC-language*) is the set of propositional letters* $\mathcal{A}_\mathcal{H} = \{p(e_1, e_2) : p \in P \text{ and } e_1, e_2 \in E\}$. *The* GMEC-language *of* $\mathcal{H}$*, denoted by* $\mathcal{L}_\mathcal{H}$*, is the modal language with propositional letters in* $\mathcal{A}_\mathcal{H}$ *and logical operators in* $\{\neg, \wedge, \vee, \Box, \Diamond\}$. *We refer to the elements of* $\mathcal{A}_\mathcal{H}$ *and* $\mathcal{L}_\mathcal{H}$ *as* atomic formulae *and* GMEC-formulae*, respectively.* □

Notice that, beyond structured notation we use for atomic formulae, $\mathcal{L}_\mathcal{H}$ is a propositional language.

We call *knowledge state* a partial (consistent) specification of the events ordering. Standard implementations of EC represent knowledge states as quasi-orders, and take their transitive closure in order to make inferences concerning MVIs. Therefore, given a GMEC-structure $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$, we interpret atomic formulae relatively to the set W_E (denoted $W_\mathcal{H}$ in this context) of the strict orderings among events in E. Given a *current state of knowledge* w, the semantics of EC is defined by the (propositional) valuation $v^w_\mathcal{H}$, which discriminates MVIs from other intervals in w.

In order for $p(e_1, e_2)$ to be an MVI relatively to the knowledge state w, (e_1, e_2) must be an interval in w, i.e. $(e_1, e_2) \in w$. Moreover, e_1 and e_2 must witness the validity of the property p at the ends of this interval by initiating and terminating p, respectively. These requirements are enforced by conditions (iii), (i) and (ii), respectively, in the definition of valuation given below. The maximality requirement is caught by the meta-predicate $nb(p, e_1, e_2, w)$ in condition (iv), which expresses the fact that the validity of an MVI must not be *broken* by any interrupting event. Any event e which is known to have happened between e_1 and e_2 in w and that initiates or terminates a property that is either p itself or a property exclusive with p interrupts the validity of $p(e_1, e_2)$.

EC has been traditionally defined by means of a set of axioms [12]. In its logic programming implementation, the valuation $v^w_\mathcal{H}$ is represented by the predicate `holds`, which relies on the predicate `broken` for testing for interrupting events (i.e. the negation of the meta-predicate nb). The original definition of these predicates will be recovered in our implementation in Section 3.

GMEC expands the scope of EC by shifting the focus from the current knowledge state–say w–to all knowledge states that are reachable from w, i.e. $Ext(w)$, and more generally to $W_\mathcal{H}$. By definition, w' is an extension of w if $w \subseteq w'$. Since $\subseteq$ is a non-strict order, $(W_\mathcal{H}, \subseteq)$ can be naturally viewed as a finite, reflexive and transitive modal frame. If we consider this frame together with the straightforward modal extension of the valuation $v^w_\mathcal{H}$ to an arbitrary knowledge state, we obtain a modal model for GMEC.

Definition 2.4 (*GMEC-model*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure. We denote as* $O_\mathcal{H}$ *and* $W_\mathcal{H}$ *the set* O_E *of quasi-orders and the set* W_E *of strict orders over* E*, respectively. We call the elements of* $O_\mathcal{H}$ *(and consequently of* $W_\mathcal{H}$*)* knowledge states. *The* GMEC-frame $\mathcal{F}_\mathcal{H}$ *of* $\mathcal{H}$ *is the frame* $(W_\mathcal{H}, \subseteq)$. *The* intended GMEC-model *of* $\mathcal{H}$ *is the modal model* $\mathcal{I}_\mathcal{H} = (W_\mathcal{H}, \subseteq, v_\mathcal{H})$*, where the valuation* $v_\mathcal{H} \subseteq W_\mathcal{H} \times \mathcal{A}_\mathcal{H}$ *is defined in such a way that* $(w, p(e_1, e_2)) \in v_\mathcal{H}$ *if and only if*

i. $e_1 \in [p\rangle$;

ii. $e_2 \in \langle p]$;

iii. $(e_1, e_2) \in w$;

iv. $nb(p, e_1, e_2, w)$*, where*

$$nb(p, e_1, e_2, w) \quad \textit{iff} \quad \neg\exists e \in E.\ \ (e_1, e) \in w \wedge (e, e_2) \in w \wedge \exists q \in P.\ ((e \in [q\rangle \vee e \in \langle q]) \wedge (]p, q[\vee p = q))).$$

The satisfiability relation is defined as follows:

$$
\begin{array}{lll}
\mathcal{I}_{\mathcal{H}}; w \models p(e_1, e_2) & \textit{iff} & (w, p(e_1, e_2)) \in v_{\mathcal{H}}; \\
\mathcal{I}_{\mathcal{H}}; w \models \neg\varphi & \textit{iff} & \mathcal{I}_{\mathcal{H}}; w \not\models \varphi; \\
\mathcal{I}_{\mathcal{H}}; w \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \mathcal{I}_{\mathcal{H}}; w \models \varphi_1 \ \textit{and} \ \mathcal{I}_{\mathcal{H}}; w \models \varphi_2; \\
\mathcal{I}_{\mathcal{H}}; w \models \varphi_1 \vee \varphi_2 & \textit{iff} & \mathcal{I}_{\mathcal{H}}; w \models \varphi_1 \ \textit{or} \ \mathcal{I}_{\mathcal{H}}; w \models \varphi_2; \\
\mathcal{I}_{\mathcal{H}}; w \models \Box\varphi & \textit{iff} & \forall w' \in W_{\mathcal{H}} \ \textit{such that} \ w \subseteq w', \mathcal{I}_{\mathcal{H}}; w' \models \varphi; \\
\mathcal{I}_{\mathcal{H}}; w \models \Diamond\varphi & \textit{iff} & \exists w' \in W_{\mathcal{H}} \ \textit{such that} \ w \subseteq w' \ \textit{and} \ \mathcal{I}_{\mathcal{H}}; w' \models \varphi.
\end{array}
$$

□

We will drop the subscripts $_{\mathcal{H}}$ whenever this does not lead to ambiguities. Moreover, given a knowledge state w in $W_{\mathcal{H}}$ and a GMEC-formula φ over $\mathcal{H}$, we write $w \models \varphi$ for $\mathcal{I}_{\mathcal{H}}; w \models \varphi$.

Notice that the definition of satisfiability given in the previous inductive definition is always *consistent*, i.e. for every knowledge state $w \in W_{\mathcal{H}}$ and formula φ it is not possible to have both $w \models \varphi$ and $w \models \neg\varphi$. In the sequel, we will take advantage of a slightly different formulation of consistency. We have the following property that can be easily proven by induction on the structure of the formula φ.

Property 2.5 (*Completeness of the satisfiability relation*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ be a GMEC-structure. For all $w \in W$ and GMEC formula φ, if $w \not\models \neg\varphi$, then $w \models \varphi$. ■

Theorem 2.6 (*GMEC and S4*)

Each thesis of S4 is a valid formula of GMEC.

To prove Theorem 2.6, it suffices to show that each axiom of S4 is a valid formula of GMEC, and that each rule of S4 preserves validity. The proof of these conditions is straigthforward, and thus omitted. The following equivalences among GMEC formulae, that will result particularly useful in Section 3, immediately follow from Theorem 2.6

Corollary 2.7 (*Some equivalent GMEC-formulae*)

Let φ, φ_1 and φ_2 be GMEC formulae. Then, for every knowledge state $w \in W$,

- $w \models \Box\neg\varphi$ *iff* $w \models \neg\Diamond\varphi$
- $w \models \Diamond\neg\varphi$ *iff* $w \models \neg\Box\varphi$
- $w \models \Box(\varphi_1 \wedge \varphi_2)$ *iff* $w \models \Box\varphi_1 \wedge \Box\varphi_2$
- $w \models \Diamond(\varphi_1 \vee \varphi_2)$ *iff* $w \models \Diamond\varphi_1 \vee \Diamond\varphi_2$
- $w \models \Box\Box\varphi$ *iff* $w \models \Box\varphi$
- $w \models \Diamond\Diamond\varphi$ *iff* $w \models \Diamond\varphi$
- $w \models \Box\Diamond\Box\Diamond\varphi$ *iff* $w \models \Box\Diamond\varphi$
- $w \models \Diamond\Box\Diamond\Box\varphi$ *iff* $w \models \Diamond\Box\varphi$

■

Observe that, unfortunately, there is no way of reducing formulae of the form $\Box(\varphi_1 \vee \varphi_2)$ and $\Diamond(\varphi_1 \wedge \varphi_2)$. An interesting consequence of Corollary 2.7 is that every modal formula φ is equivalent to a formula of the form ψ, $\Box\psi$, $\Diamond\psi$, $\Box\Diamond\psi$, $\Diamond\Box\psi$, $\Box\Diamond\Box\psi$ or $\Diamond\Box\Diamond\psi$, where the main connective of ψ is non-modal.

2.3 Properties of the Formalization

We will now give a number of results concerning the adequacy of the definition of GMEC-structure with respect to the informal concept of MVI introduced in [12], and the modal extensions defined in [1, 2, 3]. We have already shown that a satisfiable atomic formula $p(e_1, e_2)$ identifies an interval during which the property p holds. These intervals are maximal and uninterrupted, i.e. p does not hold on any superinterval or subinterval of (e_1, e_2):

Lemma 2.8 (*Satisfiable atomic formulae are MVIs*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* $w \in W$ *such that* $w \models p(e_1, e_2)$. *Then* $\forall e_1', e_2' \in E$,

a. if $(e_1', e_2') \sqsubset_w (e_1, e_2)$, *then* $w \not\models p(e_1', e_2')$;

b. if $(e_1, e_2) \sqsubset_w (e_1', e_2')$, *then* $w \not\models p(e_1', e_2')$.

Proof.

a. Assume *ab absurdum* that $(e_1', e_2') \sqsubset_w (e_1, e_2)$ and $w \models p(e_1', e_2')$. If $(e_1, e_1') \in w$, then, e_1' would violate $nb(p, e_1, e_2, w)$, and therefore $w \not\models p(e_1, e_2)$. The situation is similar if $(e_2', e_2) \in w$.

b. Assume that $(e_1, e_2) \sqsubset_w (e_1', e_2')$ and $w \models p(e_1', e_2')$. The situation is dual to the previous case.

■

In this paper, we use GMEC to investigate how the MVIs derivable within the current set of ordered pairs of events is updated due to the arrival of new ordering information. The set of MVIs can change non-monotonically in response to the acquisition of ordering data. We wish to find the laws that rule this behavior. GMEC entitles us to identify on the one hand the set of MVIs that cannot be invalidated no matter how the ordering information is updated (as far as it is consistent), and on the other hand those intervals that will possibly become MVIs depending on which ordering data are acquired. Notice that this statement must be relativized to the current set of events: we do not (and in general cannot) predict the behavior of the system as new events happenings are recorded, but we are able to draw conclusions about how the current system can evolve as the ordering information is refined.

The sets of MVIs that are necessarily and possibly valid in the current state of knowledge w correspond respectively to the $\Box$- and $\Diamond$-moded atomic formulae which are valid in w. We define the sets $MVI(w)$, $\Box MVI(w)$ and $\Diamond MVI(w)$ of respectively MVIs, necessary MVIs and possible MVIs with respect to w as follows:

$$\begin{array}{lcl} MVI(w) & = & \{p(e_1, e_2) : w \models p(e_1, e_2)\} \\ \Box MVI(w) & = & \{p(e_1, e_2) : w \models \Box p(e_1, e_2)\} \\ \Diamond MVI(w) & = & \{p(e_1, e_2) : w \models \Diamond p(e_1, e_2)\} \end{array}$$

In the following, it will be useful to view these sets as functions $MVI(\cdot)$, $\Box MVI(\cdot)$ and $\Diamond MVI(\cdot)$ of the knowledge state w.

We have that the set of necessary MVIs with respect to w will persist whatever the evolution of the ordering information will be. Similarly, each element in the set of possible MVIs of w is valid in at least one extension of w.

Lemma 2.9 (*Behavior of* $\Box MVI(\cdot)$ *and* $\Diamond MVI(\cdot)$ *with respect to* $MVI(\cdot)$)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* $w \in W$, *then*

a. if $p(e_1, e_2) \in \Box MVI(w)$, *then* $\forall w' \in Ext(w)$, $p(e_1, e_2) \in MVI(w')$;

b. if $p(e_1, e_2) \in \Diamond MVI(w)$, *then* $\exists w' \in Ext(w)$, $p(e_1, e_2) \in MVI(w')$.

Proof.

a. $p(e_1, e_2) \in \Box MVI(w)$ *iff* $w \models \Box p(e_1, e_2)$,
iff $\forall w'$ such that $w \subseteq w', w' \models p(e_1, e_2)$
iff $\forall w' \in Ext(w), w' \models p(e_1, e_2)$
iff $\forall w' \in Ext(w), p(e_1, e_2) \in MVI(w')$.

b. Similar. ■

The sets of necessary MVIs, MVIs and possible MVIs in the current state of knowledge form an inclusion chain as formally stated by Lemma 2.10.

Lemma 2.10 (*Necessary MVIs and possible MVIs enclose MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* $w \in W$, *then*

$$\Box MVI(w) \subseteq MVI(w) \subseteq \Diamond MVI(W).$$

Proof.
By the definition of the involved sets, these relations can be rewritten as follows:

a. if $w \models \Box p(e_1, e_2)$, then $w \models p(e_1, e_2)$;

b. if $w \models p(e_1, e_2)$, then $w \models \Diamond p(e_1, e_2)$.

The validity of these expressions is a direct consequence of the reflexivity of the accessibility relation of GMEC-frames. Indeed, $w \models \Box p(e_1, e_2)$ *iff* $p(e_1, e_2)$ is valid in every extension of w, in particular in w itself. Analogously, if $w \models p(e_1, e_2)$, then $w \models \Diamond p(e_1, e_2)$. ■

When the arrival of a new piece of ordering information causes a transition into a more refined state of knowledge, the current set of MVIs can be subject to two transformations. On the one hand, the update may create a new MVI by connecting an event e_1, initiating a property p, and an event e_2 terminating p. On the other hand, a new link can transform a previously innocuous event e into an interrupting event for some MVI $p(e_1, e_2)$. Therefore, the function $MVI(\cdot)$ is non-monotonic with respect to the evolution of the ordering information.

On the contrary, $\Box$- and $\Diamond$-moded atomic formulae are entities that possess a monotonic behavior: the set of necessary MVIs can only grow as the current ordering information is refined, while the set of possible MVIs shrinks monotonically as we acquire new ordering information and a smaller number of future states is viable.

Lemma 2.11 (*Monotonicity of* $\Box$*- and* $\Diamond$*-moded atomic formulae*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* w *and* w' *two states of knowledge, then*

a. if $w \subseteq w'$ *then* $\Box MVI(w) \subseteq \Box MVI(w')$;

b. if $w \subseteq w'$ *then* $\Diamond MVI(w') \subseteq \Diamond MVI(w)$.

Proof.
By the definition of $MVI(\cdot)$, $\Box MVI(\cdot)$ and $\Diamond MVI(\cdot)$, these relations can be rewritten as follows:

a. if $w \models \Box p(e_1, e_2)$, then $w' \models \Box p(e_1, e_2)$;

b. if $w' \models \Diamond p(e_1, e_2)$, then $w \models \Diamond p(e_1, e_2)$.

By the definition of GMEC-frame, where $\subseteq$ plays the role of accessibility relation, these relations hold trivially: if $w \models \Box p(e_1, e_2)$, then $p(e_1, e_2)$ is valid in every extension of w, but these comprise all extensions of w', thus $w' \models \Box p(e_1, e_2)$; similarly, if $w' \models \Diamond p(e_1, e_2)$ then $p(e_1, e_2)$ holds in an extension w^* of w', but since $w \subseteq w'$ and $\subseteq$ is transitive, w^* is an extension of w as well, and thus $w \models \Diamond p(e_1, e_2)$. ■

By combining the interpretations of Lemmas 2.10 and 2.11, we have that $\Box MVI(\cdot)$ and $\Diamond MVI(\cdot)$ constrain the variability of the set of MVIs derivable using EC. The state of minimum information corresponds to the absence of any ordering data: $\Box MVI(\cdot)$ and $MVI(\cdot)$ derive no formula, while $\Diamond MVI(\cdot)$ derives all consistent property-labelled intervals. As new ordering information arrives, $\Box MVI(\cdot)$ increases, $\Diamond MVI(\cdot)$ decreases, but $MVI(\cdot)$ always sits somewhere between them. When enough ordering information has been entered (at worst when the set of events has been completely ordered) $\Box MVI(\cdot)$ and $\Diamond MVI(\cdot)$ meet at a common value constraining $MVI(\cdot)$ to assume that same value.

The following example shows that the GMEC fragment including only atomic formulae and simply modalized atomic formulae is expressive enough to model real-world application domains. We consider the functioning of the simple beverage dispenser depicted in Figure 2.1: by setting the selector to the apple or to the orange position, apple juice or orange juice is obtained, respectively. Choosing the stop position terminates the output of juice.

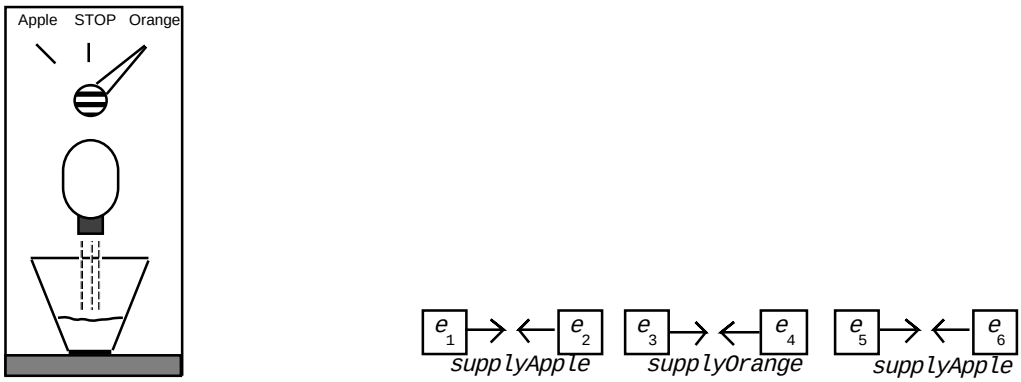


Figure 2.1: A Beverage Dispenser

We consider a scenario in which there are two events (e_1 and e_5) that initiate the property *supplyApple*, two events (e_3 and e_6) that initiate the property *supplyOrange*, and two events (e_2 and e_4) that terminate both the property *supplyApple* and the property *supplyOrange*.

This knowledge is modeled as follows in GMEC:

$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$;

$P = \{supplyApple,\ supplyOrange\}$;

$[supplyApple\rangle = \{e_1, e_5\}$;
$[supplyOrange\rangle = \{e_3, e_6\}$;
$\langle supplyApple] = \langle supplyOrange] = \{e_2, e_4\}$;
$]supplyApple, supplyOrange[$.

Suppose that, in the intended final ordering, events are ordered according to their indices. Therefore, the final situation is represented in Figure 2.1. In our example, we will consider the following sequence of ordered pairs, which arrive one at a time: (e_1, e_4); (e_1, e_6); (e_2, e_4); (e_1, e_2); (e_3, e_4); (e_4, e_5); (e_2, e_3); (e_2, e_6); (e_5, e_6). This sequence has been devised so that the complete situation shown in Figure 2.1 can be fully derived only after the last update. The 9 ordered pairs are entered into the database in sequence. The following table shows the evolution of the computation: each row corresponds to the addition of one of these ordered pairs to the database.

	MVIs derived by EC	Necessary MVIs	Possible MVIs
	$\emptyset$	$\emptyset$	$a(e_1,e_2), a(e_1,e_4), a(e_1,e_6),$ $o(e_3,e_2), o(e_3,e_4), o(e_3,e_6),$ $a(e_5,e_2), a(e_5,e_4), a(e_5,e_6)$
?- updateOrder(e_1,e_4).	$a(e_1,e_4)$	$\emptyset$	$a(e_1,e_2), a(e_1,e_4), a(e_1,e_6),$ $o(e_3,e_2), o(e_3,e_4), o(e_3,e_6),$ $a(e_5,e_2), a(e_5,e_4), a(e_5,e_6)$
?- updateOrder(e_1,e_6).	$a(e_1,e_4), a(e_1,e_6)$	$\emptyset$	$a(e_1,e_2), a(e_1,e_4), a(e_1,e_6),$ $o(e_3,e_2), o(e_3,e_4), o(e_3,e_6),$ $a(e_5,e_2), a(e_5,e_4), a(e_5,e_6)$
...	...	...	...
?- updateOrder(e_2,e_3).	$a(e_1,e_2), a(e_1,e_6), o(e_3,e_4)$	$\emptyset$	$a(e_1,e_2), a(e_1,e_6),$ $o(e_3,e_4), o(e_3,e_6), a(e_5,e_6)$
?- updateOrder(e_2,e_6).	$a(e_1,e_2), o(e_3,e_4)$	$a(e_1,e_2)$	$a(e_1,e_2), o(e_3,e_4), o(e_3,e_6),$ $a(e_5,e_6)$
?- updateOrder(e_5,e_6).	$a(e_1,e_2), o(e_3,e_4), a(e_5,e_6)$	$a(e_1,e_2), o(e_3,e_4), a(e_5,e_6)$	$a(e_1,e_2), o(e_3,e_4), a(e_5,e_6)$

The first column shows which update is being performed. The second column contains the list of the MVIs derived by EC, i.e. the result of running a generic query of the form $?-holds(periodE_i XE_t)$. For conciseness, we represented $period(e_i, supplyApple, e_t)$ as the more compact notation $a(e_i, e_t)$ and $period(e_i, supplyOrange, e_t)$ as the more compact $o(e_i, e_t)$. The third and fourth columns contains the list of necessary and possible MVIs, respectively.

Let us now move to the general case of arbitrary GMEC formulae. The following lemma stands as the basis for the treatment of the modal operators in Section 3. It shows how the satisfiability test for an arbitrary GMEC-formula having a modality as its main connective can be reduced to first testing the satisfiability of its immediate subformula in the current world and then checking the satisfiability of the original formula in the 'one-step' extensions of the current knowledge state.

Lemma 2.12 (*Unfolding modalities*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* $\varphi \in \mathcal{L}_{\mathcal{H}}$ *a GMEC-formula over* $\mathcal{H}$*, and* $w \in W$*. Then*

a. $w \models \Box\varphi$ *iff* $w \models \varphi$ *and* $\forall(e_1, e_2)$ *such that* $(e_1, e_2), (e_2, e_1) \notin w$*,* $w \uparrow \{(e_1, e_2)\} \models \Box\varphi$*;*

b. $w \models \Diamond\varphi$ *iff* $w \models \varphi$ *or* $\exists(e_1, e_2)$ *such that* $(e_1, e_2), (e_2, e_1) \notin w$*,* $w \uparrow \{(e_1, e_2)\} \models \Diamond\varphi$*.*

Proof.

First notice that if $(e_1, e_2) \notin w$ and $(e_2, e_1) \notin w$, then $w \uparrow \{(e_1, e_2)\} \in W$ since, in this case, upgrading w with (e_1, e_2) cannot violate asymmetry in any way.

Moreover, for every $w \in W$,

$$Ext(w) = \{w\} \cup \bigcup_{\substack{(e_1,e_2)\notin w \\ (e_2,e_1)\notin w}} Ext(w \uparrow \{(e_1,e_2)\}).$$

Indeed, let $w' \in Ext(w)$. Then, by definition, $w \subseteq w'$. Therefore, either $w' = w$ or there exists a pair $(e_1, e_2) \in w' \backslash w$. In the latter case, $w' \in Ext(w \uparrow \{(e_1, e_2)\})$. The opposite inclusion is straightforward.

We have now the needed tools to prove the statement of the lemma.

a. $w \models \Box\varphi$ *iff* $\forall w' \in Ext(w), w' \models \varphi$
iff $\forall w' \in \{w\} \cup \bigcup_{\substack{(e_1,e_2)\notin w \\ (e_2,e_1)\notin w}} Ext(w \uparrow \{(e_1,e_2)\}), w' \models \varphi$
iff $w \models \varphi$ and for each $e_1, e_2 \in E$ such that $(e_1, e_2) \notin w$ and $(e_2, e_1) \notin w$, it holds that for each $w' \in Ext(w \uparrow \{(e_1, e_2)\}), w' \models \varphi$
iff $w \models \varphi$ and for each $e_1, e_2 \in E$ such that $(e_1, e_2), (e_2, e_1) \notin w$, $w \uparrow \{(e_1, e_2)\} \models \Box\varphi$.

b. The proof is similar to (*a*). ■

In the sequel, we will use a different but clearly equivalent form of *a*:

$w \models \Box\varphi$ iff $w \models \varphi$ and
it is not the case that
$\exists (e_1, e_2)$ such that $(e_1, e_2), (e_2, e_1) \notin w$. $w \uparrow \{(e_1, e_2)\} \not\models \Box\varphi$.

Lemma 2.12 allows one to prove interesting properties of GMEC-models. As an example, it is possible to show that GMEC-models validate the so-called McKinsey formula $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$. Consider a GMEC-model $\mathcal{I}_{\mathcal{H}}$ and a world $w \in W_{\mathcal{H}}$ such that $w \models \Box\Diamond\phi$. By Lemma 2.12, we have that $w \models \Diamond\phi$ and for every (e_1, e_2) such that $(e_1, e_2), (e_2, e_1) \notin w$, $w \uparrow \{(e_1, e_2)\} \models \Box\Diamond\phi$. By recursively applying such an argument, we have that for all w' such that $w \subseteq w'$, $w' \models \Diamond\phi$. Since, by Definition 2.2, the set of events is finite, at last we arrive at a world w_f in which for every pair of events (e_1, e_2), it is either $(e_1, e_2) \in w_f$ or $(e_2, e_1) \in w_f$ and $w_f \models \Diamond\phi$. Here we can apply again Lemma 2.12 ($\Diamond$ part) to conclude that $w_f \models \phi$ or there exists (e_1, e_2) such that $(e_1, e_2), (e_2, e_1) \notin w$ and $w_f \uparrow \{(e_1, e_2)\} \models \Diamond\phi$. However, since w_f is final, we have that $w_f \models \phi$. Another application of Lemma 2.12 ($\Box$ part) yields $w_f \models \Box\phi$. Then, another application of it ($\Diamond$ part) leads to $w_f \models \Diamond\Box\phi$. We can then go back to w by using Lemma 2.12 ($\Diamond$ part), e.g. if $w_f = w^* \uparrow \{(e_1, e_2)\}$ for some (e_1, e_2) and $w_f \models \Diamond\Box\phi$, then we have that $w^* \models \Diamond\Box\phi$, and so on.

Next, we seek for a manner of computing necessary and possible MVIs (simply moded atomic formulae) that does not require to explore future states of knowledge. In both cases, we will be able to devise necessary and sufficient local conditions. These properties stand as the basis for the implementation of SKEC and CREC [1, 3], and will allow us to improve the naive GMEC implementation based on the result of Lemma 2.12 (semi-naive implementation).

An MVI $p(e_1, e_2)$ is undefeasible whatever ordering information is acquired if no event can interrupt it. An event e can possibly interrupt the validity of $p(e_1, e_2)$ if it initiates or terminates p or a property that is exclusive with p, and it could be consistently located between e_1 and e_2 with respect to w. The negation of this condition is expressed by the meta-predicate $nsb(p, e_1, e_2, w)$.

Lemma 2.13 (*Local condition for atomic necessity*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure. Then for any* $e_1, e_2 \in E$, $p \in P$ *and* $w \in W$, $p(e_1, e_2) \in \Box MVI(w)$ *iff the following conditions are satisfied:*

- $e_1 \in [p\rangle$,
- $e_2 \in \langle p]$,
- $(e_1, e_2) \in w$,
- $nsb(p, e_1, e_2, w)$

where $nsb(p, e_1, e_2, w)$ *stands for the expression*

$$\begin{array}{lll} \forall e \in E.\, \forall q \in P. & e = e_1 \\ & \vee & e = e_2 \\ & \vee & (e, e_1) \in w \\ & \vee & (e_2, e) \in w \\ & \vee & (e \in [q\rangle \vee e \in \langle q] \rightarrow \neg]p, q[\wedge p \neq q). \end{array}$$

Proof.

By definition, the first member of the equivalence, $p(e_1, e_2) \in \Box MVI(w)$, reduces to $w \models \Box p(e_1, e_2)$. We will take advantage of this formulation in the proof.

($\Leftarrow$) Let us proceed by contradiction. So, assume that $e_1 \in [p\rangle$, $e_2 \in \langle p]$, $(e_1, e_2) \in w$ and $nsb(p, e_1, e_2, w)$, but there exist an extension w' of w such that $w' \models p(e_1, e_2)$ does not hold, i.e. such that $nb(e_1, e_2, w')$ is false. After some logical manipulations, the latter statement rewrites to

$$\exists e \in E.\, \exists q \in P.\, ((e_1, e) \in w' \wedge (e, e_2) \in w' \wedge (e \in [q\rangle \vee e \in \langle q]) \wedge (]p, q[\vee p = q).$$

Let e' and q' witness the validity of this formula. By instantiation, we obtain:

$$(e_1, e') \in w' \wedge (e', e_2) \in w' \wedge (e' \in [q'\rangle \vee e' \in \langle q']) \wedge (]p, q'[\vee p = q') \quad (1)$$

We can instantiate the expression for $nsb(p, e_1, e_2, w)$ with these values too. The resulting formula is:

$$e' = e_1 \vee e' = e_2 \vee (e', e_1) \in w \vee (e_2, e') \in w \vee (e' \in [q'\rangle \wedge e' \in \langle q'] \rightarrow \neg]p, q'[\wedge p \neq q') \quad (2)$$

We must show that none of the alternatives in formula (2) applies. Since w' is a strict order, the validity of (1) implies that e' can be neither e_1 nor e_2. Analogously, by lemma 2.11, either $(e', e_1) \in w$ or $(e_2, e') \in w$ would violate the asymmetry of w'. Finally, the choice of q' contradicts the last alternative, i.e. that $(e' \in [q'\rangle \vee e' \in \langle q'] \rightarrow \neg]p, q'[\wedge p \neq q')$. This concludes this direction of the proof.

($\Rightarrow$) We will again proceed by contradiction. Clearly, if $e_1 \notin [p\rangle$ or $e_2 \notin \langle p]$, then we cannot obtain $w' \models p(e_1, e_2)$ in any state of knowledge w'. If $(e_1, e_2) \notin w$, then there exist extensions of w containing (e_2, e_1). Because of asymmetry, these extensions cannot contain (e_1, e_2), thus $p(e_1, e_2)$ cannot be valid in them.

Assume now that $e_1 \in [p\rangle$, $e_2 \in \langle p]$ and $(e_1, e_2) \in w$ but that $nsb(p, e_1, e_2, w)$ does not hold. Therefore, there are an event e' and a property q' such that:

$$e' \neq e_1 \wedge e' \neq e_2 \wedge (e', e_1) \notin w \wedge (e_2, e') \notin w \wedge (e' \in [q'\rangle \vee e' \in \langle q']) \wedge (]p, q'[\vee p = q').$$

Since the pair $(e_1, e_2) \in w$, there exists at least one extension w' of w such that $(e_1, e') \in w'$ and $(e', e_2) \in w'$. Therefore,

$$(e_1, e') \in w' \wedge (e', e_2) \in w' \wedge (e' \in [q'\rangle \vee e' \in \langle q']) \wedge (]p, q'[\vee p = q')$$

hence $nb(p, e_1, e_2, w')$ does not hold. This contradicts the assumed hypothesis, i.e. that $w \models \Box p(e_1, e_2)$. ■

It is worth noting that the definition of $nsb(p, e_1, e_2, w)$ is more restrictive than that of $nb(p, e_1, e_2, w)$. Let us say that e is a *critical* event for a given property p if and only if e initiates or terminates a

property q such that either $p = q$ or $]p, q[$. Condition $nb(p, e_1, e_2, w)$ states that there are no critical events $e \in E$ such that both $(e_1, e) \in w$ and $(e, e_2) \in w$, while condition $nsb(p, e_1, e_2, w)$ states that there are no critical events $e \in E$, with $e \neq e_1$ and $e \neq e_2$, such that both $(e, e_1) \notin w$ and $(e_2, e) \notin w$.

A labeled interval $p(e_1, e_2)$ might become an MVI for p in an extension of the current knowledge state w if e_1 initiates p, e_2 terminates p, the interval (e_1, e_2) is consistent with w, and there are no already known interrupting events between e_1 and e_2. More formally, we have that:

Lemma 2.14 (*Local condition for atomic possibility*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ be a GMEC-structure. Then for any $e_1, e_2 \in E$, $p \in P$ and $w \in W$, $p(e_1, e_2) \in \Diamond MVI(w)$ iff the following conditions are satisfied:

- $e_1 \in [p\rangle$,
- $e_2 \in \langle p]$,
- $(e_2, e_1) \notin w$,
- $nb(p, e_1, e_2, w)$.

Proof.

As in the previous proof, we reduce the relation $p(e_1, e_2) \in \Diamond MVI(w)$ to $w \models \Diamond p(e_1, e_2)$. We operate on this equivalent formulation.

($\Leftarrow$) Let us construct an extension w' of w such that $w' \models p(e_1, e_2)$. The state of knowledge w' is defined as $w' = (w \cup \{(e_1, e_2)\})^+$. First notice that w' is consistent (i.e. it does not violate asymmetry) since w is consistent and $(e_2, e_1) \notin w$. Then observe that $nb(p, e_1, e_2, w')$ holds by the definition of w'. Otherwise, we should be able to conclude that there is an event $e \in E$ such that $(e_1, e) \in w'$, $(e, e_2) \in w'$ and either $e \in [q\rangle$ or $e \in \langle q]$ for some property $q \in P$, with $]p, q[$ or $p = q$, but in that case, $(e_1, e) \in w$ and $(e, e_2) \in w$ contradicting the assumption that $nb(p, e_1, e_2, w)$ holds. Therefore, conditions (*i–iv*) of Definition 2.4 are satisfied w.r.t. w'; hence $w' \models p(e_1, e_2)$, and thus $w \models \Diamond p(e_1, e_2)$.

($\Rightarrow$) We proceed by contradiction. Clearly, if $e_1 \notin [p\rangle$ or $e_2 \notin \langle p]$, then we cannot obtain $w' \models p(e_1, e_2)$ in any state of knowledge w'. Analogously, if $(e_2, e_1) \in w$, then (e_2, e_1) belongs to every extension of w, forbidding in this way condition (*iii*) of ΞDefinition 2.4 to be satisfied. Finally, if $nb(p, e_1, e_2, w)$ does not hold, (i.e. there is an event $e \in E$ such that $(e_1, e) \in w$, $(e, e_2) \in w$ and $e \in [q\rangle$ or $e \in \langle q]$ for some property $q \in P$ with $]p, q[\vee p = q)$, then, by Lemma 2.11, the same condition would apply to every extension w' as well, thus $nb(p, e_1, e_2, w')$ would not hold in any extension w' of w and $p(e_1, e_2) \notin \Diamond MVI(w)$. ■

3 A Logic Programming Implementation of GMEC

In this section, we present an abstract implementation of GMEC in the language of hereditary Harrop formulae and prove its soundness and completeness with respect to the GMEC semantics presented in Section 2. In Section 3.1, we recall the definition of hereditary Harrop formulae (HH-formulae for short) and their operational semantics as a logic programming language. In Section 3.2, we define an encoding of GMEC-structures, orderings and GMEC-formulae as HH-formulae. We also give a first naive program modeling the validity relation for GMEC formulae. Section 3.3 proves the soundness and completeness of this program with respect to the notion of GMEC-model. Finally, in Section 3.4, we present an optimized (semi-naive) implementation of GMEC and prove its soundness and completeness.

3.1 Hereditary Harrop formulae

So far, the implementation language for EC has almost always been the language of Horn clauses augmented with negation-as-failure [13], which constitutes the core of the logic programming language Prolog. This traditional Prolog implementation can be easily extended to cover the propositional connectives. Moreover, we showed in [1] that a restriction of the purely modal extension of EC can be conveniently encoded in this language by taking advantage of Lemmas 2.13 and 2.14. However, when mixing arbitrarily propositional connectives and modalities, as in GMEC, a direct encoding in Prolog appears unsatisfactory. The resulting program is in fact either highly non-declarative (for the necessary presence of a large number of `assert` and `retract` statements), or extremely complex (as we experienced in [3]). In conclusion, Prolog is not adequate for a declarative description of GMEC. In particular, it makes quite difficult to prove the fundamental soundness and completeness properties.

For the implementation of GMEC, we chose the language of first-order hereditary Harrop formulae [15] augmented with negation-as-failure. Extensions of this language, with or without negation-as-failure, have been used as the underlying logic of many logic programming languages successfully proposed in the last ten years, including Miller's λProlog [14], Gabbay's N-Prolog [7], Pfenning's Elf [18] and Hodas and Miller's Lolli [10].

Hereditary Harrop formulae extend Horn clauses by allowing the presence of implication and universal quantification in goal formulae. The former feature will give us a declarative means of temporarily augmenting the program with new facts and performing in this manner a form of hypothetical reasoning. Universal quantification in goals provides a powerful tool for data and program abstraction: it allows, for instance, a purely declarative definition of abstract data types and modules. We will not take advantage of this last feature.

We will now describe the syntax and the operational semantics of hereditary Harrop formulae. We assume the reader to be familiar with Prolog [13].

The language of hereditary Harrop formulae is a subset of intuitionistic first-order predicate logic. Formulae in this language are functionally subdivided in program formulae and goal formulae depending on whether they can appear as program clauses or they can only be used in queries. We use the syntactic variables D and G respectively to refer to these formulae. Program and goal formulae are mutually defined according to the following formal grammar, where A ranges over atomic formulae:

$$\begin{array}{rcl@{\quad}l} D & ::= & A \mid \top \mid D_1 \wedge D_2 \mid G \rightarrow A \mid \forall x.D & (\textit{Program formulae}) \\ G & ::= & A \mid \top \mid G_1 \wedge G_2 \mid D \rightarrow G \mid \forall x.G & \\ & & \mid \bot \mid G_1 \vee G_2 \mid \exists x.G & (\textit{Goal formulae}) \end{array}$$

Syntactically, hereditary Harrop formulae differ from Horn clauses only for the admissibility of implication and universal quantification in goal formulae (items 4 and 5 in the definition of G): as soon as we get rid of these productions, we obtain a language that is equivalent to Horn clauses. In order to represent negation-as-failure, we augment the definition of goal formulae with expressions of the form *not* G. A *hereditary Harrop clause* is a closed program formula of the form $\forall\vec{x}.(G \rightarrow A)$, where $\forall\vec{x}$ represents a possibly empty sequence of universal quantifications; A and G are called the *head* and the *body* of the clause, respectively. A closed formula of the form $\forall\vec{x}.A$ is called a *fact* and is considered as a clause with an empty body (i.e. $\forall\vec{x}.(\top \rightarrow A)$). Any program formula can be transformed into a set of clauses. In the following, we will use the terms D-formula and G-formula as synonyms of program formula and goal formula, respectively.

We will describe the semantics of hereditary Harrop formulae with negation-as-failure by means of two judgments called *positive* and *negative sequents* and denoted $\mathcal{P} \Longrightarrow G$ and $\mathcal{P} \not\Longrightarrow G$, respectively, where $\mathcal{P}$ is a set of D-formulae and G a G-formula. Negative sequents are needed for defining negation-as-failure. In both cases, $\mathcal{P}$ and G are called the *program* and the *goal* of the sequent respectively. If c is a clause and $\mathcal{P}$ is a program, we abbreviate $\mathcal{P} \cup \{c\}$ as $\mathcal{P}, c$. As we said, any program is equivalent, modulo elementary logical manipulations, to a program consisting uniquely of clauses. We write $\mathcal{P}^c$ for the *clausal form* of the program $\mathcal{P}$.

Hereditary Harrop formulae constitute the biggest sublanguage of first-order logic that is complete with respect to *uniform proofs* [15]. Uniform provability views logical connectives in goal formulae as search directives for the construction of derivations and clauses as partial definitions of atomic formulae. The goal part of a sequent is decomposed up to the level of atomic formulae, and only then the program part is accessed in order to retrieve a clause defining this atom. The computation fails when trying to solve undefined instances of atomic formulae.

The non-deterministic search for a proof of the goal G from the program $\mathcal{P}$ corresponds to the construction of a derivation for the positive sequent $\mathcal{P} \Longrightarrow G$ according to the rules to be defined below. Every derivation tree for $\mathcal{P} \Longrightarrow G$ built in this manner constitutes a *proof* of G from $\mathcal{P}$. Therefore, G is *provable* from $\mathcal{P}$ if there exists a proof-tree for the positive sequent $\mathcal{P} \Longrightarrow G$.

Conversely, G is not provable from $\mathcal{P}$ if there is no proof-tree for $\mathcal{P} \Longrightarrow G$. In terms of (uniform) proof search, non-provability can come in two flavors: either every attempt of building a derivation for $\mathcal{P} \Longrightarrow G$ generates a sequent $\mathcal{P}' \Longrightarrow G'$ to which no rule is applicable, or an infinite tree can be obtained by the application of the derivation rules, and in this case the search does not terminate. In the first case, we say that this sequent is *finitely non-provable*. In the second case, we say that the sequent is *divergent*.

When trying to find a proof for a negated goal *not* G from the program $\mathcal{P}$, we want to show that there is no proof of G from $\mathcal{P}$, i.e. that $\mathcal{P} \Longrightarrow G$ is non provable. It will become evident from the examples below that, in general, diverging sequents cannot be finitely recognized. Therefore, we are reduced to showing that $\mathcal{P} \Longrightarrow G$ is finitely non-provable. The (failed) proof-trees constructed during a search for a proof of $\mathcal{P} \Longrightarrow G$ are not accessible for this purpose. Instead, we internalize them and model finite non-provability by means of negative sequents, $\mathcal{P} \not\Longrightarrow G$ in this case. Again, a derivation tree for $\mathcal{P} \not\Longrightarrow G$ constitutes a proof of this sequent.

The derivability rules for positive and negative sequents have dual definitions. Moreover, the proof-search semantics of negated goals makes them mutually recursive. The complete definition is given in Figure 3.2. The rules that do not apply to Horn clauses are outlined. Notice that the rules **exist**$-$ and **atom**$-$ are non standard since some of the involved parameters (the term t and the clause $\forall\vec{x}.(G \rightarrow A')$ respectively) are subject to extensional universal quantification. Therefore, **exist**$-$ can be viewed as a rule with an infinite number of premisses (Harland has shown in [8] that it is sufficient to consider a finite set of *representations*). Similarly **atom**$-$ is better seen as a rule with a variable number of premisses depending on the number of matching clauses.

Let us now give some examples that better illustrate the distinction among provable, finitely non-provable and diverging sequents. These notions apply also to negative sequents and are defined similarly to the positive case.

- Let $\mathcal{P}_1 = \{a, a \rightarrow b\}$. The clausal form of $\mathcal{P}_1$ is $\mathcal{P}_1^c = \{\top \rightarrow a, a \rightarrow b\}$. The sequent $\mathcal{P}_1 \Longrightarrow b$ is provable by applying in sequence the rules **atom+**, **atom+** and **true+**. However, $\mathcal{P}_1 \not\Longrightarrow b$ fails after two applications of **atom**$-$ (therefore it is finitely non-provable). On the other hand, $\mathcal{P}_1 \Longrightarrow c$ fails immediately while $\mathcal{P}_1 \not\Longrightarrow c$ succeeds by rule **atom**$-$.

- Let $\mathcal{P}_2 = \{a \rightarrow a\}$. Then both $\mathcal{P}_2 \Longrightarrow a$ and $\mathcal{P}_2 \not\Longrightarrow a$ diverge by infinite applications of the rules **atom+** and **atom**$-$, respectively. It is easy to notice that these sequents do not have any derivations since each step reproduces the original sequent. However, a simple loop-detection mechanism is not sufficient in most cases. Consider for instance a first-order variant of this example: $\mathcal{P}_2' = \{\forall x.(a(f(x)) \rightarrow a(x)\}$. Then, the sequents $\mathcal{P}_2' \Longrightarrow a(v)$ and $\mathcal{P}_2' \not\Longrightarrow a(v)$ diverge in the same manner, but at each stage the sequent to be proved is different. As a less trivial example, consider a non-terminating program that computes the decimal expansion of π.

- Finally, let $\mathcal{P}_3 = \{a, a \rightarrow a\}$. Clearly $\mathcal{P}_3 \Longrightarrow a$ is derivable. Notice that this sequent has infinitely many proofs, as well as a diverging derivation. On the other hand, $\mathcal{P}_3 \not\Longrightarrow a$ is not derivable since, after applying rule **atom**$-$, there is no way to proceed with the branch corresponding to a. Notice however that the resulting (failed) proof-tree is infinite.

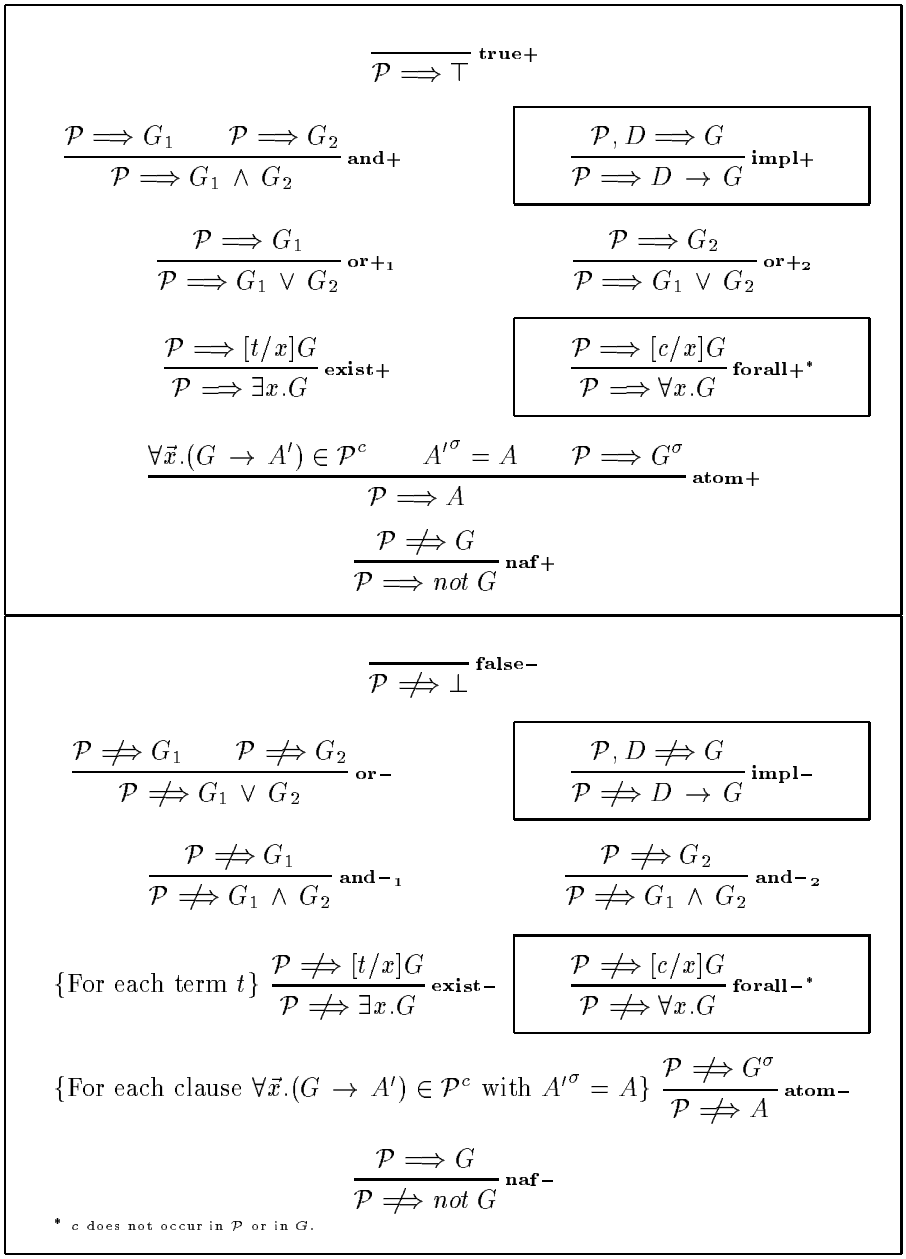


Figure 3.2: Sequent derivation rules for hereditary Harrop formulae with negation-as-failure

We will now state the duality between positive and negative sequents. First, since we defined negative sequents with the aim of formalizing finite non-provability, it should not be possible that both the positive and the negative sequents involving the same program and goal are provable. Harland has proved the following result for a similar rule system [8, 9].

Property 3.1 (*Consistency of positive and negative sequents*)
For given program $\mathcal{P}$ and goal G, either $\mathcal{P} \Longrightarrow G$ or $\mathcal{P} \not\Longrightarrow G$ is not derivable. ■

This property can be sharpened by considering finite non-provability. We have indeed that a positive

sequent is finitely non-provable *iff* the corresponding negative sequent is derivable. The dual property obtained by flipping the adjectives positive and negative holds as well.

Property 3.2 (*Duality of positive and negative sequents over finite derivations*)
Let $\mathcal{P}$ and G be a program and a goal respectively. Then:

- *$\mathcal{P} \Longrightarrow G$ is finitely non-provable iff $\mathcal{P} \not\Longrightarrow G$ is provable;*
- *$\mathcal{P} \not\Longrightarrow G$ is finitely non-provable iff $\mathcal{P} \Longrightarrow G$ is provable.* ■

We will take advantage of this result as follows. Let $p(\mathcal{P}, G)$ be a property of a given program $\mathcal{P}$ and a goal G. Assume that we are able to prove that $p(\mathcal{P}, G)$ *iff* $\mathcal{P} \Longrightarrow G$ is derivable. Then, if we know that $\mathcal{P} \Longrightarrow G$ has finite derivations only, we obtain as an immediate consequence that $\neg p(\mathcal{P}, G)$ *iff* $\mathcal{P} \Longrightarrow$ *not* G is derivable.

Whenever the problem at hand is characterized by finite derivations, negation-as-failure *behaves as* classical negation. The presence of this operator does not turn the logic of HH-formulae into a classical formalism, not even in this case. Indeed, implication still maintains its original intuitionistic flavor, as supported by the operational reading of this connective. In particular, the G-formula $D \rightarrow G$ is not equivalent to *not* $D \vee G$

We conclude this section by defining a concrete syntax for the language of hereditary Harrop formulae. We use identifiers beginning with lower case letters (e.g., `must`, `before`, ...) for constants and symbols beginning with uppercase letters for implicitly quantified variables (e.g., `Ei`, `P`, ...). We write terms and atoms in curried form (e.g., `(before Ei Et)` for the binary predicate `before` applied to the variables `Ei` and `Et`). The unary operator `not` is reserved to represent negation-as-failure when used in a goal formula (it will be convenient to overload it in Section 3.2 to model object level negation in a term position). The constants `true` and `fail` are reserved for the logical symbols $\top$ and $\bot$ respectively. We represent the logical operators $\wedge$, $\vee$ and $\rightarrow$ as the infix symbols `,` (comma), `;` (semicolon) and `=>` respectively, and the quantifiers $\forall x.$ and $\exists x.$ as `forall[X]` and `exist[X]` respectively. In a program position, we represent $\rightarrow$ as `:-` with the antecedent and the consequent reversed. We follow the usually accepted convention to drop the leading universal quantifiers when representing a clause (and in general a D-formula) in the concrete syntax.

3.2 Encoding of GMEC as hereditary Harrop formulae

The aim of this section is twofold. We will first give a precise encoding of GMEC into the language of hereditary Harrop formulae. Then we will show a naive implementation of GMEC and give an informal overview of its features. The soundness and completeness of this encoding will be proved in the next section. Section 3.4 analyzes a more refined version of this implementation.

We define a family of representation functions $\ulcorner \cdot \urcorner$ that relate the mathematical entities we have been using in Section 2 to the terms of the logic programming language we have chosen for the implementation. Specifically, we will need to encode GMEC-structures, the associated orderings, and the GMEC-language. In the remainder of this section, we will refer to the GMEC-structure $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$.

In order to represent $\mathcal{H}$, we need to give an encoding of the entities that constitute it. For this purpose, we first specify the functions $\ulcorner \cdot \urcorner^E$ and $\ulcorner \cdot \urcorner^P$ that give the concrete syntax of individual events and properties, respectively. We explicitly assume that these functions are injective, i.e. that every event e in E (property p in P) has a representation that is different from that of all other events (resp. properties). Moreover, we want $\ulcorner \cdot \urcorner^E$ and $\ulcorner \cdot \urcorner^P$ to give distinct representations to events and properties. The exact definition of these functions is problem-specific.

The injectivity of the representation functions on events and properties enables us to utilize the respective inverse functions, $\llcorner \cdot \lrcorner^E$ and $\llcorner \cdot \lrcorner^P$, whenever they are defined. Notice indeed that $\llcorner \cdot \lrcorner^E$ and $\llcorner \cdot \lrcorner^P$

```
% -------- Equality
X = X.                                                    (1)

% -------- Transitive closure of knowledge states
before E1 E2 :-                                           (2)
     beforeFact E1 E2.

before E1 E2 :-                                           (3)
     beforeFact E1 E, before E E2.

% -------- Propositional formulae
holds (period Ei P Et) :-                                 (4)
     happens Ei, initiates Ei P,
     happens Et, terminates Et P,
     before Ei Et,
     not (broken Ei P Et).

broken Ei P Et :-                                         (5)
     happens E,
     before Ei E, before E Et,
     (initiates E Q ; terminates E Q),
     (exclusive P Q; P = Q).

holds (not X) :-                                          (6)
     not (holds X).

holds (and X Y) :-                                        (7)
     holds X, holds Y.

holds (or X Y) :-                                         (8)
     holds X; holds Y.

% -------- Modal formulae
holds (must X) :-                                         (9)
     holds X,
     not (fails_must X).

fails_must X :-                                           (10)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          not (holds (must X)).

holds (may X) :-                                          (11)
     holds X.

holds (may X) :-                                          (12)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          holds (may X).
```

Figure 3.3: `GMEC`: a naive implementation of GMEC.

cannot be defined for all terms t. As a matter of convenience, we take the liberty of writing ill-formed expressions of the form $\llcorner t \lrcorner_E \in E$ for a generic t, assigning them the truth value *false* whenever t is not in the range of $\ulcorner \cdot \urcorner^E$.

The next step consists in defining the translation maps for $[\cdot\rangle$, $\langle\cdot]$ and $]\cdot,\cdot[$. We represent these relations by means of the binary predicates `initiates`, `terminates` and `exclusive`, respectively. The traditional formulations of EC give an explicit representation to the occurrences of events. We utilize the unary predicate `happens` for this purpose. The corresponding representation functions are defined as follows:

- $\ulcorner [\cdot\rangle \urcorner^I = \{\texttt{initiates}\ \ulcorner e \urcorner^E\ \ulcorner p \urcorner^P : e \in E, p \in P, \text{ and } e \in [p\rangle\}$;
- $\ulcorner \langle\cdot] \urcorner^T = \{\texttt{terminates}\ \ulcorner e \urcorner^E\ \ulcorner p \urcorner^P : e \in E, p \in P, \text{ and } e \in \langle p]\}$;
- $\ulcorner]\cdot,\cdot[\urcorner^X = \{\texttt{exclusive}\ \ulcorner p \urcorner^P\ \ulcorner q \urcorner^P : p, q \in P \text{ and }]p, q[\}$;
- $\ulcorner E \urcorner^H = \{\texttt{happens}\ \ulcorner e \urcorner^E : e \in E\}$.

At this point, we define the representation of the GMEC-structure $\mathcal{H}$ by taking the union of the representations of its constituent entities:

$$\ulcorner \mathcal{H} \urcorner^S = \ulcorner E \urcorner^H \cup \ulcorner [\cdot\rangle \urcorner^I \cup \ulcorner \langle\cdot] \urcorner^T \cup \ulcorner]\cdot,\cdot[\urcorner^X .$$

In Section 2, we assumed that the ordering information of a GMEC problem was specified by means of strict orders in W. When integrating GMEC into practical applications, e.g. [3], this assumption turns out to be inadequate since, in general, the host system will simply pass the raw ordering data to

the GMEC module as they are recorded. Therefore, we choose to represent this kind of information as our knowledge states and to reconstruct the corresponding strict ordering as needed. We assume the information source to be reliable, and thus the raw ordering information constitutes a quasi-order in O. We use the binary predicate `beforeFact` to represent the atomic ordered pairs contained in a quasi-order $o \in O$. The function $\ulcorner \cdot \urcorner^O$ relates a knowledge state to its concrete syntax. It is defined as follows:

$$\ulcorner o \urcorner^O = \{\texttt{beforeFact } \ulcorner e_1 \urcorner^E \ \ulcorner e_2 \urcorner^E : (e_1, e_2) \in o\}.$$

The last entity we need to represent is the GMEC-language of $\mathcal{H}$. We encode the formulae in $\mathcal{L}_\mathcal{H}$ as terms in the language of hereditary Harrop formulae. Specifically, we use the ternary function symbol `period` to represent atomic formulae and the constants `not`, `and`, `or`, `must` and `may`, with the obvious arities, as the concrete syntax of the logical symbols of GMEC: $\neg$, $\wedge$, $\vee$, $\Box$ and $\Diamond$, respectively. The representation function $\ulcorner \cdot \urcorner^L$ for GMEC-formulae is specified by the following recursive definition, based on the structure of the formula in $\mathcal{L}_\mathcal{H}$ being represented:

- $\ulcorner p(e_1, e_2) \urcorner^L = \texttt{period } \ulcorner e_1 \urcorner^E \ \ulcorner p \urcorner^P \ \ulcorner e_2 \urcorner^E$
- $\ulcorner \neg \varphi \urcorner^L = \texttt{not } \ulcorner \varphi \urcorner^L$
- $\ulcorner \varphi_1 \wedge \varphi_2 \urcorner^L = \texttt{and } \ulcorner \varphi_1 \urcorner^L \ \ulcorner \varphi_2 \urcorner^L$
- $\ulcorner \varphi_1 \vee \varphi_2 \urcorner^L = \texttt{or } \ulcorner \varphi_1 \urcorner^L \ \ulcorner \varphi_2 \urcorner^L$
- $\ulcorner \Box \varphi \urcorner^L = \texttt{must } \ulcorner \varphi \urcorner^L$
- $\ulcorner \Diamond \varphi \urcorner^L = \texttt{may } \ulcorner \varphi \urcorner^L$

Notice that we have overloaded the symbol `not`. However, its position dictates its use: within a term, it represents the negation of $\mathcal{L}_\mathcal{H}$, and at the predicate level it stands as the negation-as-failure operator. In order to simplify the notation, we will write the previously defined translation maps as $\ulcorner \cdot \urcorner$, whenever the omitted subscript is easily deducible from the context.

Figure 3.3 shows an implementation of GMEC in the language of HH-formulae. We call this program the *naive* implementation of GMEC, and refer to it as `GMEC`. Clause (1) models object level equality. Clauses (2) and (3) define the predicate `before` that reconstructs the transitive closure of the ordering information currently stored in the program. The remaining clauses show the actual implementation of GMEC. We use the unary predicate `holds` to represent the validity of a GMEC-formula with respect to the GMEC-structure and the knowledge state represented in the program. Said in a different way, we aim at representing the judgment $\mathcal{I}_\mathcal{H}; o^+ \models \varphi$ by means of the relation $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{holds } \ulcorner \varphi \urcorner$.

Clauses (4) and (5) implement the definition of modal valuation of the standard GMEC-model given in Definition 2.4; the latter clause corresponds to the negation of the meta-predicate *nb*. These clauses coincide with the standard Prolog axiomatization of EC [12]. Clauses (6-8) map the object-level propositional connectives to the corresponding meta-level operators.

Clauses (9-10) define `holds` for $\Box$-moded GMEC-formulae. They implement directly the statement of the remark following Lemma 2.12. In order to check that the formula $\Box\varphi$ holds in the current state of knowledge, first we check φ locally and then we ascertained that there is no future knowledge state where $\Box\varphi$ does not hold. Clause (10) attempts to find a counterexample to this requirement, i.e. a proper extension of the current world (i.e. a state of knowledge that orders two currently unrelated events e_1 and e_2) where $\Box\varphi$ fails to hold. No such knowledge state must exist for the body of clause (9) to hold. Notice the essential use of implication in the goal position in these cases.

The remaining clauses deal with GMEC-formulae having $\Diamond$ as their main connective in a similar manner. Note that the implementation of `holds` for GMEC-formulae involving modalities requires the exhaustive exploration of all extensions of the current knowledge state. This approach is clearly expensive, and for this reason we qualify `GMEC` as the *naive* implementation of GMEC. An enhanced program for GMEC that takes these properties into account is analyzed in detail in section 3.4.

In section 3.1, we described hereditary Harrop formulae as an extension to Horn clauses permitting the use of implication and universal quantification in goal positions. Being the latter connective available, it is tempting replace clauses (9-10) by

```
holds (must X) :-                                                    (*)
     holds X,
     forall[E1, E2]
         (happens E1, happens E2,
          not (before E1 E2),
          not (before E2 E1),
          beforeFact E1 E2 => holds (must X)).
```

implementing in this way the statement of lemma 2.12 directly, instead of taking the complicated detours dictated by the subsequent remark. Unfortunately, this clause is not a faithful transcription of the lemma. The bug originates from the confusion between two forms of universal quantification.

When a universal goal of the form $\forall x.G(x)$ is encountered, it is reduced to the goal $G(c)$, where c is a *new* constant. Solving this goal requires to work abstractly with the generic individual c only. Therefore, if this goal succeeds, $G(t)$ holds for every term t. This form of universal quantification is called *intensional*. In our setting, solving the body of clause $(*)$ requires to invent two new events, say e_1^* and e_2^*, and use them to solve the embedded goal. However, $e_1^*, e_2^* \notin E$, therefore the subgoals `happens` $\ulcorner e_1^* \urcorner$ and `happens` $\ulcorner e_2^* \urcorner$ will never succeed.

This is obviously not the behavior that we have in mind. We would rather want the variables `E1` and `E2` to be instantiated to *all* events in E in turn. Abstracting away from our problem, we may model this situation by means of the formula $\forall x \in S.\, G(x)$, where S is some (recursive) set. This form of universal quantification is called *extensional*. This quantifier cannot be represented within plain hereditary Harrop formulae. However, the presence of classical negation (modeled to some extent by negation-as-failure) allows to recover it as soon as we manage to represent the relation $x \in S$ by a predicate (as in our example). Then, we rewrite the previous formula as $\neg\exists x.\, (x \in S \wedge \neg G(x))$. This formula is in turn equivalent to

$$(\forall x.\, ((x \in S \wedge \neg G(x)) \rightarrow p')) \rightarrow \neg p',$$

where p' is a new atomic formula. Notice that the quantifier is now in a program position; therefore, it will not be solved intensionally. As soon as we substitute logical negation ($\neg$) with negation-as-failure (*not*), we obtain a formula that is acceptable in our framework. These are precisely the steps that led to the displayed formulation of clauses (9-10), where `fails_must` is used as the accessory atomic formula.

3.3 Soundness and Completeness Results

In this section, we show that `GMEC` is a faithful implementation of the semantics given in Section 2.2 for GMEC. This statement is formalized in the soundness and completeness theorem (Theorem 3.10) that concludes the section. This result is accomplished in a number of steps: we present here only the most important ones; their proofs, together with auxiliary lemmas, can be found in the Appendix. First we need to prove that `before` is a sound and complete implementation of the transitive closure over knowledge states, then we show that the implementation of atomic formulae is sound and complete, and finally we will be able to freely mix boolean connectives and modal operators.

We begin with a lemma about the properties of `before`. When only ordering information is concerned, we do not need to refer to the representation of the underlying GMEC-structure, but only implicitly to the representation of events. First, we show that the HH-formula `before` $\ulcorner e_1 \urcorner$ $\ulcorner e_2 \urcorner$ is provable precisely when (e_1, e_2) is in the transitive closure of the current knowledge state. Moreover, the goal `before` $\ulcorner e_1 \urcorner$ $\ulcorner e_2 \urcorner$ finitely fails exactly when (e_1, e_2) is not in the transitive closure of the current knowledge state.

The second part of this lemma will be of extreme importance when dealing with negative sequents since `before` is the only predicate, besides `holds`, that has a recursive definition, and therefore that could diverge.

Lemma 3.5 (*Soundness and completeness of* `before` *with respect to transitive closure*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then for any* $e_1, e_1 \in E$

a. $\texttt{GMEC}, \ulcorner o \urcorner \Longrightarrow \texttt{before } \ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner$ *iff* $(e_1, e_2) \in o^+$;
b. $\texttt{GMEC}, \ulcorner o \urcorner \Longrightarrow \texttt{not (before } \ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner\texttt{)}$ *iff* $(e_1, e_2) \notin o^+$.

On the basis of this result, we address the problem of proving that the clauses for atomic GMEC-formulae implement the semantics of MVIs. We start by proving a lemma that states that the predicate `broken` behaves like the negation of the meta-predicates *nb*.

Lemma 3.6 (*Correspondence between* `broken` *and nb*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then*

a. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{broken } \ulcorner e_1 \urcorner\ \ulcorner p \urcorner\ \ulcorner e_2 \urcorner$ *iff* $\neg nb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$;
b. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{not (broken } \ulcorner e_1 \urcorner\ \ulcorner p \urcorner\ \ulcorner e_2 \urcorner\texttt{)}$ *iff* $nb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$.

At this point, we have all the tools we need to prove that the implementation of `holds` on bare atomic formulae behaves isomorphically to the satisfiability relation on these formulae. Therefore, GMEC provides an effective implementation of MVIs.

Theorem 3.7 (`GMEC` *computes MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then*

a. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{holds (period } \ulcorner e_1 \urcorner\ \ulcorner p \urcorner\ \ulcorner e_2 \urcorner\texttt{)}$ *iff* $p(e_1, e_2) \in MVI(o^+)$;
b. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{not (holds (period } \ulcorner e_1 \urcorner\ \ulcorner p \urcorner\ \ulcorner e_2 \urcorner\texttt{))}$ *iff* $p(e_1, e_2) \notin MVI(o^+)$.

We conclude this section by stating its main result, namely, soundness and completeness of `GMEC` with respect to the GMEC-frame semantics.

Theorem 3.10 (*Soundness and completeness of* `GMEC` *with respect to the GMEC-frame semantics*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* o *a state of knowledge and* φ *and GMEC-formula, then*

a. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{holds } \ulcorner \varphi \urcorner$ *iff* $o^+ \models \varphi$;
b. $\texttt{GMEC}, \ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow \texttt{not (holds } \ulcorner \varphi \urcorner\texttt{)}$ *iff* $o^+ \not\models \varphi$.

3.4 A Semi-Naive Implementation of GMEC

Theorem 3.10 establishes a strong connection between the GMEC semantics and the hereditary Harrop program `GMEC` displayed in Figure 3.3, providing in this way a computational flavor to the Generalized Modal Event Calculus presented in Section 2. Although this is a valuable theoretical property, it looses most of its practical appeal as soon as we give a close look at the treatment of the modal operators in `GMEC`. Indeed, checking the validity of a goal having $\Box$ as its main connective (clauses (9) and (10)) triggers the exploration of all the states of knowledge reachable from the current ordering information (unless failure occurs). The situation is not better in the case of $\Diamond$-moded formulae (clauses (11-12)): "only" an arbitrarily large subset of the extension of the current state of knowledge must be examined.

```
% -------- Equality
X = X.                                                    (1')

% -------- Transitive closure of knowledge states
before E1 E2 :-                                           (2')
     beforefact E1 E2.

before E1 E2 :-                                           (3')
     beforefact E1 E, before E E2.

% -------- Propositional formulae
holds (period Ei P Et) :-                                 (4')
     happens Ei, initiates Ei P,
     happens Et, terminates Et P,
     before Ei Et,
     not (broken Ei P Et).

broken Ei P Et :-                                         (5')
     happens E,
     before Ei E, before E Et,
     (initiates E Q; terminates E Q),
     (exclusive P Q; P = Q).

holds (not X) :-                                          (6')
     not (holds X).

holds (and X Y) :-                                        (7')
     holds X, holds Y.

holds (or X Y) :-                                         (8')
     holds X; holds Y.
```

Figure 3.4: **GMEC+**, a semi-naive implementation of GMEC (*part I*).

It is easy to figure out that the cardinality of the set of extensions of a given state of knowledge is in general exponential in the number of events.

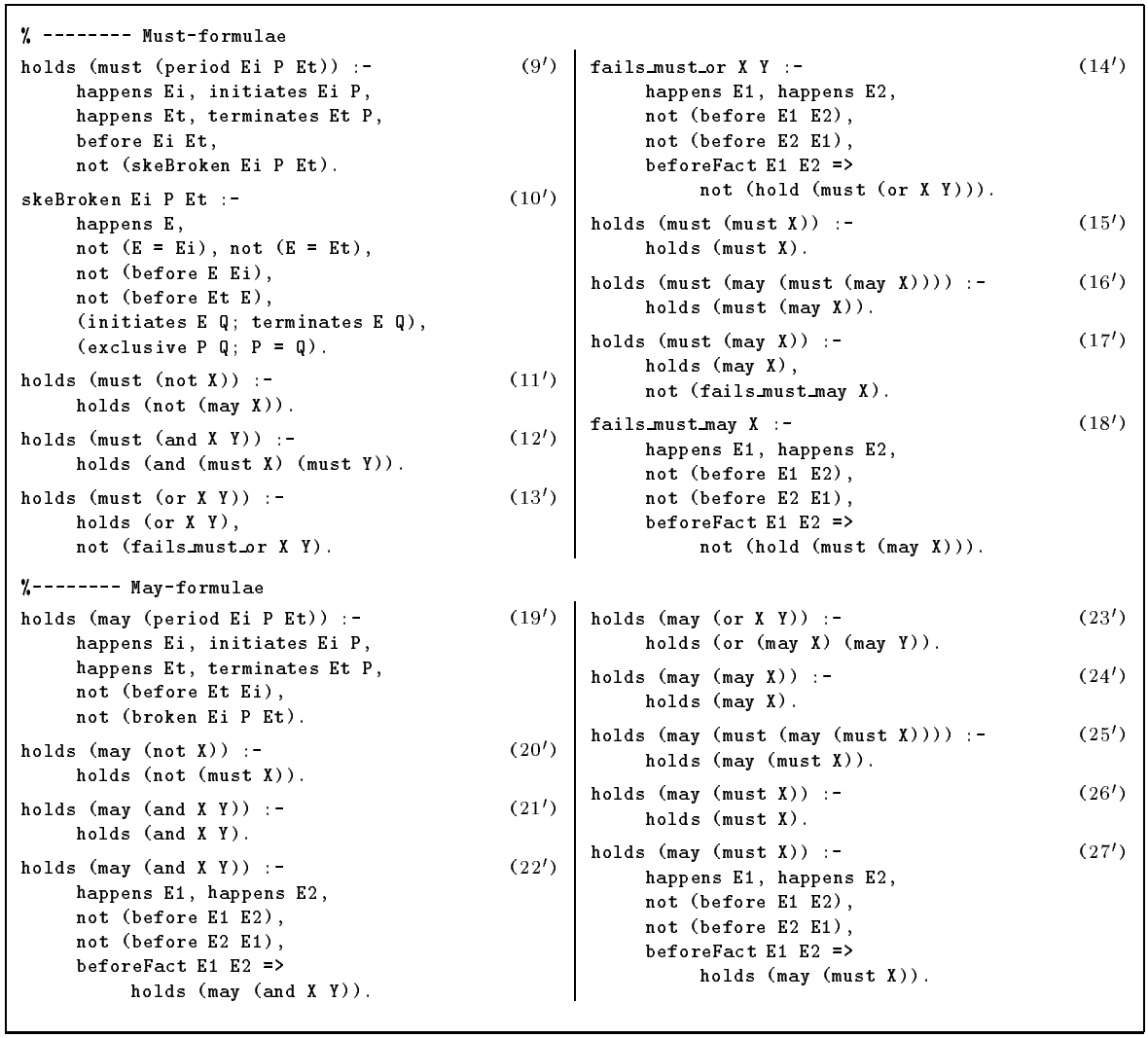
In this section, we solve these problems, up to a certain extent, by providing an alternative implementation for the GMEC semantics. We foresay that we cannot completely avoid the exhaustive exploration of the set of possible future knowledge states. However, the resulting decision procedure will be search-free in a number of cases that are likely to occur in real applications (this is the case, for instance, of the beverage system).

The key idea behind our enhanced implementation of GMEC is to take into account the meta-properties of our framework for the modal event calculus. First, we exploit the intrinsic properties of GMEC. In particular, Lemmas 2.13 and 2.14 suggest a local method for checking the validity of atomic formulae preceded by a single occurrence of a modal operator. Remember that the definition of the functions $\Box MVI(\cdot)$ and $\Diamond MVI(\cdot)$ relies on formulae of this form. Being able to compute the value of these functions locally is clearly of crucial importance for practical applications. Second, we can take advantage of the equivalences that hold in GMEC (Property 2.7). Although they occasionally permit to eliminate occurrences of a modal operator, we will mainly use these equivalences as rewriting rules to push the modalities as close to the atomic formulae as possible, with the goal of using Lemmas 2.13 and 2.14 whenever possible. Alternatively, we could have used the equivalences of Corollary 2.7 to precompile a GMEC-formula into a form on which these lemmas can be applied directly.

This technique cannot be applied systematically. In particular, we know from Section 2.2 that formulae of the form $\Box(\varphi' \vee \varphi'')$, and dually $\Diamond(\varphi' \wedge \varphi'')$, cannot be reduced. Moreover, the formulae $\Box\Diamond\varphi$ and $\Diamond\Box\varphi$ are reducible only for particular φs. In these cases, and only in these cases, the actual exploration of the extensions of the current knowledge state cannot be avoided.

On the basis of these considerations, we will now describe a second (semi-naive) implementation of GMEC in the language of hereditary Harrop formulae. The enhanced program, that we call **GMEC+**, shares with **GMEC** the encoding presented in Section 3.2 for the various entities at hand. Moreover, for the sake of simplicity, we use the same names as in Figure 3.3 for predicates performing the same functionalities. This program is presented in Figures 3.4–3.5. Clauses (1'-8') in Figure 3.4 do not undergo any change.

The top part of Figure 3.5 illustrates the definition of `holds` for $\Box$-moded GMEC-formulae $\Box\varphi$. In order to apply the previous observations, we need to look at the main connective of φ. Clauses (9') and



```
% -------- Must-formulae
holds (must (period Ei P Et)) :-                                    (9′)
     happens Ei, initiates Ei P,
     happens Et, terminates Et P,
     before Ei Et,
     not (skeBroken Ei P Et).

skeBroken Ei P Et :-                                                (10′)
     happens E,
     not (E = Ei), not (E = Et),
     not (before E Ei),
     not (before Et E),
     (initiates E Q; terminates E Q),
     (exclusive P Q; P = Q).

holds (must (not X)) :-                                             (11′)
     holds (not (may X)).

holds (must (and X Y)) :-                                           (12′)
     holds (and (must X) (must Y)).

holds (must (or X Y)) :-                                            (13′)
     holds (or X Y),
     not (fails_must_or X Y).

fails_must_or X Y :-                                                (14′)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          not (hold (must (or X Y))).

holds (must (must X)) :-                                            (15′)
     holds (must X).

holds (must (may (must (may X)))) :-                                (16′)
     holds (must (may X)).

holds (must (may X)) :-                                             (17′)
     holds (may X),
     not (fails_must_may X).

fails_must_may X :-                                                 (18′)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          not (hold (must (may X))).

%-------- May-formulae
holds (may (period Ei P Et)) :-                                     (19′)
     happens Ei, initiates Ei P,
     happens Et, terminates Et P,
     not (before Et Ei),
     not (broken Ei P Et).

holds (may (not X)) :-                                              (20′)
     holds (not (must X)).

holds (may (and X Y)) :-                                            (21′)
     holds (and X Y).

holds (may (and X Y)) :-                                            (22′)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          holds (may (and X Y)).

holds (may (or X Y)) :-                                             (23′)
     holds (or (may X) (may Y)).

holds (may (may X)) :-                                              (24′)
     holds (may X).

holds (may (must (may (must X)))) :-                                (25′)
     holds (may (must X)).

holds (may (must X)) :-                                             (26′)
     holds (must X).

holds (may (must X)) :-                                             (27′)
     happens E1, happens E2,
     not (before E1 E2),
     not (before E2 E1),
     beforeFact E1 E2 =>
          holds (may (must X)).
```

Figure 3.5: **GMEC+**, a semi-naive implementation of GMEC (*part II*).

(10′) deal with the case where φ is atomic by implementing the statement of Lemma 2.13, with (10′) corresponding to the negation of the meta-predicate *nsb*. Clauses (11′-12′, 15′-16′) implement some of the reductions described by property 2.7. The other clauses deal with the remaining patterns for φ by means of the brute-force approach derived from the remark following Lemma 2.12. They are instances of clauses (9-10) of **GMEC**. Notice that clause (17′) subsumes clause (16′). Therefore, the latter ought to be given precedence over the former.

The lower part of Figure 3.5 shows the treatment of GMEC-formulae having $\Diamond$ as their main connective. The underlying idea is similar to the previous case. Notice that clause (26′) subsumes clause (25′).

We have extensively investigated in [1, 3] two axiomatic variants of the Event Calculus based on clauses (9′-10′) and (19′, 5′) respectively. These calculi, called respectively the *Skeptical Event Calculus* (*SKEC*) and the *Credulous Event Calculus* (*CREC*), now emerge as a by-product of the

broader notion of Generalized Modal Event Calculus.

We want now to prove that `GMEC+` is a faithful implementation of the GMEC semantics presented in Section 2.2. In order to achieve this goal, we need to process `GMEC+` through the same steps applied to `GMEC` in Section 3.3. Fortunately we can borrow from that Section Lemmas 3.3, 3.4, 3.5, 3.6 and Theorem 3.7 since on the one hand the clauses of `GMEC` involved in these statements are present also in `GMEC+`, and on the other hand, they are not subject to interferences from the new clauses. This claim, which validity can be easily checked, will save us a lot of work.

Our first endeavor will be to prove that the predicate `skeBroken` behaves like the negation of the meta-predicate *nsb*. The statement and the proof of this result recall Lemma 3.6, according to which `broken` is a sound and complete implementation of *nb*.

Lemma 3.11 (*Correspondence between* `skeBroken` *and nsb*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and o a state of knowledge, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `skeBroken` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$ *iff* $\neg nsb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$;

b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `not (skeBroken` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`)` *iff* $nsb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$.

We will now prove that `holds` applied to the encoding of atomic formulae preceded by one occurrence of a modal operator behaves isomorphically to the satisfiability relation for these formulae. Therefore, `GMEC+` provides an effective implementation of MVIs (by Theorem 3.7), necessary MVIs and possible MVIs.

We first consider $\Box$-moded atomic formulae and make explicit their relation to necessary MVIs. The proof of this statement relies on the previous lemma.

Theorem 3.12 (`GMEC+` *computes necessary MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and o a state of knowledge, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `holds (must (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`))` *iff* $p(e_1, e_2) \in \Box MVI(o^+)$;

b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `not (holds (must (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`)))` *iff* $p(e_1, e_2) \notin \Box MVI(o^+)$.

A similar result holds for possible MVIs, formalized as the function $\Diamond MVI(\cdot)$. Indeed, `holds` constitutes a decision procedure for the validity relation for $\Diamond$-moded atomic formulae.

Theorem 3.13 (`GMEC+` *computes possible MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and o a state of knowledge, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `holds (may (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`))` *iff* $p(e_1, e_2) \in \Diamond MVI(o^+)$;

b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `not (holds (may (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`)))` *iff* $p(e_1, e_2) \notin \Diamond MVI(o^+)$.

Finally, we can prove that a formula is valid in the GMEC semantics if and only if the goal obtained by encoding it and plugging it as the argument of `holds` is derivable in `GMEC+`. Moreover, a goal of this form has only finite derivations, therefore `holds` captures also the unsatisfiability of a GMEC-formula.

Theorem 3.14 (*Soundness and completeness of* `GMEC+` *with respect to the GMEC-frame semantics*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure, o a state of knowledge and* φ *and GMEC-formula, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `holds` $\ulcorner\varphi\urcorner$ *iff* $o^+ \models \varphi$;

b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner,\ulcorner o\urcorner \Longrightarrow$ `not (holds` $\ulcorner\varphi\urcorner$`)` *iff* $o^+ \not\models \varphi$.

4 Conclusions and further developments

This paper proposed and formally analyzed GMEC, a modal extension of EC to compute current, necessary and possible MVIs in a context where the ordering of events is relative, partial and incremental. Unlike previous modal extensions of EC (e.g., MEC [1]), GMEC supports free mixing of boolean connectives and modal operators. The paper presented two sound and complete implementations of GMEC as logic programs in the language of hereditary Harrop formulae.

Maybe more important than the results themselves is the method we adopted to achieve them. First, we provided a precise semantic formalization in order to capture the intuitions underlying EC. In this way, we could prove properties of EC (and subsequentially of GMEC) rather than claim them. Second, we used a proof-theoretic approach for proving the faithfulness of our implementations with respect to the behavior of GMEC, as expressed by the semantics.

We are developing this work in several directions. On the one hand, we are considering the possibility of dealing with more complex specifications of the ordering information such as non-committed data (e.g. disjunctive orderings) and possibly inconsistent orderings. On the other hand, we are comparing `GMEC` with classical modal logics such as Sobocinski's K1.1 which is characterized by the class of all finite partial orderings, i.e. by the class of finite frames whose accessibility relation is reflexive, transitive and antisymmetric [11]. We actually proved the following theorem that allows us to strenghten the result of Theorem 2.6

Theorem 4.1 (*GMEC and K1.1*)
Each thesis of K1.1 is a valid formula of GMEC.

Since an axiom system for K1.1 can be obtained by adding to the axiom system for S4 the axiom $\Box(\Box(\phi \rightarrow \Box\phi) \rightarrow \phi) \rightarrow \phi$, in view of Theorem 2.6, to prove Theorem 4.1 we only need to show that such an additional axiom is a valid formula of GMEC. The proof is quite straigthforward, and thus omitted.

Theorem 4.1 allows us to immediately derive a number of interesting properties of GMEC-models. For instance, the validity of the McKinsey formula in GMEC-models (that we explicitly proved in Section 2) is a consequence of Theorem 4.1. In particular, it allows us to establish the following equivalences among GMEC formulae:

Corollary 4.2 (*Further equivalent GMEC-formulae*)
Let φ be a GMEC formula. Then, for every knowledge state $w \in W$,

- $w \models \Box\Diamond\Box\varphi$ *iff* $w \models \Box\Diamond\varphi$
- $w \models \Diamond\Box\Diamond\varphi$ *iff* $w \models \Diamond\Box\varphi$ ■

Pairing Corollaries 2.7 and 4.2, we can conclude that each GMEC-formula ϕ is logically equivalent to a GMEC-formula of one of the following forms: ψ, $\Box\psi$, $\Diamond\psi$, $\Box\Diamond\psi$, $\Diamond\Box\psi$, where the outermost logical symbol of ψ is not a modal operator. We are currently revising the semi-naive implementation of GMEC to further improve its performance by exploiting the equivalences of Corollary 4.2.

Acknowledgments

The work of the first author was partially supported by NFS grant CCR-9303383. The work of the second and third author was partially supported by the CNR project *Ambienti e strumenti per la gestione di informazioni temporali.*

We would like to thank James Harland for the valuable discussions concerning the syntactic formulation of negation-as-failure presented in Section 3.1. Thanks also to Kristof Van Belleghem for the useful remarks about the expressive power of Modal Event Calculus (the GMEC fragment including atomic formulae and simply modalized atomic formulae).

References

[1] I. Cervesato, L. Chittaro, A. Montanari: "A Modal Calculus of Partially Ordered Events in a Logic Programming Framework", *Proc. of ICLP'95: 12th International Conference on Logic Programming*, Kanegawa, Japan, MIT Press, 1995, 299–313.

[2] I. Cervesato, A. Montanari, A. Provetti: "On the Non-Monotonic Behavior of Event Calculus for Deriving Maximal Time Intervals", *Interval Computations*, 3(2), 1993, 83–119.

[3] L. Chittaro, A. Montanari, A. Provetti: "Skeptical and Credulous Event Calculi for Supporting Modal Queries", *Proc. of ECAI'94: 11th European Conference on Artificial Intelligence*, A. Cohn (ed.), John Wiley & Sons, 1994, 361–365.

[4] L. Chittaro, A. Montanari, I. Cervesato: "Speeding up temporal reasoning by exploiting the notion of kernel of an ordering relation", *Proc. of TIME'95: 2nd International Workshop on Temporal Representation and Reasoning*, S. Goodwin, H. Hamilton (eds.), University of Regina (Canada), 1995, 73–80.

[5] T. Dean, M. Boddy: "Reasoning about partially ordered events", *Artificial Intelligence*, 36, 1988, 375–399.

[6] M. Denecker, L. Missiaen, M. Bruynooghe: "Temporal reasoning with abductive Event Calculus", *Proc. of ECAI'92: 10th European Conference on Artificial Intelligence*, B. Neumann (ed.), John Wiley & Sons, 1992, 384–388.

[7] D.M. Gabbay, U. Reyle: "N-Prolog: an Extension of Prolog with Hypothetical Implication. I", *Journal of Logic Programming*, 1(4), 1984, 319–355.

[8] J. Harland. *On Hereditary Harrop Formulae as a basis for Logic Programming.* PhD thesis, University of Edinburgh, UK, 1991.

[9] J. Harland. Toward a static proof system for negation as failure. Technical Report CITRI/TR-92-49, Department of Computer Science, University of Melbourne, Australia, 1992.

[10] J. Hodas and D. Miller. Logic programming in a fragment of intuitionistic linear logic. *Journal of Information and Computation*, 110(2):327–365, 1994.

[11] G. Hughes, M. Cresswell: "*An Introduction to Modal Logic*", Methuen, London, 1968.

[12] R. Kowalski, M. Sergot: "A Logic-based Calculus of Events", *New Generation Computing*, 4, Ohmsha Ltd and Springer-Verlag, 1986, 67–95.

[13] J.W. Lloyd: "*Foundations of Logic Programming*" (2nd ed.), Springer-Verlag, 1987.

[14] D. Miller, G. Nadathur: "An overview of λProlog", *Proc. of ICSLP'88: 5th International Conference and Symposium on Logic Programming*, K.A. Bowen, R. Kowalski (eds.), Cambridge, MA, 1988, 810–827, MIT Press. North-Holland.

[15] D. Miller, G. Nadathur, F. Pfenning, A. Scedrov: "Uniform proofs as a foundation for logic programming", *Annals of Pure and Applied Logic* 51, 1991, 125–157, North-Holland.

[16] D.C. Moffat, G.D. Ritchie: "Modal Queries about Partially-ordered Plans", *Journal of Expt. Theor. Artificial Intelligence*, 2, 1990, 341–368.

[17] A. Montanari, L. Chittaro, and I. Cervesato: "A general Modal Framework for the Event Calculus and its Skeptical and Credulous Variants", *Proc. of ECAI'96: 12th European Conference on Artificial Intelligence*, W. Wahlster (ed.), John Wiley & Sons, 1996, 33–37.

[18] F. Pfenning. Elf: A language for logic definition and verified metaprogramming. In *Proceedings of the 7th Symposium on Logic in Computer Science – LICS'89*, pages 702–712, Monterey, CA, 1989. IEEE Computer Society Press.

[19] M.P. Shanahan: "Prediction is Deduction but Explanation is Abduction", *Proc. of IJCAI'89: 11th International Joint Conference on Artificial Intelligence*, Detroit, 1989, 1055–1050.

A Proofs and Auxiliary Lemmas from Section 3

Lemma 3.3 (*Soundness and completeness of* = *w.r.t. equality for events*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* $e_1, e_2 \in E$ *and* o *a state of knowledge. Then*

a. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner$ *is derivable* *iff* $e_1 = e_2$;

b. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (}\ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner\texttt{)}$ *is derivable* *iff* $e_1 \neq e_2$.

Proof.

a. ($\Rightarrow$) Being the goal $\ulcorner e_1\urcorner$ = $\ulcorner e_2\urcorner$ atomic, the last rule applied must have been **atom+** with clause (1) and substitution $\sigma = \{X \mapsto \ulcorner e_1\urcorner, X \mapsto \ulcorner e_2\urcorner\}$. This substitution is well-formed *iff* $\ulcorner e_1\urcorner = \ulcorner e_2\urcorner$. Therefore, we have that $e_1 = e_2$ by the injectivity of the representation function $\ulcorner\cdot\urcorner^E$.
($\Leftarrow$) If $e_1 = e_2$, a derivation of $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner$ is obtained by application of rules **atom+** and **true+**.

b. ($\Rightarrow$) By the uniform provability property, the last inference rule applied is **naf+**. Therefore, the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner$ is provable. By property 3.1, the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner$ has no derivation. Finally, by (a), $e_1 \neq e_2$.
($\Leftarrow$) If $e_1 \neq e_2$, we have that $\ulcorner e_1\urcorner \neq \ulcorner e_2\urcorner$ since $\ulcorner\cdot\urcorner^E$ is injective. Therefore, rule **atom−** succeeds with no premisses for the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner$. Therefore, by rule **naf+**, $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (}\ulcorner e_1\urcorner \texttt{ = } \ulcorner e_2\urcorner\texttt{)}$ is derivable. ■

Lemma 3.4 (*Soundness and completeness of* = *w.r.t. equality for properties*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* $p_1, p_2 \in P$ *and* o *a state of knowledge. Then*

a. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \ulcorner p_1\urcorner \texttt{ = } \ulcorner p_2\urcorner$ *is derivable* *iff* $p_1 = p_2$;

b. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (}\ulcorner p_1\urcorner \texttt{ = } \ulcorner p_2\urcorner\texttt{)}$ *is derivable* *iff* $p_1 \neq p_2$.

Proof.
Similar to the proof of previous lemma. ■

Lemma 3.5 (*Soundness and completeness of* `before` *w.r.t. transitive closure*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then for any* $e_1, e_2 \in E$

a. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ *is derivable* *iff* $(e_1, e_2) \in o^+$;

b. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\texttt{)}$ *is derivable* *iff* $(e_1, e_2) \notin o^+$.

Proof.
For the sake of exemplification, we provide a rigorous proof of this simple statement. The proofs given in the rest of the section will be more sketchy. However, it should be clear to the reader how to rewrite them in a similar style. Indeed, in order to limit the length of these proofs, we will mainly focus on the critical steps, that correspond to the applications of rules **atom+** and **atom−**. The application of the remaining rules will often be maintained implicit in the informal arguments used to chain critical rules.

Throughout this and many of the subsequent inductive proofs, we will rely on the following strict schema. The cases of the induction are treated in dedicated paragraphs headed with an identifying label. Within each paragraph, the proof is organized in a series of lines consisting of three zones. On the left, we have a counter used for referencing. The central field contains a formal relation that is claimed to hold. The right part of each line provides a justification of this claim. Each step is in general justified with respect to the previous line (to the statement of the theorem in the case of the first line). Occasionally, the justification will refer to one or more non-immediate predecessors of the current line. In these cases, we take advantage of the counter. In certain occasions, we will have to follow alternative courses in the proof, and each should be proved in order for the overall proof to be correct. We use bullets (•) to identify the first line of each alternative, and indent the subsequent lines.

a. ($\Rightarrow$) We proceed by induction on the structure of a derivation tree for the positive sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$. Since $\texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ is atomic, the last rule applied must have been **atom+**. The only program formulas that match this atom are clauses (2) and (3). Therefore, the proof can proceed in two ways:

[1]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	assumption
[2]	• $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	by rule **atom+** on [1] and clause (2),
[3]	$(\texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner) \in \ulcorner o\urcorner$	by rule **atom+** on [2] and since no rule for `beforeFact` is defined in `GMEC` or $\ulcorner\mathcal{H}\urcorner$,
[4]	$(e_1, e_2) \in o$	by definition of $\ulcorner\cdot\urcorner^O$,
[5]	$(e_1, e_2) \in o^+$	by definition of transitive closure;
[6]	• $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner\texttt{,}\ \texttt{before}\ \ulcorner e\urcorner\ \ulcorner e_2\urcorner$	by rule **atom+** on [1] and clause (3), for some event e,
[7]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner$	by rule **and+** on [6] (left branch),
[8]	$(\texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner) \in \ulcorner o\urcorner$	by rule **atom+** on [7] and since no rule for `beforeFact` is defined in `GMEC` or $\ulcorner\mathcal{H}\urcorner$,
[9]	$(e_1, e) \in o$	by definition of $\ulcorner\cdot\urcorner^O$,
[10]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e\urcorner\ \ulcorner e_2\urcorner$	by rule **and+** on [6] (right branch),
[11]	$(e, e_2) \in o^+$	by induction hypothesis on [10],
[12]	$(e_1, e_2) \in o^+$	by definition of transitive closure on [9, 11].

($\Leftarrow$) Let $\sigma = e'_1 \ldots e'_l$, with $e'_1 = e_1$ and $e'_l = e_2$ be a sequence of events such that, for $i = 1 \ldots l-1$, $(e'_i, e'_{i+1}) \in o$, proving in this way that $(e_1, e_2) \in o^+$. We conduce the proof by induction on the length l of this sequence.

Case $l = 1$:

[1]	$(e_1, e_2) \in o$	assumption
[2]	$(\texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner) \in \ulcorner o\urcorner$	by definition of $\ulcorner\cdot\urcorner^O$,
[3]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	by rules **true+** and **atom+**,
[4]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	by rules **atom+** on [3] and clause (2).

Case $l > 1$: Then $\sigma = e_1, e \ldots e_2$ with $(e_1, e) \in o$ and $(e, e_2) \in o^+$. Thus

[1]	$(e_1, e) \in o$ *and* $(e, e_2) \in o^+$	assumption
[2]	$(e_1, e) \in o$	conjunct from [1],
[3]	$(\texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner) \in \ulcorner o\urcorner$	by definition of $\ulcorner\cdot\urcorner^O$,
[4]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner$	by rules **true+** and **atom+**,
[5]	$(e, e_2) \in o^+$	conjunct from [1],
[6]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e\urcorner\ \ulcorner e_2\urcorner$	by induction hypothesis,
[7]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner e\urcorner\texttt{,}\ \texttt{before}\ \ulcorner e\urcorner\ \ulcorner e_2\urcorner$	by rule **and+** on [4, 6],
[8]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	by rule **atom+** on [7] and clause (2).

b. ($\Rightarrow$) The last rule applied in a derivation of $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not}\ (\texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner)$ must have been **naf+**. Therefore, the negative sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ has a derivation. Now, by property 3.1, the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ is not derivable. Thus, by (*a*), $(e_1, e_2) \notin o^+$.
($\Leftarrow$) By property 3.2, it is enough to show that, whenever $(e_1, e_2) \notin o^+$, the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ is finitely non-provable. We show a stronger property, i.e. that the search for a proof of a sequent of this form must terminate (either with success, as in (*a*), or with failure).
Assume *ab absurdum* that the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before}\ \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ has an infinite derivation. Being the goal atomic, this sequent must result from the application of rule **atom+** to either clause (2) or clause (3), which define `before`. Being the former a fact in program `GMEC`, we must discard this alternative: the derivation would otherwise terminate after one application of rule **true+**. Therefore, rule **atom+** has been used on clause (3) and the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact}\ \ulcorner e_1\urcorner\ \ulcorner \hat{e}_1\urcorner\texttt{,}\ \texttt{before}\ \ulcorner \hat{e}_1\urcorner\ \ulcorner e_2\urcorner$

for some event $\hat{e}_1 \in E$. By an application of rule **and+**, we reduce this sequent to $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact } \ulcorner e_1\urcorner\ \ulcorner\hat{e}_1\urcorner$ and $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } \ulcorner\hat{e}_1\urcorner\ \ulcorner e_2\urcorner$. By definition of $\ulcorner\cdot\urcorner^{O}$, the former corresponds to $(e_1, \hat{e}_1) \in o$. The latter is a reinstantiation of our original problem.

By iterating this reasoning pattern *ad infinitum*, we conclude that the recursive clause (3) must have been applied infinitely many time for the original sequent to have an infinite derivation. In particular, the sequents $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact } \ulcorner\hat{e}_i\urcorner\ \ulcorner\hat{e}_{i+1}\urcorner$ are derivable for an infinite sequence of events $\{\hat{e}_i\}_{i\in\omega}$ (with $\hat{e}_0 = e_1$). Thus $(\hat{e}_i, \hat{e}_{i+1}) \in o$ for all $i \in \omega$. At this point, we must remember that E is finite. Therefore, there are two distinct indices i, j with $i < j$ such that $\hat{e}_i = \hat{e}_j$. Then, by definition of transitive closure, we have that $(\hat{e}_i, \hat{e}_j) \in o^+$, but this violates the irreflexivity of o^+. ■

Lemma 3.6 (*Correspondence between* `broken` *and nb*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ be a GMEC-structure and o a state of knowledge, then

a. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{broken } \ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$ *iff* $\neg nb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$;

b. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (broken } \ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner\texttt{)}$ *iff* $nb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$.

Proof.

a. ($\Rightarrow$) Assume that the sequent $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{broken } \ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$ is derivable. By rule **atom+** on clause (5) and a number of applications of rule **and+**, we are left with the sequents below. For the sake of conciseness, we display the proof in a tabular form: the left column displays the derived sequents, the corresponding meta-mathematical property is shown in the central column, and the right column contains a justification of this correspondence.

$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e\urcorner$	$e \in E$	by definition of $\ulcorner E\urcorner$,
$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } \ulcorner e_1\urcorner\ \ulcorner e\urcorner$	$(e_1, e) \in o^+$	by lemma 3.5,
$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } \ulcorner e\urcorner\ \ulcorner e_2\urcorner$	$(e, e_2) \in o^+$	by lemma 3.5,
$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{initiates } \ulcorner e\urcorner\ \ulcorner q\urcorner;$ $\texttt{terminates } \ulcorner e\urcorner\ \ulcorner q\urcorner$	$(e \in [q\rangle$ $\vee\, e \in \langle q])$	by definition of $\ulcorner[\cdot\rangle\urcorner$ and of $\ulcorner\langle\cdot]\urcorner$,
$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{exclusive } \ulcorner p\urcorner\ \ulcorner q\urcorner;$ $\ulcorner p\urcorner \texttt{=} \ulcorner q\urcorner$	$(e \in]p, q[$ $\vee\, p = q)$	by definition of $\ulcorner]\cdot,\cdot[\urcorner$ and lemma 3.4.

We need to take the conjunction of the items in the central column in order to obtain a statement equivalent to $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{broken } \ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$:

$$(e_1, e) \in o^+ \ \wedge\ (e, e_2) \in o^+ \ \wedge\ (e \in [q\rangle \ \vee\ e \in \langle q]) \ \wedge\ (]p, q[\vee p = q).$$

We now abstract over the event e and the property q and obtain the formula

$$\exists e \in E.\, \exists q \in P.\, ((e_1, e) \in o^+ \ \wedge\ (e, e_2) \in o^+ \ \wedge\ (e \in [q\rangle \ \vee\ e \in \langle q]) \ \wedge\ (]p, q[\vee p = q))$$

that is equivalent, after some logical manipulations, to $nb(p, e_1, e_2, o^+)$.

($\Leftarrow$) Assume now that $\neg nb(p, e_1, e_2, o^+)$ is valid in $\mathcal{H}$, i.e. that

$$\exists e \in E.\, ((e_1, e) \in o^+ \ \wedge\ (e, e_2) \in o^+ \ \wedge\ \exists q \in P.\, ((e \in [q\rangle \ \vee\ e \in \langle q]) \ \wedge\ (]p, q[\vee p = q))).$$

Let e' and q' be such e and q respectively. Then, by instantiation, we obtain:

$$(e_1, e') \in o^+ \ \wedge\ (e', e_2) \in o^+ \ \wedge\ (e' \in [q'\rangle \ \vee\ e' \in \langle q']) \ \wedge\ (]p, q'[\vee p = q').$$

Each conjunct, plus the fact that $e' \in E$, can be immediately rewritten as a valid sequent. We use conventions similar to the ones adopted in the first part of this proof.

$e' \in E$	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e'\urcorner$	by definition of $\ulcorner E\urcorner$,

$(e_1, e') \in o^+$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `before` $\ulcorner e_1\urcorner\ \ulcorner e'\urcorner$	by lemma 3.5,
$(e', e_2) \in o^+$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `before` $\ulcorner e'\urcorner\ \ulcorner e_2\urcorner$	by lemma 3.5,
$(e' \in [q\rangle$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `initiates` $\ulcorner e'\urcorner\ \ulcorner q\urcorner$;	by definition of $\ulcorner[\cdot\rangle\urcorner$
$\vee\, e' \in \langle q])$	`terminates` $\ulcorner e'\urcorner\ \ulcorner q\urcorner$	and of $\ulcorner\langle\cdot]\urcorner$,
$(e \in]p, q[$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `exclusive` $\ulcorner p\urcorner\ \ulcorner q\urcorner$;	by definition of $\ulcorner]\cdot,\cdot[\urcorner$
$\vee\, p = q)$	$\ulcorner p\urcorner = \ulcorner q\urcorner$	and lemma 3.4.

We have proved in this way every goal in the body of clause (5). Thus, by a number of applications of rule **and+** and an application of rule **atom+**, the head of this clause is valid, i.e.

GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `broken` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$.

b. ($\Rightarrow$) By property 3.1 and (*a*).

($\Leftarrow$) By rule **naf+** and property 3.2, we are reduced to proving that GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `broken` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$ has only (possibly failed) finite derivations. Assume *ab absurdum* that there is an infinite derivation of this sequent. The last inference rules applied in this derivation must be **atom+** and **and+**. Therefore, one of the atomic formulas in the body of rule (5) must have an infinite derivation. Clearly, only predicates having a recursive definition are candidate. The only predicate having this property is `before`, but by lemma 3.5 this sequent has finite derivations only. ■

Theorem 3.7 (GMEC *computes MVIs*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ be a GMEC-structure and o a state of knowledge, then

a. GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `holds (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$) *iff* $p(e_1, e_2) \in MVI(o^+)$;

b. GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `not (holds (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$`))` *iff* $p(e_1, e_2) \notin MVI(o^+)$.

Proof.

a. ($\Rightarrow$) Assume that GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `holds (period` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$). We prove that, under this hypothesis, $(p(e_1, e_2), o^+) \in v_{\mathcal{H}}$; the thesis will follow by the definitions of validity and of the function $MVI(\cdot)$.

By applying rule **atom+** on clause (4), and then rule **and+**, we get reduced to proving the following relations, where, as in the proof of lemma 3.6, the left and central columns stand in an *if-and-only-if* relation justified by the right column.

GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` $\ulcorner e_1\urcorner$	$e_1 \in E$	by definition of $\ulcorner E\urcorner$,
GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `initiates` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner$	$e_1 \in [p\rangle$	by definition of $\ulcorner[\cdot\rangle\urcorner$,
GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` $\ulcorner e_2\urcorner$	$e_2 \in E$	by definition of $\ulcorner E\urcorner$,
GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `terminates` $\ulcorner e_2\urcorner\ \ulcorner p\urcorner$	$e_2 \in \langle p]$	by definition of $\ulcorner\langle\cdot]\urcorner$,
GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `before` $\ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$	$(e_1, e_2) \in o^+$	by lemma 3.5,
GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `not (broken` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner\ \ulcorner e_2\urcorner$)	$nb(p, e_1, e_2, o^+)$	by lemma 3.6.

Now, it suffices to notice that the second, fourth, fifth and sixth relation on the right-hand side correspond respectively to the conditions (*i*), (*ii*), (*iii*) and (*iv*) of the definition of evaluation. Therefore $(p(e_1, e_2), o^+) \in v_{\mathcal{H}}$, thus $\mathcal{I}_{\mathcal{H}}; o^+ \models p(e_1, e_2)$ and finally $p(e_1, e_2) \in MVI(o^+)$.

($\Leftarrow$) Assume that $p(e_1, e_2) \in MVI(o^+)$. Therefore, by definition, $(p(e_1, e_2), o^+) \in v_{\mathcal{H}}$, i.e.

$e_1 \in [p\rangle \wedge e_2 \in \langle p] \wedge (e_1, e_2) \in o^+ \wedge nb(p, e_1, e_2, o^+)$.

Each conjunct and the fact that $e_1, e_2 \in E$ can be related to sequent derivations by reversing the previous construction:

$e_1 \in E$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` $\ulcorner e_1\urcorner$	by definition of $\ulcorner E\urcorner$,
$e_1 \in [p\rangle$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `initiates` $\ulcorner e_1\urcorner\ \ulcorner p\urcorner$	by definition of $\ulcorner[\cdot\rangle\urcorner$,
$e_2 \in E$	GMEC, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` $\ulcorner e_2\urcorner$	by definition of $\ulcorner E\urcorner$,

$e_2 \in \langle p]$	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{terminates } \ulcorner e_2\urcorner \ \ulcorner p\urcorner$	by definition of $\ulcorner\langle\cdot]\urcorner$,
$(e_1, e_2) \in o^+$	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } \ulcorner e_1\urcorner \ \ulcorner e_2\urcorner$	by lemma 3.5,
$nb(p, e_1, e_2, o^+)$	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (broken } \ulcorner e_1\urcorner \ \ulcorner p\urcorner \ \ulcorner e_2\urcorner\texttt{)}$	by lemma 3.6.

Therefore, we have derivations for all the atomic formulas in the body of clause 4. By some applications of rule **and+** and then of rule **atom+**, we produce a derivation for the sequent

$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds (period } \ulcorner e_1\urcorner \ \ulcorner p\urcorner \ \ulcorner e_2\urcorner\texttt{)}$

b. ($\Rightarrow$) By property 3.1 and (a).

($\Leftarrow$) As in the proof of lemma 3.6, it suffices to prove that the sequent:

$$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds (period } \ulcorner e_1\urcorner \ \ulcorner p\urcorner \ \ulcorner e_2\urcorner\texttt{)}$$

has only (possibly failed) finite derivations. The last inference rule applied during the search for a proof of this sequent must be **atom+** on clause (4). Therefore, it has an infinite derivation if and only if an atomic subgoal in the body of this clause has an infinite derivation. However, by lemmas 3.5 and 3.6, and the definition of $\ulcorner\mathcal{H}\urcorner$, every such subgoal is finitely provable or unprovable. ■

Lemma 3.8 (*Soundness of* GMEC *w.r.t. the GMEC-frame semantics*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* o *a state of knowledge and* φ *and GMEC-formula, then*

a. if $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$, *then* $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi$;

b. if $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$, *then* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi$.

Proof.

Since the definition of the predicate `holds` contains recursive calls in the context of negation-as-failure (clauses (6), (9) and (10)), the statements (a) and (b) depend on each other. Therefore, we need to use a proof technique somewhat more elaborated than in the case of the previous results.

Indeed we will prove the two statements simultaneously by mutual induction. The inductive argument is on the ordered pair consisting of the number of connectives in the formula φ and height of the derivation trees for the sequents

a. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$ *and*

b. $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$.

Technically, this corresponds to a nested induction over the structure of φ and on the structure of the two sequent derivations.

For the sake of readability, we use singly framed labels to denote the proof cases for (a) and double frames for the proof cases for (b).

$\boxed{\varphi = p(e_1, e_2)}$ and $\boxed{\boxed{\varphi = p(e_1, e_2)}}$

The result follows by theorem 3.7.

$\boxed{\varphi = \neg\varphi'}$

[1]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (holds } \ulcorner\varphi'\urcorner\texttt{)}$	by rule **atom+** on clause (6),
[2]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$	by rule **naf+**,
[3]	$\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$	by induction hypothesis (b),
[4]	$\mathcal{I}_{\mathcal{H}}; o^+ \models \neg\varphi'$	by definition of $\models$.

$\boxed{\boxed{\varphi = \neg\varphi'}}$

[1]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (holds } \ulcorner\varphi'\urcorner\texttt{)}$	by rule **atom**$-$ on clause (6),
[2]	$\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$	by rule **naf**$-$,
[3]	$\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$	by induction hypothesis (a),
[4]	$\mathcal{I}_{\mathcal{H}}; o^+ \not\models \neg\varphi'$	by the consistency of $\models$.

$\boxed{\varphi = \varphi' \wedge \varphi''}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner, \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rules **atom+** on clause (7),
[2] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ by rule **and+** on [1],
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ by induction hypothesis (a),
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rule **and+** on [1],
[5] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ by induction hypothesis (a),
[6] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi' \wedge \varphi''$ by definition of $\models$ on [3, 5].

$\boxed{\boxed{\varphi = \varphi' \wedge \varphi''}}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner, \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rules **atom−** on clause (7),
[2] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ by rule **and**$-_1$ on [1],
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ by induction hypothesis (b),
[4] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rule **and**$-_2$ on [1],
[5] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ by induction hypothesis (b),
[6] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi' \wedge \varphi''$ by the consistency of $\models$ on [3, 5].

$\boxed{\varphi = \varphi' \vee \varphi''}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner; \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rules **atom+** on clause (8),
[2] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ by rule **or+**$_1$ on [1],
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ by induction hypothesis (a),
[4] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rule **or+**$_2$ on [1],
[5] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ by induction hypothesis (a),
[6] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi' \vee \varphi''$ by definition of $\models$ on [3, 5].

$\boxed{\boxed{\varphi = \varphi' \vee \varphi''}}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner; \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rules **atom−** on clause (8),
[2] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ by rule **or−** on [1],
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ by induction hypothesis (b),
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ by rule **or−** on [1],
[5] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ by induction hypothesis (b),
[6] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi' \vee \varphi''$ by the consistency of $\models$ on [3, 5].

$\boxed{\varphi = \Box\varphi'}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner,$ by rule **atom+** on clause (9),
$\texttt{not (fails\_must}\ \ulcorner\varphi'\urcorner\texttt{)}$
[2] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ by rule **and+** on [1],
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ by induction hypothesis (a),
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (fails\_must}\ \ulcorner\varphi'\urcorner\texttt{)}$ by rule **and+** on [1],
[5] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{fails\_must}\ \ulcorner\varphi'\urcorner$ by rules **naf+**,
[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens}\ t_1,$ by rule **atom−** on clause (10); since the variables E1 and E2 are implicitly quantified in front of the clause, this relation should hold for all terms t_1 and t_2,
$\texttt{happens}\ t_2,$
$\texttt{not (before}\ t_1\ t_2\texttt{)},$
$\texttt{not (before}\ t_2\ t_1\texttt{)},$
$\texttt{beforeFact}\ t_1\ t_2$
$\texttt{=> not (holds (must}\ \ulcorner\varphi'\urcorner\texttt{))}$
[7] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens}\ t_1$ by rule **and**$-_1$ on [6],
[8] $\llcorner t_1 \lrcorner \notin E$ by rule **atom−** and definition of $\ulcorner E\urcorner$,
[9] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens}\ t_2$ by rules **and**$-_2$ and **and**$-_1$ on [6],

[10] $\llcorner t_2 \lrcorner \notin E$ — by rule **atom**$-$ and definition of $\ulcorner E \urcorner$,

[11] • GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `not (before` t_1 t_2`)` — by rules **and**$-_2$ and **and**$-_1$ on [6],

[12] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `before` t_1 t_2 — by rules **naf**$-$,

[13] $(\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+$ — by lemma 3.5,

[14] • GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `not (before` t_2 t_1`)` — by rules **and**$-_2$ and **and**$-_1$ on [6],

[15] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `before` t_2 t_1 — by rules **naf**$-$,

[16] $(\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+$ — by lemma 3.5,

[17] • GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `beforeFact` t_1 t_2 `=> not (holds (must` $\ulcorner \varphi' \urcorner$`))` — by rule **and**$-_2$ on [6],

[18] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \urcorner \not\Longrightarrow$ `not (holds (must` $\ulcorner \varphi' \urcorner$`))` — by rule **impl**$-$ and the definition of $\ulcorner \cdot \urcorner^O$,

[19] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \urcorner \Longrightarrow$ `holds (must` $\ulcorner \varphi' \urcorner$`)` — by rule **naf**$-$,

[20] $\mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Box\varphi'$ — by induction hypothesis (a), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[21] $\forall t_1, t_2.\quad (\llcorner t_1 \lrcorner \notin E \vee \llcorner t_2 \lrcorner \notin E \vee (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+ \vee (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+ \vee \mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Box\varphi')$ — by taking the disjunction of [8, 10, 13, 16, 20],

[22] $\neg\exists t_1, t_2.\quad (\llcorner t_1 \lrcorner \in E \wedge \llcorner t_2 \lrcorner \in E \wedge (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \notin o^+ \wedge (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Box\varphi')$ — by logical equivalences,

[23] $\neg\exists e_1, e_2 \in E.\quad ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi')$ — by definition of $\ulcorner \cdot \urcorner^E$,

[24] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\varphi'$ — by combining [3] and [23] and lemma 2.12.

$\boxed{\boxed{\varphi = \Box\varphi'}}$

[1] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `holds` $\ulcorner \varphi' \urcorner$`, not (fails_must` $\ulcorner \varphi' \urcorner$`)` — by rule **atom**$-$ on clause (9),

[2] • GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `holds` $\ulcorner \varphi' \urcorner$ — by rule **and**$-_1$ on [1]

[3] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ — by induction hypothesis (b),

[4] • GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \not\Longrightarrow$ `not (fails_must` $\ulcorner \varphi' \urcorner$`)` — by rule **and**$-_1$ on [1]

[5] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `fails_must` $\ulcorner \varphi' \urcorner$ — by rule **naf**$-$,

[6] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `happens` t_1`, happens` t_2`, not (before` t_1 t_2`), not (before` t_2 t_1`), beforeFact` t_1 t_2 `=> not (holds (must` $\ulcorner \varphi' \urcorner$`))` — by rule **atom**$+$ on clause (10), for some term t_1 and t_2,

[7] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `happens` t_1 — by rule **and**$+$ on [6],

[8] $t_1 = \ulcorner e_1 \urcorner$ with $e_1 \in E$ — by definition of $\ulcorner E \urcorner$,

[9] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `happens` t_2 — by rule **and**$+$ on [6],

[10] $t_2 = \ulcorner e_2 \urcorner$ with $e_2 \in E$ — by definition of $\ulcorner E \urcorner$,

[11] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `not (before` $\ulcorner e_1 \urcorner$ $\ulcorner e_2 \urcorner$`)` — by rule **and**$+$ on [6],

[12] $(e_1, e_2) \notin o^+$ — by lemma 3.5,

[13] GMEC, $\ulcorner \mathcal{H} \urcorner, \ulcorner o \urcorner \Longrightarrow$ `not (before` $\ulcorner e_2 \urcorner$ $\ulcorner e_1 \urcorner$`)` — by rule **and**$+$ on [6],

[14] $(e_2, e_1) \notin o^+$ by lemma 3.5,

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `beforeFact` $\ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ `=> not (holds (must` $\ulcorner\varphi'\urcorner$`))` by rule **and+** on [6],

[16] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \Longrightarrow$ `not (holds (must` $\ulcorner\varphi'\urcorner$`))` by rule **impl+** and the definition of $\ulcorner\cdot\urcorner^O$,

[17] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \not\Longrightarrow$ `holds (must` $\ulcorner\varphi'\urcorner$`)` by rule **naf+**,

[18] $\mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi'$ by induction hypothesis (b), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[19] $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi')$ by taking the conjunction of [8, 10, 12, 14, 18],

[20] $\mathcal{I}_\mathcal{H}; o^+ \not\models \Box\varphi'$ by Lemma 2.12 on [3, 19].

$\boxed{\varphi = \Diamond\varphi'}$

[1] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `holds` $\ulcorner\varphi'\urcorner$ by rules **atom+** on clause (11),

[2] $\mathcal{I}_\mathcal{H}; o^+ \models \varphi'$ by induction hypothesis (a),

[3] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` t_1`,` `happens` t_2`,` `not (before` t_1 t_2`),` `not (before` t_2 t_1`),` `beforeFact` t_1 t_2 `=> holds (may` $\ulcorner\varphi'\urcorner$`)` by rule **atom+** on clause (12), for some term t_1 and t_2,

[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` t_1 by rule **and+** on [3],

[5] $t_1 = \ulcorner e_1\urcorner$ with $e_1 \in E$ by definition of $\ulcorner E\urcorner$,

[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `happens` t_2 by rule **and+** on [3],

[7] $t_2 = \ulcorner e_2\urcorner$ with $e_2 \in E$ by definition of $\ulcorner E\urcorner$,

[8] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `not (before` $\ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$`)` by rule **and+** on [3],

[9] $(e_1, e_2) \notin o^+$ by lemma 3.5,

[10] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `not (before` $\ulcorner e_2\urcorner\ \ulcorner e_1\urcorner$`)` by rule **and+** on [3],

[11] $(e_2, e_1) \notin o^+$ by lemma 3.5,

[12] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ `beforeFact` $\ulcorner e_1\urcorner\ \ulcorner e_2\urcorner$ `=> holds (may` $\ulcorner\varphi'\urcorner$`)` by rule **and+** on [3],

[13] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \Longrightarrow$ `holds (may` $\ulcorner\varphi'\urcorner$`)` by rule **impl+** and the definition of $\ulcorner\cdot\urcorner^O$,

[14] $\mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi'$ by induction hypothesis (b), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[15] $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi')$ by taking the conjunction of [5, 7, 9, 11, 14],

[16] $\mathcal{I}_\mathcal{H}; o^+ \models \Diamond\varphi'$ by lemma 2.12 on [2, 15].

$\boxed{\boxed{\varphi = \Diamond\varphi'}}$

[1] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow$ `holds` $\ulcorner\varphi'\urcorner$ by rules **atom−** on clause (11),

[2] $\mathcal{I}_\mathcal{H}; o^+ \not\models \varphi'$ by induction hypothesis (b),

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow$ `happens` t_1`,` `happens` t_2`,` `not (before` t_1 t_2`),` `not (before` t_2 t_1`),` `beforeFact` t_1 t_2 `=> holds (may` $\ulcorner\varphi'\urcorner$`)` by rule **atom−** on clause (12); since the variables `E1` and `E2` are implicitly quantified in front of the clause, this relation should hold for all terms t_1 and t_2,

[4] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_1$ — by rule **and**$-_1$ on [3],

[5] $\llcorner t_1 \lrcorner \notin E$ — by rule **atom**$-$ and definition of $\ulcorner E\urcorner$,

[6] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_2$ — by rules **and**$-_2$ and **and**$-_1$ on [3],

[7] $\llcorner t_2 \lrcorner \notin E$ — by rule **atom**$-$ and definition of $\ulcorner E\urcorner$,

[8] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_1\ t_2\texttt{)}$ — by rules **and**$-_2$ and **and**$-_1$ on [3],

[9] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_1\ t_2$ — by rules **naf**$-$,

[10] $(\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+$ — by lemma 3.5,

[11] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_2\ t_1\texttt{)}$ — by rules **and**$-_2$ and **and**$-_1$ on [3],

[12] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_2\ t_1$ — by rules **naf**$-$,

[13] $(\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+$ — by lemma 3.5,

[14] • $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{beforeFact } t_1\ t_2$ $\texttt{=> holds (may } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **and**$-_2$ on [3],

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\urcorner \not\Longrightarrow \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **impl**$-$ and the definition of $\ulcorner\cdot\urcorner^O$,

[16] $\mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Diamond\varphi'$ — by induction hypothesis (b), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[17] $\forall t_1, t_2.\quad (\llcorner t_1 \lrcorner \notin E \vee \llcorner t_2 \lrcorner \notin E \vee (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+ \vee (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+ \vee \mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Diamond\varphi')$ — by taking the disjunction of [5, 7, 10, 13, 16],

[18] $\neg\exists t_1, t_2.\quad (\llcorner t_1 \lrcorner \in E \wedge \llcorner t_2 \lrcorner \in E \wedge (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \notin o^+ \wedge (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Diamond\varphi')$ — by logical equivalences,

[19] $\neg\exists e_1, e_2 \in E.\quad ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi')$ — by definition of $\ulcorner\cdot\urcorner^E$,

[20] $\mathcal{I}_\mathcal{H}; o^+ \not\models \Diamond\varphi'$ — by combining [2] and [19] and lemma 2.12. ■

Lemma 3.9 (*Completeness of* GMEC *w.r.t. the GMEC-frame semantics*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* o *a state of knowledge and* φ *and GMEC-formula, then*

a. if $\mathcal{I}_\mathcal{H}; o^+ \models \varphi$, *then* $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$;

b. if $\mathcal{I}_\mathcal{H}; o^+ \not\models \varphi$, *then* $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi\urcorner$.

Proof.

As in the previous lemma, we need to cope with the two statements simultaneously. Therefore, we proceed by a nested mutual induction on the structure of the formula φ and the cardinality of $Ext(o^+)$.

We rely on essentially the same conventions as in the proof of lemma 3.8. The two proofs are essentially dual.

$\boxed{\varphi = p(e_1, e_2)}$ and $\boxed{\boxed{\varphi = p(e_1, e_2)}}$

The desired result follows by theorem 3.7.

$\boxed{\varphi = \neg\varphi'}$

[1] $\mathcal{I}_\mathcal{H}; o^+ \not\models \varphi'$ — by definition of $\models$,

[2] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$ — by induction hypothesis (b),

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (holds } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **naf**$+$,

[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\neg\varphi'\urcorner$ — by rule **atom**$+$ on clause (6).

$\boxed{\varphi = \neg\varphi'}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ — by the consistency of $\models$,
[2] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ — by induction hypothesis (a),
[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (holds}\ \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **naf**–,
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\neg\varphi'\urcorner$ — by rule **atom**– on clause (6).

$\boxed{\varphi = \varphi' \wedge \varphi''}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ *and* $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ — by definition of $\models$,
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ — conjunct in [1]
[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ — by induction hypothesis (a),
[4] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ — conjunct in [1]
[5] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ — by induction hypothesis (a),
[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{, holds}\ \ulcorner\varphi''\urcorner$ — by rule **and**+ on [3, 5],
[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi' \wedge \varphi''\urcorner$ — by rule **atom**+ on clause (7).

$\boxed{\varphi = \varphi' \wedge \varphi''}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ *or* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ — by the consistency of $\models$,
[2] • $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ — subcase of [1]
[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ — by induction hypothesis (b),
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{, holds}\ \ulcorner\varphi''\urcorner$ — by rule **and**–$_1$,
[5] • $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ — subcase of [1]
[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ — by induction hypothesis (b),
[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{, holds}\ \ulcorner\varphi''\urcorner$ — by rule **and**–$_2$,
[8] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi' \wedge \varphi''\urcorner$ — by rules **atom**– on [4, 7] and clause (7).

$\boxed{\varphi = \varphi' \vee \varphi''}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ *or* $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ — by definition of $\models$,
[2] • $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ — subcase of [1]
[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ — by induction hypothesis (a),
[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{; holds}\ \ulcorner\varphi''\urcorner$ — by rule **or**+$_1$,
[5] • $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi''$ — subcase of [1]
[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ — by induction hypothesis (a),
[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{; holds}\ \ulcorner\varphi''\urcorner$ — by rule **or**+$_2$,
[8] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds}\ \ulcorner\varphi' \vee \varphi''\urcorner$ — by rules **atom**+ on [4, 7] and clause (8).

$\boxed{\varphi = \varphi' \vee \varphi''}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ *and* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ — by the consistency of $\models$,
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ — conjunct in [1]
[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner$ — by induction hypothesis (b),
[4] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi''$ — conjunct in [1]
[5] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi''\urcorner$ — by induction hypothesis (b),
[6] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi'\urcorner\texttt{; holds}\ \ulcorner\varphi''\urcorner$ — by rule **or**– on [3, 5],
[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds}\ \ulcorner\varphi' \vee \varphi''\urcorner$ — by rule **atom**– on clause (8).

$\boxed{\varphi = \Box\varphi'}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\varphi'$ — assumption
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ — by Lemma 2.12 on [1],

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$ — by induction hypothesis (a),

[4] $\neg\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi')$ — by Lemma 2.12 on [1],

[5] $\neg\exists t_1, t_2.\ (\llcorner t_1 \lrcorner \in E \wedge \llcorner t_2 \lrcorner \in E \wedge (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \notin o^+ \wedge (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Box\varphi')$ — by definition of $\ulcorner\cdot\urcorner^E$

[6] $\forall t_1, t_2.\ (\llcorner t_1 \lrcorner \notin E \vee \llcorner t_2 \lrcorner \notin E \vee (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+ \vee (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+ \vee \mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Box\varphi')$ — by logical equivalences

[7] • $\llcorner t_1 \lrcorner \notin E$ — subcase of [6]

[8] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_1$ — by rule **atom**− and definition of $\ulcorner E\urcorner$,

[9] • $\llcorner t_2 \lrcorner \notin E$ — subcase of [6]

[10] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_2$ — by rule **atom**− and definition of $\ulcorner E\urcorner$,

[11] • $(\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+$ — subcase of [6]

[12] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_1\ t_2$ — by lemma 3.5,

[13] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_1\ t_2\texttt{)}$ — by rules **naf**−,

[14] • $(\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+$ — subcase of [6]

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_2\ t_1$ — by lemma 3.5,

[16] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_2\ t_1\texttt{)}$ — by rules **naf**−,

[17] • $\mathcal{I}_\mathcal{H}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Box\varphi'$ — subcase of [6]

[18] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\urcorner \Longrightarrow \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ — by induction hypothesis (a), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[19] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\urcorner \not\Longrightarrow \texttt{not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rule **naf**−,

[20] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{beforeFact } t_1\ t_2\ \texttt{=> not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rule **impl**− and the definition of $\ulcorner\cdot\urcorner^O$,

[21] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_1\texttt{,}$ $\texttt{happens } t_2\texttt{,}$ $\texttt{not (before } t_1\ t_2\texttt{),}$ $\texttt{not (before } t_2\ t_1\texttt{),}$ $\texttt{beforeFact } t_1\ t_2$ $\texttt{=> not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rules **and**−$_1$ and **and**−$_2$ on [8, 10, 13, 16, 20],

[22] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{fails\_must } \ulcorner\varphi'\urcorner$ — by rule **atom**− on clause (10); this relation should hold for all terms t_1 and t_2 since the variables `E1` and `E2` are implicitly quantified in front of the clause,

[23] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (fails\_must } \ulcorner\varphi'\urcorner\texttt{)}$ — by rules **naf**+,

[24] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\Box\varphi'\urcorner$ — by rules **and**+ on [3, 23] and **atom**+ on clause (9).

$\boxed{\varphi = \Box\varphi'}$

[1] $\mathcal{I}_\mathcal{H}; o^+ \not\models \varphi'$ *or* $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_\mathcal{H}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi')$ — by Lemma 2.12,

[2] • $\mathcal{I}_\mathcal{H}; o^+ \not\models \varphi'$ — subcase of [1]

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$ — by induction hypothesis (b),

[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$, $\texttt{not (fails\_must } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **and**$-_1$,

[5] • $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi')$ — subcase of [1]

[6] $e_1 \in E$ — conjunct in [5]

[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_1\urcorner$ — by definition of $\ulcorner E\urcorner$,

[8] $e_2 \in E$ — conjunct in [5]

[9] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_2\urcorner$ — by definition of $\ulcorner E\urcorner$,

[10] $(e_1, e_2) \notin o^+$ — conjunct in [5]

[11] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\texttt{)}$ — by lemma 3.5,

[12] $(e_2, e_1) \notin o^+$ — conjunct in [5]

[13] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (before } \ulcorner e_2\urcorner\ \ulcorner e_1\urcorner\texttt{)}$ — by lemma 3.5,

[14] $\mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \not\models \Box\varphi'$ — conjunct in [5]

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \not\Longrightarrow \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ — by induction hypothesis (b),, since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o

[16] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \Longrightarrow \texttt{not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rule **naf+**,

[17] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner \texttt{ => not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rule **impl+** and the definition of $\ulcorner\cdot\urcorner^O$,

[18] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_1\urcorner\texttt{, happens } \ulcorner e_2\urcorner\texttt{, not (before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\texttt{), not (before } \ulcorner e_2\urcorner\ \ulcorner e_1\urcorner\texttt{), beforeFact } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner \texttt{ => not (holds (must } \ulcorner\varphi'\urcorner\texttt{))}$ — by rule **and+** on [7, 9, 11, 13, 17],

[19] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{fails\_must } \ulcorner\varphi'\urcorner$ — by rule **atom+** on clause (10), with `E1` and `E2` instantiated to $\ulcorner e_1\urcorner$ and $\ulcorner e_2\urcorner$ respectively,

[20] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (fails\_must } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **naf**$-$,

[21] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner\texttt{, not (fails\_must } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **and**$-_2$,

[22] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\Box\varphi'\urcorner$ — by rules **atom**$-$ on [4, 21] and clause (9).

$\boxed{\varphi = \Diamond\varphi'}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ *or* $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi')$ — by lemma 2.12,

[2] • $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi'$ — subcase of [1]

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$ — by induction hypothesis (a),

[4] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\Diamond\varphi'\urcorner$ — by rule **atom+** on clause (11),

[5] • $\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi')$ — subcase of [1]

[6] $e_1 \in E$ — conjunct in [5]

[7] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_1\urcorner$ — by definition of $\ulcorner E\urcorner$,

[8] $e_2 \in E$ — conjunct in [5]

[9] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_2\urcorner$ — by definition of $\ulcorner E\urcorner$,

[10] $(e_1, e_2) \notin o^+$ — conjunct in [5]

[11] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\texttt{)}$ — by lemma 3.5,

[12] $(e_2, e_1) \notin o^+$ — conjunct in [5]

[13] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{not (before } \ulcorner e_2\urcorner\ \ulcorner e_1\urcorner\texttt{)}$ — by lemma 3.5,

[14] $\mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi'$ — conjunct in [5]

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (e_1, e_2)\urcorner \Longrightarrow \texttt{holds (may } \ulcorner\varphi'\urcorner\texttt{)}$ — by induction hypothesis (a), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[16] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{beforeFact } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\ \texttt{=> holds (may } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **impl+** and the definition of $\ulcorner\cdot\urcorner^O$,

[17] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{happens } \ulcorner e_1\urcorner\texttt{, happens } \ulcorner e_2\urcorner\texttt{, not (before } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\texttt{), not (before } \ulcorner e_2\urcorner\ \ulcorner e_1\urcorner\texttt{), beforeFact } \ulcorner e_1\urcorner\ \ulcorner e_2\urcorner\ \texttt{=> holds (may } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **atom+** on [7, 9, 11, 13, 16] and clause (12), with $\ulcorner e_1\urcorner$ and $\ulcorner e_2\urcorner$ substituted for the variables E1 and E2 respectively,

[18] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds } \ulcorner\Diamond\varphi'\urcorner$ — by rules **atom+** on clause (12).

$\boxed{\varphi = \Diamond\varphi'}$

[1] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Diamond\varphi'$ — assumption

[2] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi'$ — by lemma 2.12 on [1],

[3] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{holds } \ulcorner\varphi'\urcorner$ — by induction hypothesis (b),

[4] $\neg\exists e_1, e_2 \in E.\ ((e_1, e_2) \notin o^+ \wedge (e_2, e_1) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (e_1, e_2)\}^+ \models \Diamond\varphi')$ — by lemma 2.12 on [1],

[5] $\neg\exists t_1, t_2.\ (\llcorner t_1 \lrcorner \in E \wedge \llcorner t_2 \lrcorner \in E \wedge (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \notin o^+ \wedge (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \notin o^+ \wedge \mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \models \Diamond\varphi')$ — by definition of $\ulcorner\cdot\urcorner^E$,

[6] $\forall t_1, t_2.\ (\llcorner t_1 \lrcorner \notin E \vee \llcorner t_2 \lrcorner \notin E \vee (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+ \vee (\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+ \vee \mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Diamond\varphi')$ — by logical equivalences,

[7] • $\llcorner t_1 \lrcorner \notin E$ — subcase of [6]

[8] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_1$ — by rule **atom−** and definition of $\ulcorner E\urcorner$,

[9] • $\llcorner t_2 \lrcorner \notin E$ — subcase of [6]

[10] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{happens } t_2$ — by rule **atom−** and definition of $\ulcorner E\urcorner$,

[11] • $(\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner) \in o^+$ — subcase of [6]

[12] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_1\ t_2$ — by lemma 3.5,

[13] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_1\ t_2\texttt{)}$ — by rules **naf−**,

[14] • $(\llcorner t_2 \lrcorner, \llcorner t_1 \lrcorner) \in o^+$ — subcase of [6]

[15] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{before } t_2\ t_1$ — by lemma 3.5,

[16] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{not (before } t_2\ t_1\texttt{)}$ — by rules **naf−**,

[17] • $\mathcal{I}_{\mathcal{H}}; \{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+ \not\models \Diamond\varphi'$ — subcase of [6]

[18] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\urcorner \not\Longrightarrow \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ — by induction hypothesis (b), since $\{o \uparrow (\llcorner t_1 \lrcorner, \llcorner t_2 \lrcorner)\}^+$ is a proper extension of o,

[19] $\texttt{GMEC}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow \texttt{beforeFact } t_1\ t_2\ \texttt{=> holds (may } \ulcorner\varphi'\urcorner\texttt{)}$ — by rule **impl−** and the definition of $\ulcorner\cdot\urcorner^O$,

[20]	GMEC, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \not\Longrightarrow$ happens t_1, happens t_2, not (before t_1 t_2), not (before t_2 t_1), beforeFact t_1 t_2 => holds (may $\ulcorner\varphi'\urcorner$)	by rules **and**$-_1$ and **and**$-_2$ on [8, 10, 13, 16, 19]; this is provable for all terms t_1 and t_2,
[21]	GMEC, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \not\Longrightarrow$ holds $\ulcorner\Diamond\varphi'\urcorner$	by rule **atom**$-$ on clauses (11) and (12) for [3] and [20] respectively. ■

Theorem 3.10 (*Soundness and completeness of* GMEC *w.r.t. the GMEC-frame semantics*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* o *a state of knowledge and* φ *and GMEC-formula, then*

a. GMEC, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ holds $\ulcorner\varphi\urcorner$ *iff* $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi$;

b. GMEC, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not (holds $\ulcorner\varphi\urcorner$) *iff* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi$.

Proof.

By rules **naf+** and **naf**$-$, the second statement can be rewritten as

b′. GMEC, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \not\Longrightarrow$ holds $\ulcorner\varphi\urcorner$ *iff* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi$.

It suffices now to apply lemmas 3.8 and 3.9 to the two directions of (a) and (b') to prove the theorem. ■

Lemma 3.11 (*Correspondence between* skeBroken *and nsb*)

Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then*

a. GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ skeBroken $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$ *iff* $\neg nsb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$;

b. GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not (skeBroken $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$) *iff* $nsb(p, e_1, e_2, o^+)$ *holds in* $\mathcal{H}$.

Proof.

We proceed as in the proof of lemma 3.6.

a. ($\Rightarrow$) By unfolding clause ($5'$), we obtain the following relations.

GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ happens $\ulcorner e\urcorner$	$e \in E$	by definition of $\ulcorner E\urcorner$,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not ($\ulcorner e\urcorner$ = $\ulcorner e_1\urcorner$)	$e \neq e_1$	by lemma 3.3,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not ($\ulcorner e\urcorner$ = $\ulcorner e_2\urcorner$)	$e \neq e_2$	by lemma 3.3,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not (before $\ulcorner e\urcorner$ $\ulcorner e_1\urcorner$)	$(e, e_1) \notin o^+$	by lemma 3.5,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ not (before $\ulcorner e_2\urcorner$ $\ulcorner e\urcorner$)	$(e_2, e) \notin o^+$	by lemma 3.5,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ initiates $\ulcorner e\urcorner$ $\ulcorner q\urcorner$; terminates $\ulcorner e\urcorner$ $\ulcorner q\urcorner$	$(e \in [q\rangle$ $\vee\ e \in \langle q])$	by definition of $\ulcorner[\cdot\rangle\urcorner$ and of $\ulcorner\langle\cdot]\urcorner$,
GMEC+, $\ulcorner\mathcal{H}\urcorner$, $\ulcorner o\urcorner \Longrightarrow$ exclusive $\ulcorner p\urcorner$ $\ulcorner q\urcorner$; $\ulcorner p\urcorner$=$\ulcorner q\urcorner$	$(e \in]p, q[$ $\vee\ p = q)$	by definition of $\ulcorner]\cdot,\cdot[\urcorner$ and lemma 3.4.

By taking the conjunction of the formulas displayed in the central column, we have:

$$e \neq e_1 \wedge e \neq e_2 \wedge (e, e_1) \notin o^+ \wedge (e_2, e) \notin o^+ \wedge ((e \in [q\rangle \vee e \in \langle q]) \wedge (]p, q[\vee p = q))$$

By abstracting over e and q, we obtain

$$\begin{aligned} \exists e \in E.\, \exists q \in P.\quad & e \neq e_1 \\ \wedge\quad & e \neq e_2 \\ \wedge\quad & (e, e_1) \notin o^+ \\ \wedge\quad & (e_2, e) \notin o^+ \\ \wedge\quad & ((e \in [q\rangle \vee e \in \langle q]) \wedge (]p, q[\vee p = q)) \end{aligned}$$

that is equivalent, after some logical manipulations, to $nsb(p, e_1, e_2, o^+)$.

($\Leftarrow$) Similarly to the situation encountered in the proof of lemma 3.6, this direction of the proof follows by simply reversing the reasoning pattern just used. We omit it.

b. ($\Leftarrow$) By property 3.1 and (*a*).
($\Rightarrow$) This direction follows by property 3.2 since the only calls in clause ($10'$) that invoke recursive definitions involve the predicate `before`, that has only finite derivations, by lemma 3.5. ■

Theorem 3.12 (`GMEC+` *computes necessary MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ holds (must (period` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`))` *iff* $p(e_1, e_2) \in \Box MVI(o^+)$;
b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ not (holds (must (period` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`)))` *iff* $p(e_1, e_2) \notin \Box MVI(o^+)$.

Proof.
We proceed as in the proof of theorem 3.7.

a. ($\Rightarrow$) Assume that `GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ holds (must (period` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`))` is derivable. We will prove that $e_1 \in [p\rangle$, $e_2 \in \langle p]$, $(e_1, e_2) \in o^+$ and $nsb(p, e_1, e_2, o^+)$ are entailed by this hypothesis. The thesis will follow by lemma 2.13.
By unfolding clause ($9'$) we obtain the following relations:

`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ happens` $\ulcorner e_1\urcorner$	$e_1 \in E$	by definition of $\ulcorner E\urcorner$,
`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ initiates` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$	$e_1 \in [p\rangle$	by definition of $\ulcorner[\cdot\rangle\urcorner$,
`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ happens` $\ulcorner e_2\urcorner$	$e_2 \in E$	by definition of $\ulcorner E\urcorner$,
`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ terminates` $\ulcorner e_2\urcorner$ $\ulcorner p\urcorner$	$e_2 \in \langle p]$	by definition of $\ulcorner\langle\cdot]\urcorner$,
`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ before` $\ulcorner e_1\urcorner$ $\ulcorner e_2\urcorner$	$(e_1, e_2) \in o^+$	by lemma 3.5,
`GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ not (skeBroken` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`)`	$nb(p, e_1, e_2, o^+)$	by lemma 3.11.

The central column contains all the hypotheses needed for the application of lemma 2.13.

($\Leftarrow$) As in the proof of theorem 3.7, this direction follows by simply reversing the reasoning pattern just used. We omit it.

b. ($\Leftarrow$) By property 3.1 and (*a*).
($\Rightarrow$) By the definition of $\ulcorner\mathcal{H}\urcorner$ and lemmas 3.5 and 3.11, clause ($9'$) cannot start a diverging derivation. The desired result follows from property 3.2. ■

Theorem 3.13 (`GMEC+` *computes possible MVIs*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure and* o *a state of knowledge, then*

a. `GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ holds (may (period` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`))` *iff* $p(e_1, e_2) \in \Diamond MVI(o^+)$;
b. `GMEC+,`$\ulcorner\mathcal{H}\urcorner$`,`$\ulcorner o\urcorner$ `⟹ not (holds (may (period` $\ulcorner e_1\urcorner$ $\ulcorner p\urcorner$ $\ulcorner e_2\urcorner$`)))` *iff* $p(e_1, e_2) \notin \Diamond MVI(o^+)$.

Proof.
Similar to the proofs of theorems 3.7 and 3.12. ■

Theorem 3.14 (*Soundness and completeness of* `GMEC+` *w.r.t. the GMEC-frame semantics*)
Let $\mathcal{H} = (E,\ P,\ [\cdot\rangle,\ \langle\cdot],\ ]\cdot,\cdot[)$ *be a GMEC-structure,* o *a state of knowledge and* φ *and GMEC-formula, then*

a. GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ holds $\ulcorner\varphi\urcorner$ *iff* $\mathcal{I}_{\mathcal{H}}; o^+ \models \varphi$;

b. GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ not (holds $\ulcorner\varphi\urcorner$) *iff* $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \varphi$.

Proof.

As in theorem 3.10, (a) and (b) must be proved simultaneously in each direction. We will only sketch the proof for the forward direction ($\Rightarrow$). The present discussion together with the detailed proof of the analogous case treated as lemma 3.9 should suffice to the intrepid reader to reconstruct this long proof in its entirety.

The forward direction of the proof requires the techniques exploited in the proof of lemma 3.8, with the only difference that we need to distinguish finer proof cases for the modal formulas. More precisely, whenever the main connective of a formula is modal, we must consider the main connective of its immediate subformula.

For the sake of conciseness, we will perform the proof only for cases where the main connective is $\Box$. Again, we leave the rest of the proof to the valiant reader (the cases for $\Diamond$ are similar, and whenever the main connective is non-modal, the analogous cases in the proof of lemma 3.8 apply unchanged).

We are performing a mutual nested induction on the structure of the formula φ and of the derivations for the sequents

a. GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ holds $\ulcorner\varphi\urcorner$ *and*

b'. GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow$ holds $\ulcorner\varphi\urcorner$.

Again, we use single frames to label proof cases for (a), and double frames for proof cases for (b').

$\boxed{\varphi = \Box p(e_1, e_2)}$ and $\boxed{\boxed{\varphi = \Box p(e_1, e_2)}}$

The result follows by theorem 3.12.

$\boxed{\varphi = \Box\neg\varphi'}$

[1]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner$ $\Longrightarrow$ holds (not (may $\ulcorner\varphi'\urcorner$))	by rule **atom+** on clause $(11')$,
[2]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ not (holds $\ulcorner\Diamond\varphi'\urcorner$)	by rule **atom+** on clause $(6')$,
[3]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow$ holds $\ulcorner\Diamond\varphi'\urcorner$	by rule **naf+**,
[4]	$\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Diamond\varphi'$	by induction hypothesis (b),
[5]	$\mathcal{I}_{\mathcal{H}}; o^+ \models \neg\Diamond\varphi'$	by definition of $\models$,
[6]	$\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\neg\varphi'$	by property 2.7.

$\boxed{\boxed{\varphi = \Box\neg\varphi'}}$

[1]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner$ $\not\Longrightarrow$ holds (not (may $\ulcorner\varphi'\urcorner$))	by rule **atom−** on clause $(11')$,
[2]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \not\Longrightarrow$ not (holds $\ulcorner\Diamond\varphi'\urcorner$)	by rule **atom−** on clause $(6')$,
[3]	GMEC+, $\ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow$ holds $\ulcorner\Diamond\varphi'\urcorner$	by rule **naf−**,
[4]	$\mathcal{I}_{\mathcal{H}}; o^+ \models \Diamond\varphi'$	by induction hypothesis (a),
[5]	$\mathcal{I}_{\mathcal{H}}; o^+ \not\models \neg\Diamond\varphi'$	by the consistency of $\models$,
[6]	$\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Box\neg\varphi'$	by property 2.7.

$\boxed{\varphi = \Box(\varphi' \wedge \varphi'')}$ and $\boxed{\boxed{\varphi = \Box(\varphi' \wedge \varphi'')}}$

Similarly to the previous case, clause $(12')$ is used to push the modality inside the formula. Then the technique seen in the proof of lemma 3.8 for the cases concerning conjunction is applied. Finally, we appeal to property 2.7 to restore φ by pushing $\Box$ out as its main connective.

$\boxed{\varphi = \Box(\varphi' \vee \varphi'')}$ and $\boxed{\boxed{\varphi = \Box(\varphi' \vee \varphi'')}}$

Take verbatim the proof cases for $\Box$ from the proof of lemma 3.8 changing simply the reference to clause (9) and (10) to references to clauses $(13')$ and $(14')$ respectively. Clearly the structure of the subformula $\varphi' \vee \varphi''$ needs not to be expanded.

$\boxed{\varphi = \Box\Box\varphi'}$

[1] $\texttt{GMEC+}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ by rule **atom+** on clause $(15')$,
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\varphi'$ by induction hypothesis (a),
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\Box\varphi'$ by property 2.7.

$\boxed{\boxed{\varphi = \Box\Box\varphi'}}$

[1] $\texttt{GMEC+}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow\!\!\!\!\!/\ \texttt{holds (must } \ulcorner\varphi'\urcorner\texttt{)}$ by rule **atom−** on clause $(15')$,
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Box\varphi'$ by induction hypothesis (b),
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Box\Box\varphi'$ by property 2.7.

$\boxed{\varphi = \Box\Diamond\varphi'}$

Both clauses $(17')$ and $(16')$ can have been used by rule **atom+** as the last derivation step. In the first case, we simply need to transpose the corresponding proof case for $\Box$ from the proof of lemma 3.8. The second case applies only if $\varphi = \Box\Diamond\Box\Diamond\varphi''$. We have the following derivation:

[1] $\texttt{GMEC+}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow \texttt{holds (must (may } \ulcorner\varphi'\urcorner\texttt{))}$ by rule **atom+** on clause $(16')$,
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\Diamond\varphi''$ by induction hypothesis (a),
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \models \Box\Diamond\Box\Diamond\varphi''$ by property 2.7.

$\boxed{\boxed{\varphi = \Box\Diamond\varphi'}}$

We must again distinguish two cases, based on the structure of φ'. If this formula is not of the form $\Box\Diamond\varphi''$, we behave as in the corresponding proof case for $\Box$-moded formulas in the proof of lemma 3.8.

Otherwise, the last rule applied must be **atom−** on clauses $(17')$ and $(16')$. The branch concerning the first clause is handled again as the second proof case for $\Box$ from the proof of lemma 3.8. The branch referring to the second clause is instead handled as follows:

[1] $\texttt{GMEC+}, \ulcorner\mathcal{H}\urcorner, \ulcorner o\urcorner \Longrightarrow\!\!\!\!\!/\ \texttt{holds (must (may } \ulcorner\varphi'\urcorner\texttt{))}$ subcase generated by clause $(16')$
[2] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Box\Diamond\varphi''$ by induction hypothesis (b),
[3] $\mathcal{I}_{\mathcal{H}}; o^+ \not\models \Box\Diamond\Box\Diamond\varphi''$ by property 2.7. ■

Throw this page away

1
2
3
4
5
6
7
8
9
10
11
12
1′
2′
3′
4′
5′
6′
7′
8′
9′
10′
11′
12′
13′
14′
15′
16′
17′
18′
19′
20′
21′
22′
23′
24′
25′
26′
27′