Maximum Cardinality Search and Chordal Graphs

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Definitions:

- A graph G is chordal if every cycle of length > 3 has a chord, that is, an edge between vertices of the cycle that is not itself in the cycle.
- A vertex v is simplicial if $N(v)$, the neighbors of v, form a complete subgraph.
- An ordering $v_1, \dots, v_n$ of the vertices of G is a simplicial elimination ordering if v_i is simplicial in the subgraph $v_1, \dots, v_i$, for all $i \leq n$.
- The maximum cardinality search algorithm works as follows: Initialize $W \leftarrow V$ where $V = V(G)$, and set $\mathsf{weight}(v) = 0$ for all $v \in V$. For each $i = 1, \dots n$, let u be a node with maximal weight in W, set $v_i \leftarrow u$, and increment the weight of all neighbors of u in W by one. Then remove u from W and repeat. The resulting order is the sequence $v_1, \dots, v_n$.

Theorem 1. Performing a maximum cardinality search on a graph G produces a simplicial elimination ordering for G if and only if G is chordal.

Suppose G is not chordal. Then I claim there is no simplicial elimination ordering of G, so the maximum cardinality search cannot return one. Suppose that we have such an ordering. Let $s_0, s_1, \dots, s_k = s_0$ be a cycle of length $k > 3$ with no chord, and suppose s_i appears last in the ordering. Then s_{i-1} and s_{i+1} are not equal (since $k > 2$) and are not adjacent (since $k > 3$ and otherwise this would be a chord), so s_i is not simplicial in the subgraph $v_1, \dots, v_m = s_i$.

For the converse, suppose that G is chordal, and suppose that $v_1, \dots, v_n$ is the ordering returned by the MCS algorithm. Let $V_i = \{v_1, \dots, v_i\}$, and let $v \prec w$ mean that v appears before w in the ordering. Let "v_i is simplicial" denote that v_i is simplicial in its subgraph V_i, that is, whenever $a, b \prec v_i$ and $a, b \in N(v_i)$ then $a \in N(b)$.

Then for each $i \leq n$, after the loop has executed i times, the following loop invariant holds (with subscripts on W and weight to indicate values depending on the loop counter):

1. $u = v_i$ (if $i > 0$)
2. $W_i = V - V_i$
3. $v_1, \dots, v_i$ are distinct, and have been set to their final values

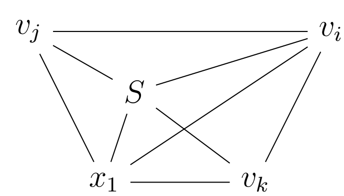
4. $\mathsf{weight}_i(w) = |N(w) \cap V_i|$ for all $w \in W_i$
5. v_i is simplicial
6. Let S be a clique of simplicial vertices, and let $N(S)$ denote the vertices incident on every vertex of S. If $a \prec v_i$, and $a, v_i \in N(S) - S$, and there is a path $a \sim v_i$ in $N(S) - S$, then there is a path $a \sim v_i$ in $N(S) - S$ all of whose vertices are below v_i in the order.

Parts 1-4 are immediate from the definition of the algorithm. From part 3 we obtain that $v_1. \ldots, v_n$ is an ordering of V, so $a \prec b$ is a total order on V.

Proof of part 5. Suppose v_i is not simplicial in V_i. Then there exists $v_j, v_k \in N(v_i)$ with $j < k < i$, but $v_j \notin N(v_k)$, and a clique S (possibly empty) incident on v_i, v_j, v_k and below v_k. By part 5 of the induction hypothesis at each $s \prec v_i$, s is simplicial, so by part 6 at $k < i$, since v_j, v_i, v_k is a path in $N(S) - S$, there is a path $v_j = x_0, \ldots, x_m = v_k$ in $N(S) - S$ where each x_a is below v_k, and necessarily $m > 1$.

Among all such configurations choose one such that $|S|$ is maximal, and among these choose one such that m is minimal. Then $v_i, v_j = x_0, \ldots, x_m = v_k, v_i$ is a cycle of size > 3.

- If there was a chord $x_a \in N(x_b)$ where $a + 1 < b$, then there would be a shorter path $v_j = x_0, \ldots, x_a, x_b, \ldots, x_m = v_k$.
- If there was a chord $x_a \in N(v_i)$ where $0 < a < m-1$, then $x_a, \ldots, x_m = v_k$ is a shorter path satisfying the conditions, since $x_a, v_k \in N(v_i)$ and $x_a \notin N(v_k)$.
- If there was a chord $x_a \in N(v_i)$ where $1 < a < m$, then $v_j = x_0, \ldots, x_a$ is a shorter path satisfying the conditions, since $v_j, x_a \in N(v_i)$ and $v_j \notin N(x_a)$.
- Otherwise, $a = 1$ and $m = 2$, that is, we have the configuration:



  But then $S \cup \{x_1\}$ is a larger clique incident on v_i, v_j, v_k, contradicting maximality.

□

Let S be a clique of simplicial vertices, and let $N(S)$ denote the vertices incident on every vertex of S. If $a \prec v_i$, and $a, v_i \in N(S) - S$, and there is a path $a \sim v_i$ in $N(S) - S$, then there is a path $a \sim v_i$ in $N(S) - S$ all of whose vertices are below v_i in the order.

Proof of part 6. If $a \in N(v_i)$ then we are done. Otherwise, if there exists $b \prec v_i$ in $V - S$ adjacent to v_i, then for each $s \in S$:

- If $s \prec v_i$, then since $b, s \in N(v_i)$ and $b, s \prec v_i$ and v_i is simplicial (using part 5 at i), b is adjacent to s.

- If $a \prec v_i \prec s$, then $a, v_i \in N(s)$ and s is simplicial so $a \in N(v_i)$, contradiction.

Thus $b \in N(S) - S$, so by the inductive hypothesis at the larger of a or b there is a path $a \sim b$ in $N(S) - S$ below $\max(a, b)$, and this extends to a path $a \sim b - v_i$.

Otherwise, v_i has no neighbors in $V_{i-1} - S$, so $\mathsf{weight}_{i-1}(v_i) \leq |S \cap V_{i-1}|$. But if $a \sim b - v_i$ is the path from a to v_i, we must have $a \prec v_i \prec b$, so there is a first transition $a \sim c - d \sim v_i$ such that $c \prec v_i \prec d$. Then $\mathsf{weight}_{i-1}(d) \geq |S \cap V_{i-1}| + 1$ since $d \in N(S) - S$ and $c \in N(d)$, but since $d \in W_{i-1}$ as well, $\mathsf{weight}_{i-1}(d) \leq \mathsf{weight}_{i-1}(v_i)$ by maximality of v_i. This is a contradiction. □

Thus v_i is simplicial in V_i for each i, so $v_1, \ldots, v_n$ is a simplicial elimination ordering.