Designing extensible, domain-specific languages for mathematical diagrams

Katherine Ye

with Keenan Crane
Jonathan Aldrich
Josh Sunshine

Off the Beaten Track '17

Source: "An Algorithm to Generate Impossible Art?", Danny Tarlow

(A proposal for)

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Designing extensible, domain-specific languages for mathematical diagrams

(EXTENDED REMIX)

(45 mins)

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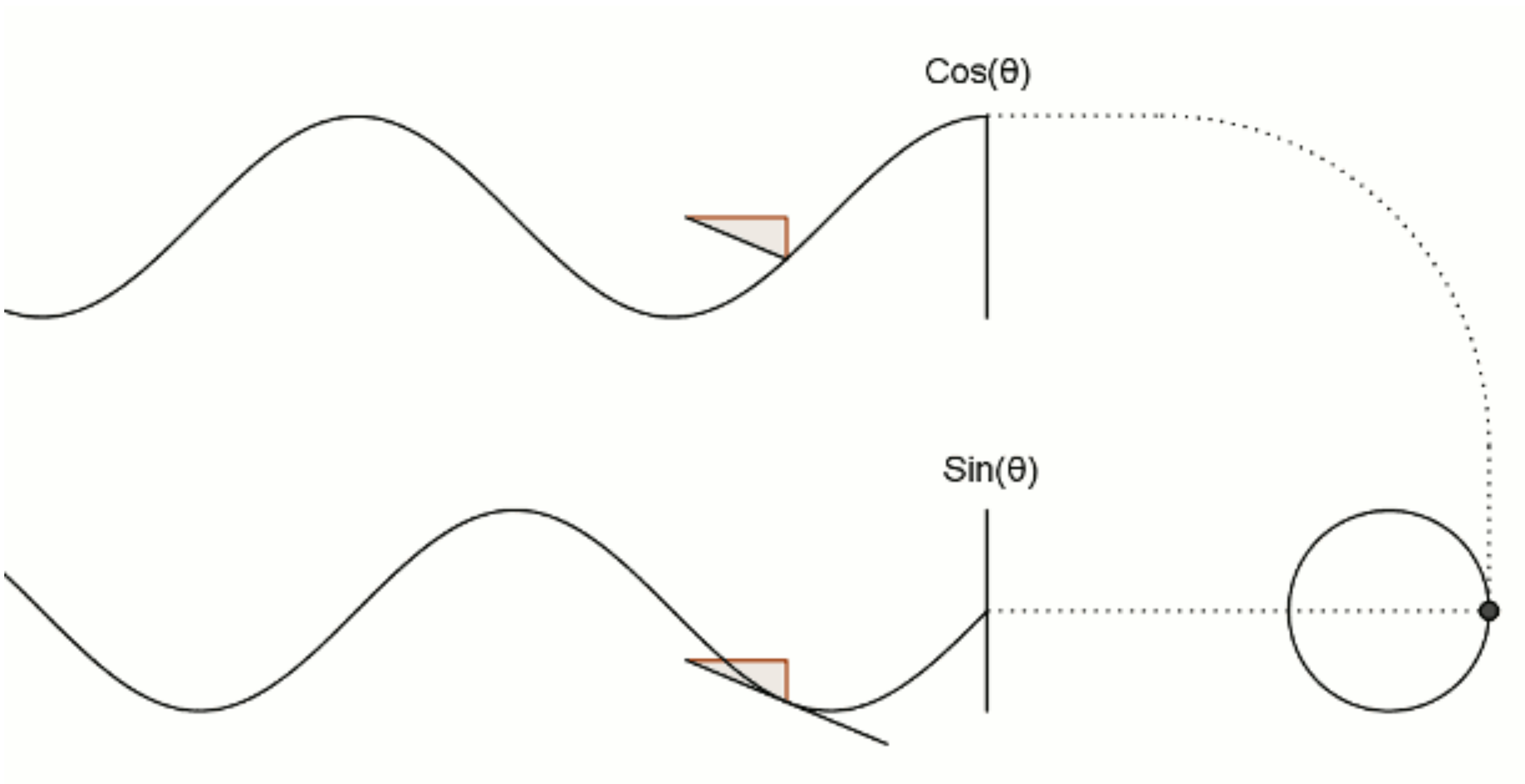
Motivation

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$


Cos(θ)
Sin(θ)

MEAN VALUE THEOREMS

5.7 Definition Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

5.8 Theorem *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

The analogous statement for local minima is of course also true.

Proof Choose δ in accordance with Definition 5.7, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

5.9 Theorem *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \qquad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b). \tag{12}$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$.

If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum (Theorem 4.16). By (12), $x \in (a, b)$, and Theorem 5.8 shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a *generalized mean value theorem*; the following special case is usually referred to as "the" mean value theorem:

5.10 Theorem *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof Take $g(x) = x$ in Theorem 5.9.

5.11 Theorem *Suppose f is differentiable in (a, b).*

(a) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(b) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(c) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Proof All conclusions can be read off from the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

which is valid, for each pair of numbers x_1, x_2 in (a, b), for *some* x between x_1 and x_2.

THE CONTINUITY OF DERIVATIVES

We have already seen [Example 5.6(*b*)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Theorem 4.23). The precise statement follows.

5.12 Theorem *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

A similar result holds of course if $f'(a) > f'(b)$.

Proof Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attains its minimum on $[a, b]$ (Theorem 4.16) at some point x such that $a < x < b$. By Theorem 5.8, $g'(x) = 0$. Hence $f'(x) = \lambda$.

New submissions for Fri, 13 Jan 17

[1] arXiv:1701.03119 [pdf, ps, other]

How to Quantize n Outputs of a Binary Symmetric Channel to $n-1$ Bits?

Wasim Huleihel, Or Ordentlich

Comments: 5 pages, submitted to ISIT 2017

Subjects: **Information Theory (cs.IT)**

Suppose that Y^n is obtained by observing a uniform Bernoulli random vector X^n through a binary symmetric channel with crossover probability α. The "most informative Boolean function" conjecture postulates that the maximal mutual information between Y^n and any Boolean function b(X^n) is attained by a dictator function. In this paper, we consider the "complementary" case in which the Boolean function is replaced by $f:\{0,1\}^n \mapsto \{0,1\}^{n-1}$, namely, an $n-1$ bit quantizer, and show that $I(f(X^n);Y^n) \leq (n-1)\cdot(1-h(\alpha))$ for any such f. Thus, in this case, the optimal function is of the form $f(x^n)=(x_1,\ldots,x_{n-1})$.

[2] arXiv:1701.03120 [pdf, ps, other]

The fourth moment theorem on the Poisson space

Christian Döbler, Giovanni Peccati

Comments: 31 pages

Subjects: **Probability (math.PR)**

We prove an exact fourth moment bound for the normal approximation of random variables belonging to the Wiener chaos of a general Poisson random measure. Such a result -- that has been elusive for several years -- shows that the so-called `fourth moment phenomenon', first discovered by Nualart and Peccati (2005) in the context of Gaussian fields, also systematically emerges in a Poisson framework. Our main findings are based on Stein's method, Malliavin calculus and Mecke-type formulae, as well as on a methodological breakthrough, consisting in the use of carr\'e-du-champ operators on the Poisson space for controlling residual terms associated with add-one cost operators. Our approach can be regarded as a successful application of Markov generator techniques to probabilistic approximations in a non-diffusive framework: as such, it represents a significant extension of the seminal contributions by Ledoux (2012) and Azmoodeh, Campese and Poly (2014). To demonstrate the flexibility of our results, we also provide some novel bounds for the Gamma approximation of non-linear functionals of a Poisson measure.

[3] arXiv:1701.03124 [pdf, ps, other]

Totaro's Question for Adjoint Groups of Types A_{1} and A_{2n}

Reed Gordon-Sarney

Comments: 7 pages

Subjects: **Algebraic Geometry (math.AG)**

Let G be a smooth connected linear algebraic group over a field k, and let X be a G-torsor. Totaro asked: if X admits a zero-cycle of degree $d \geq 1$, then does X have a closed \'etale point of degree dividing d? We give an affirmative answer for absolutely simple classical adjoint groups of types A_1 and A_{2n} over fields of characteristic $\neq 2$.

[4] arXiv:1701.03131 [pdf, other]

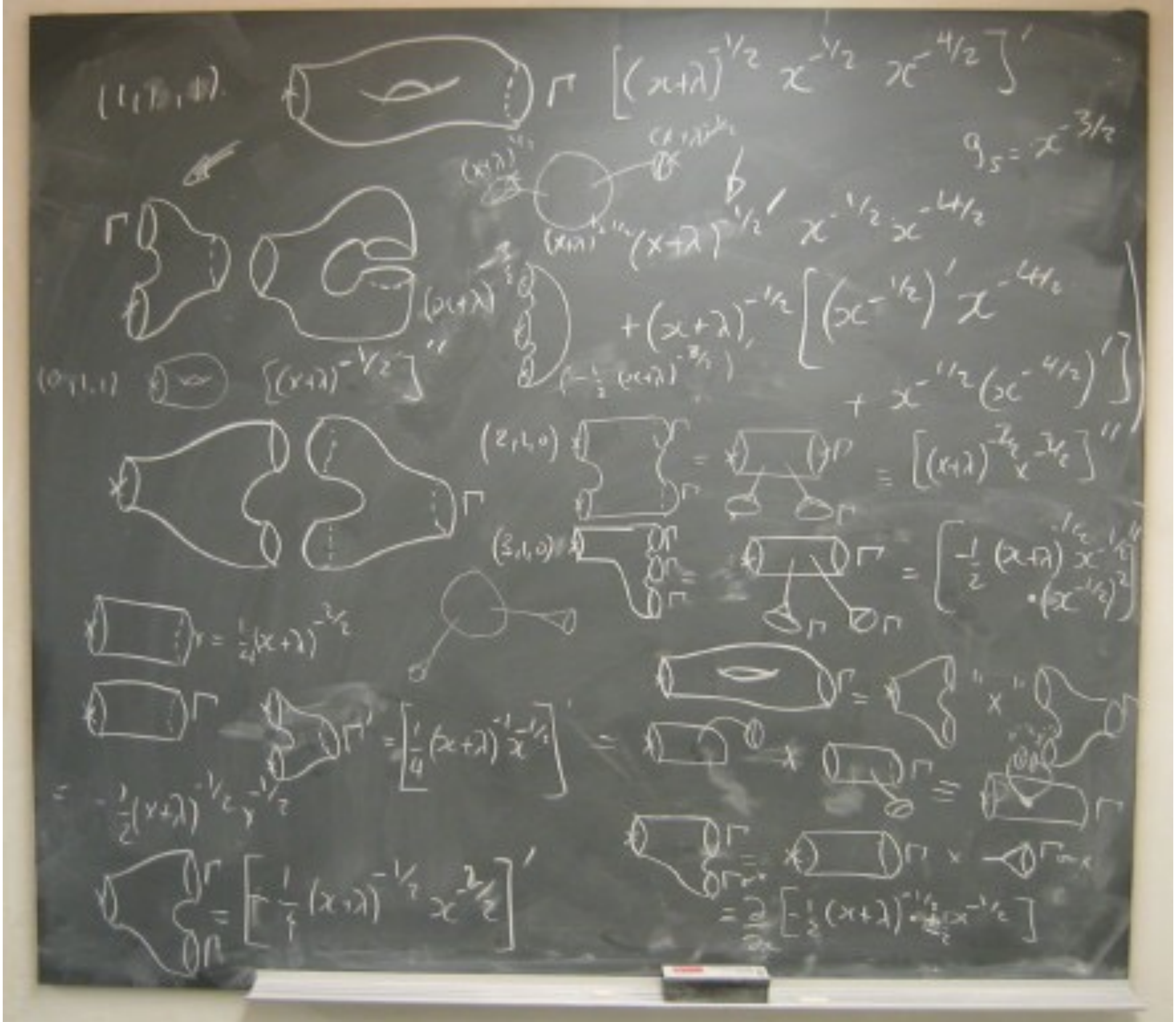
Classification of irregular free boundary points for non-divergence type equations with discontinuous coefficients

Serena Dipierro, Aram Karakhanyan, Enrico Valdinoci

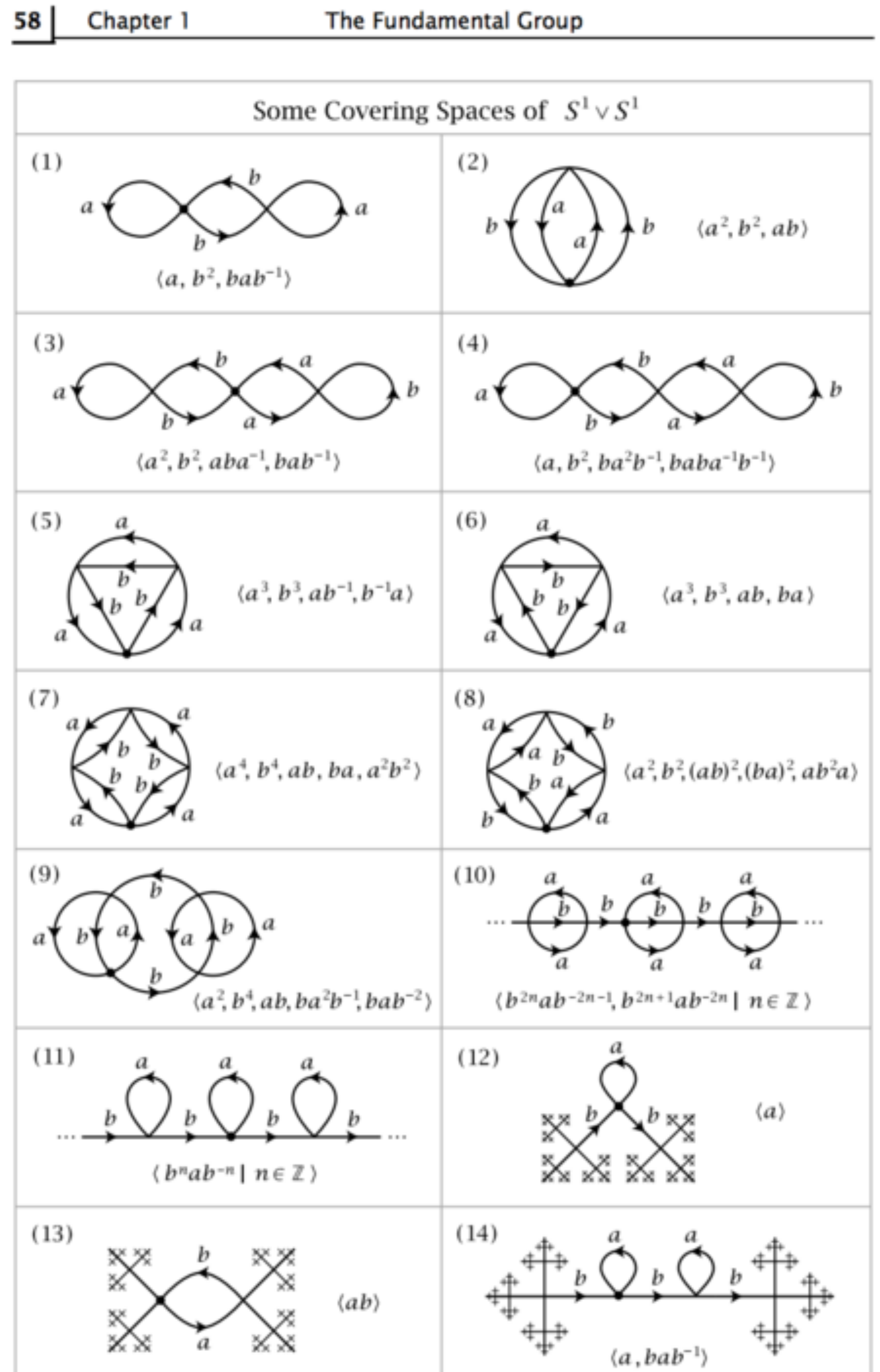
Subjects: **Analysis of PDEs (math.AP)**

We provide an integral estimate for a non-divergence (non-variational) form second order elliptic equation $a_{ij}u_{ij}=u^p$, $u \geq 0$, $p \in [0,1)$, with bounded discontinuous coefficients a_{ij} having small BMO norm. We consider the simplest discontinuity of the

Sometimes people do illustrate mathematics beautifully.


g_s = x^{-3/2}

illustrating groups (with their notations)



Source:
Algebraic Topology
Hatcher

composing different types of diagrams

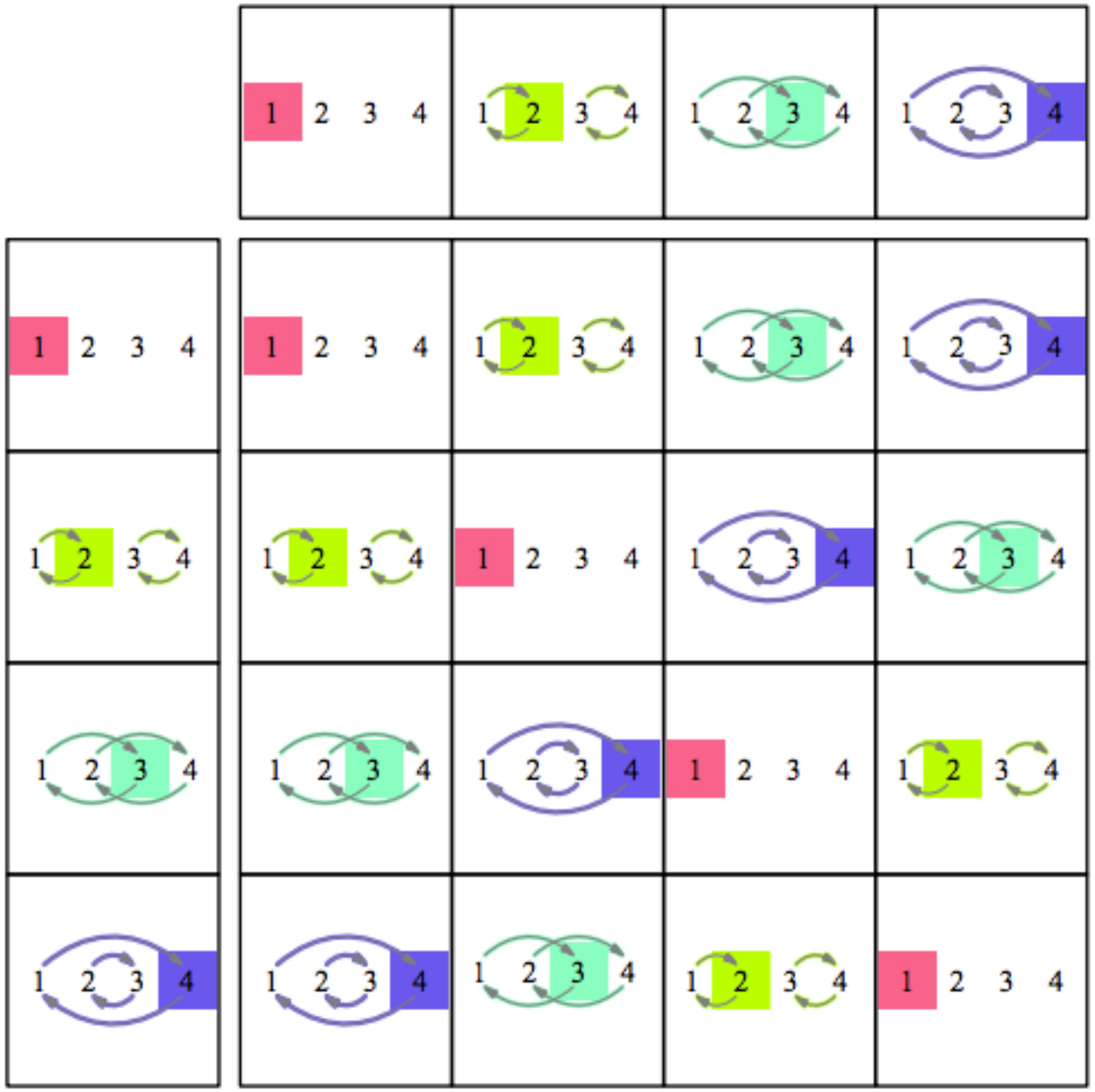


Figure 5.32. A multiplication table made up of the permutations created in Figure 5.31. Each cell highlights the destination to which the permutation sends 1, using the corresponding color from Figure 5.31, emphasizing what the colors of the arrows already showed: The two tables contain the same pattern.

Source:
Visual Group Theory

Problem:

Drawing good diagrams requires a tremendous amount of effort and expertise.

People have very powerful facilities for taking in information visually… and thinking spatially.

William Thurston
Fields medalist

People have very powerful facilities for taking in information visually… and thinking spatially.

On the other hand, they do not have a very good built-in facility for inverse vision, that is, turning an internal spatial understanding back into a two-dimensional image.

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On the other hand, they do not have a very good built-in facility for inverse vision, that is, turning an internal spatial understanding back into a two-dimensional image.

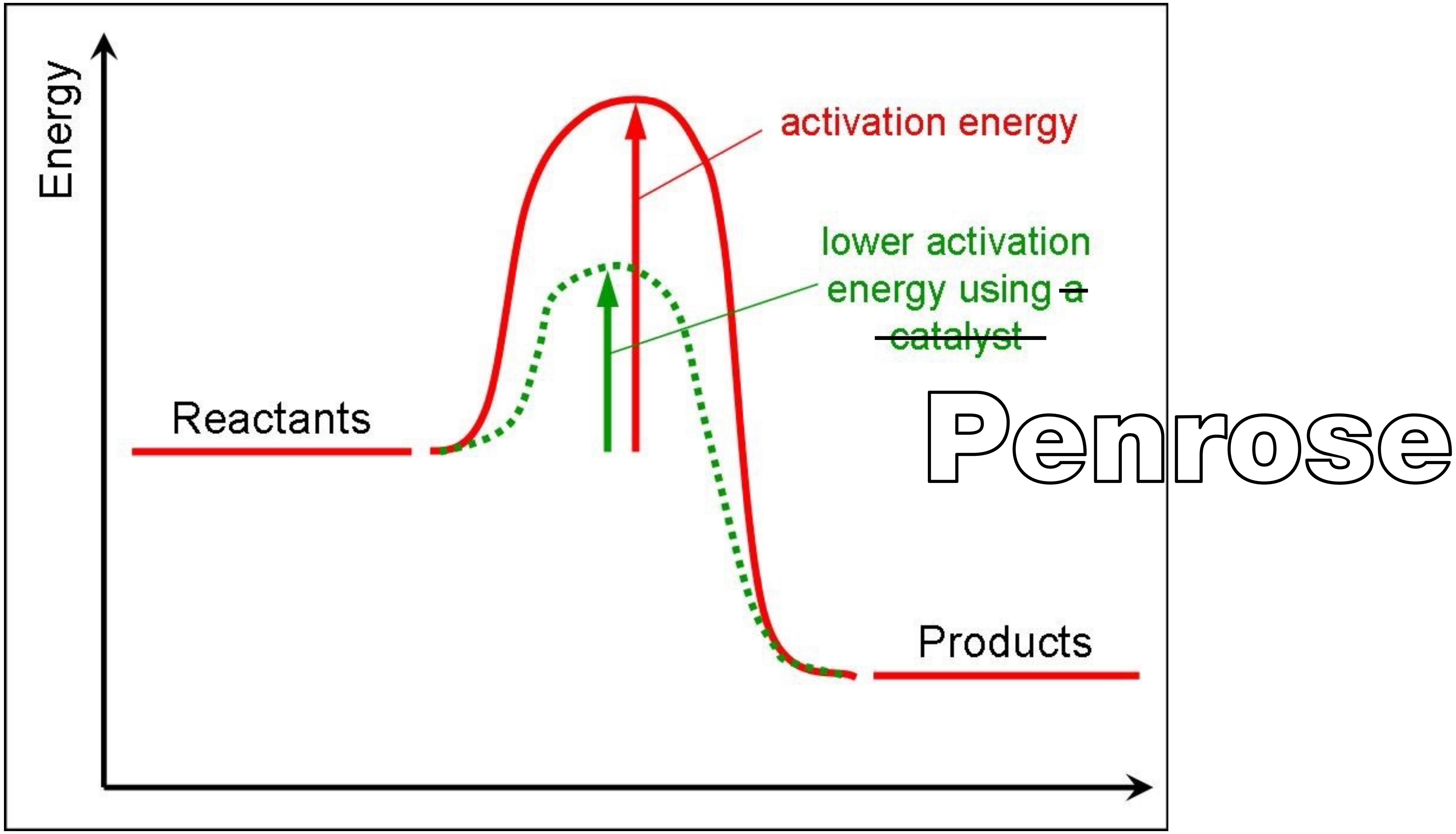
Consequently, mathematicians usually have fewer and poorer figures in their papers and books than in their heads.

William Thurston
Fields medalist

What if mathematicians had *more* and *richer* figures in their papers and books than in their heads?

Penrose

Source: “An Algorithm to Generate Impossible Art?”, Danny Tarlow



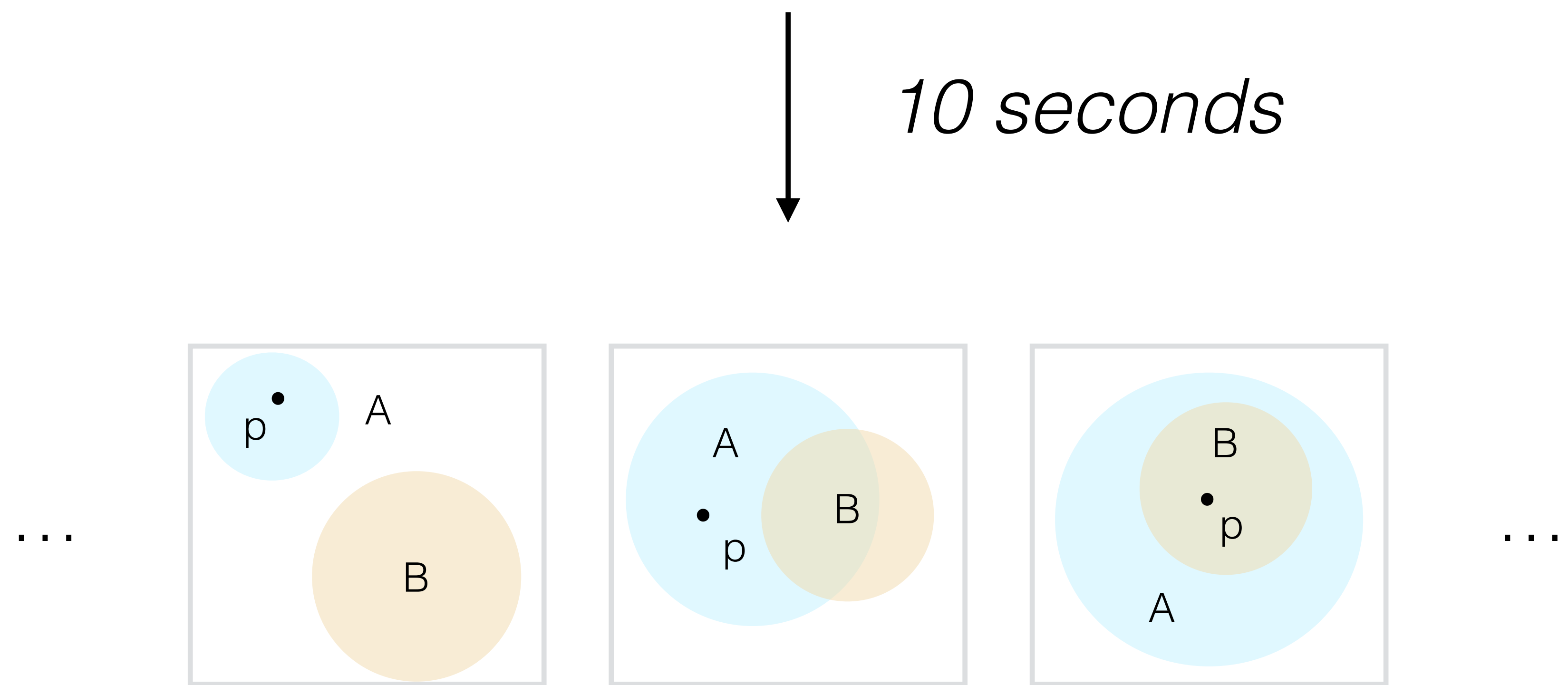
Source: https://sallyarjoon.wordpress.com/2013/04/01/enzymes/

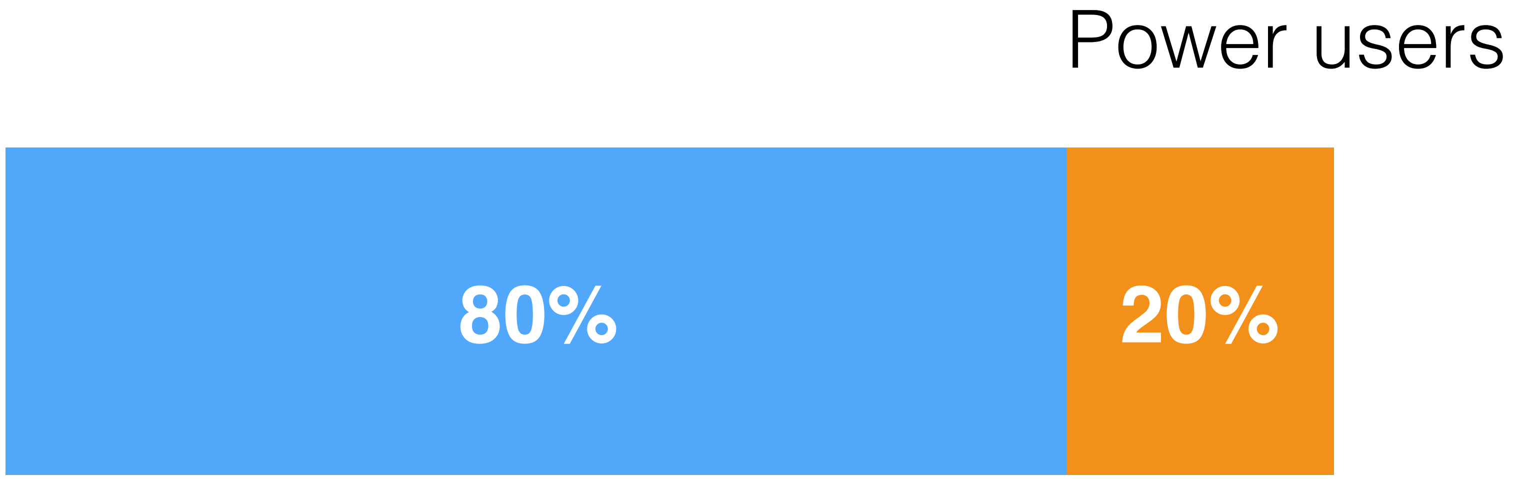
Casual users

“Let A and B be sets, and p be a point in A.”

↓ *10 seconds*

“Let A and B be sets, and p be a point in A.”

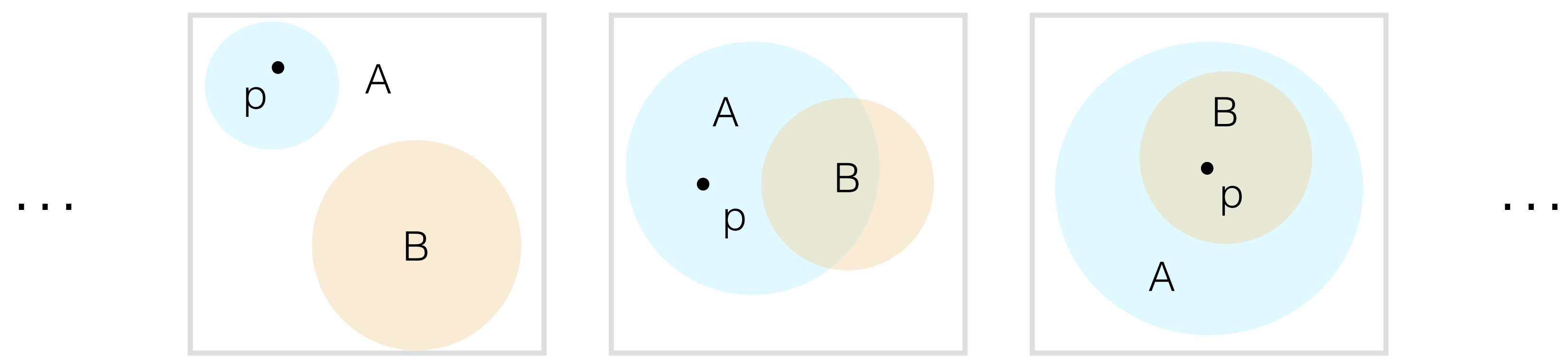




Power users
80%
20%

"Let A and B be sets, and p be a point in A."

2 hours

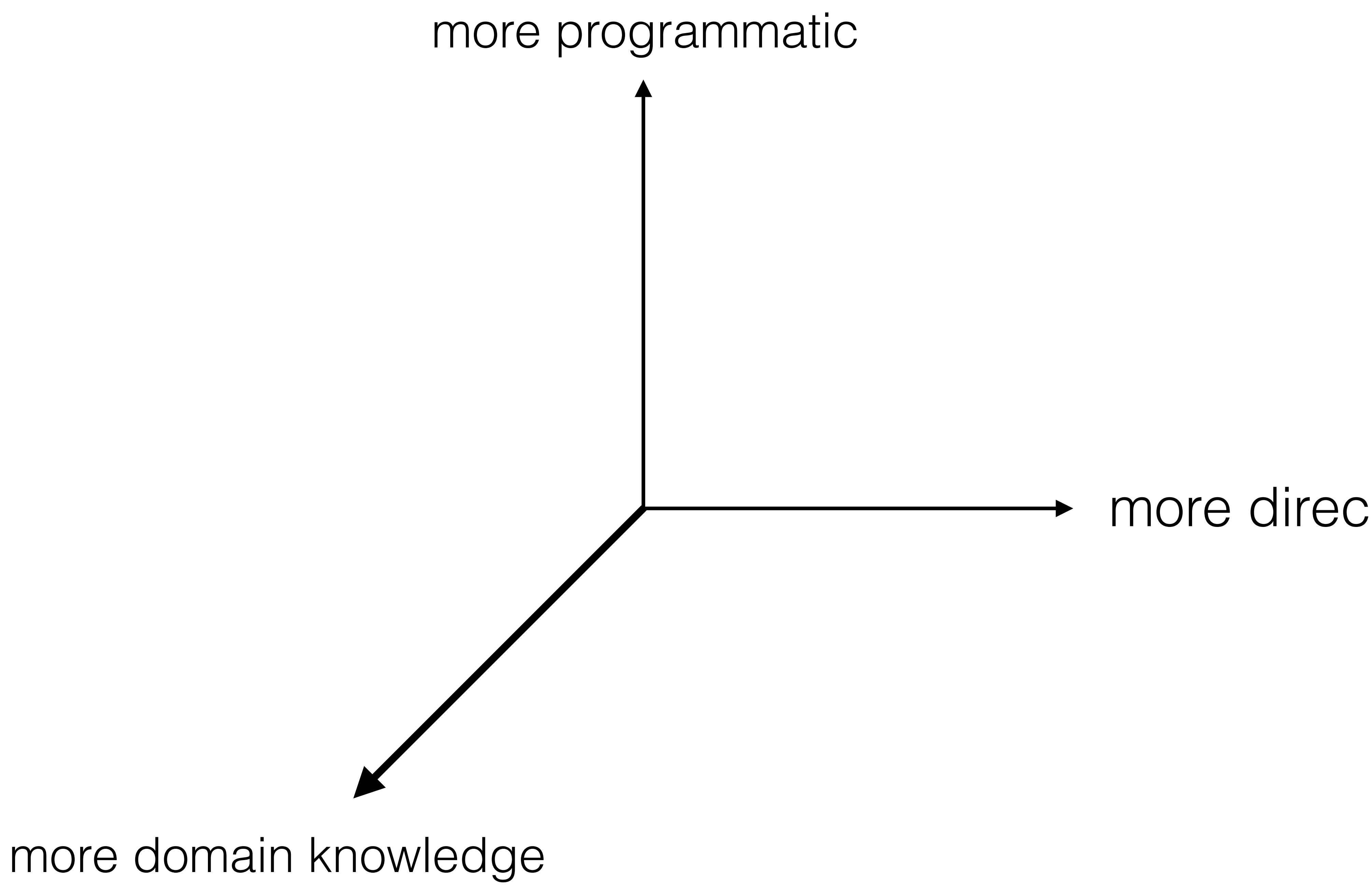


Existing solutions for diagramming

Factors in choosing a tool:

Programmatic,
direct manipulation,
domain knowledge

The design space



Writing a program for a diagram

$$K_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$$

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↓

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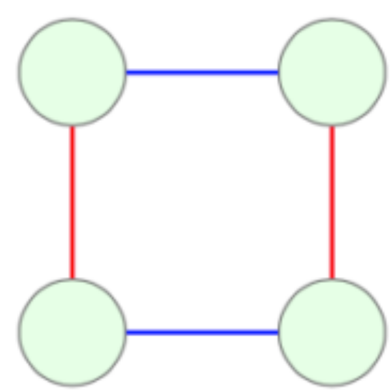

Figure 1: Cayley diagram of the Klein 4-group

$$K_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$$

TikZ

```
% node style
circ/.style={circle, draw=black!60, fill=green!10, minimum size=7mm}],

% nodes
\node[circ] (gen_bl) {};
\node[circ] (gen_tl) [above=of gen_bl] {};
\node[circ] (gen_br) [right=of gen_bl] {};
\node[circ] (gen_tr) [above=of gen_br, right=of gen_tl] {};

% \node[circ] (gen_bl);
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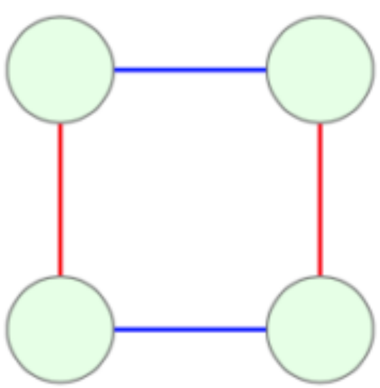
% \node[circ] (gen_tr) [above=of gen_br, right=of gen_tl];
% \node[circ] (gen_br) [right=of gen_bl];

% draw lines between nodes
\draw[-][red, thick] (gen_tl.south) -- (gen_bl.north);
\draw[-][blue, thick] (gen_tl.east) -- (gen_tr.west);
\draw[-][blue, thick] (gen_bl.east) -- (gen_br.west);
\draw[-][red, thick] (gen_tr.south) -- (gen_br.north);
```



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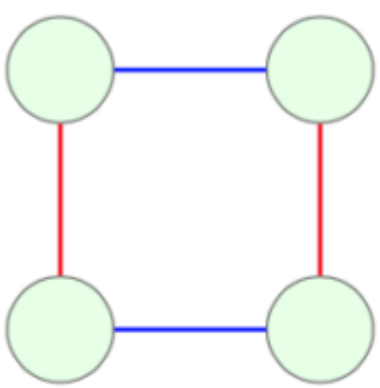

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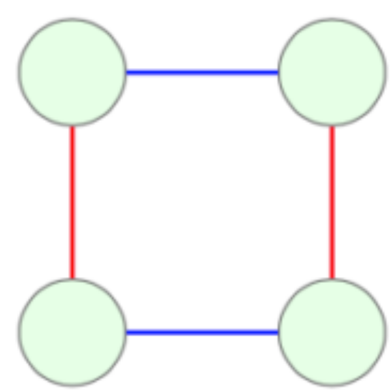
```

node style

node positions



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```

node style

node positions

line style + positions

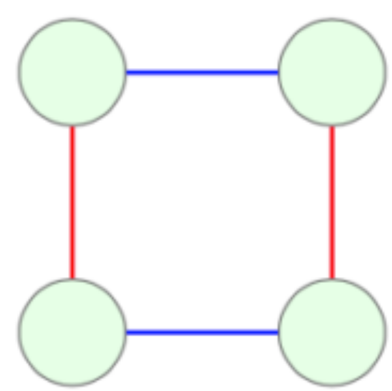
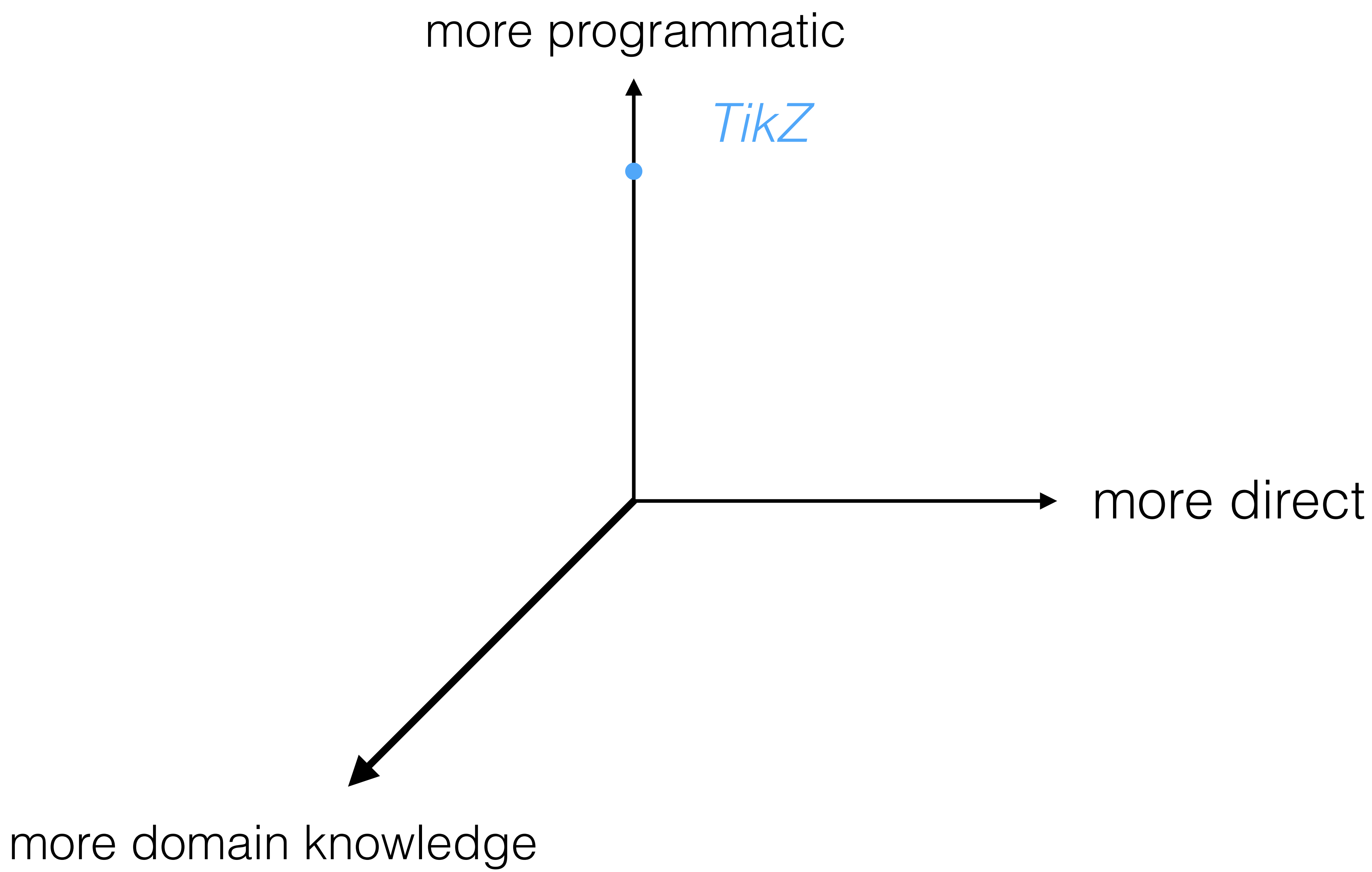


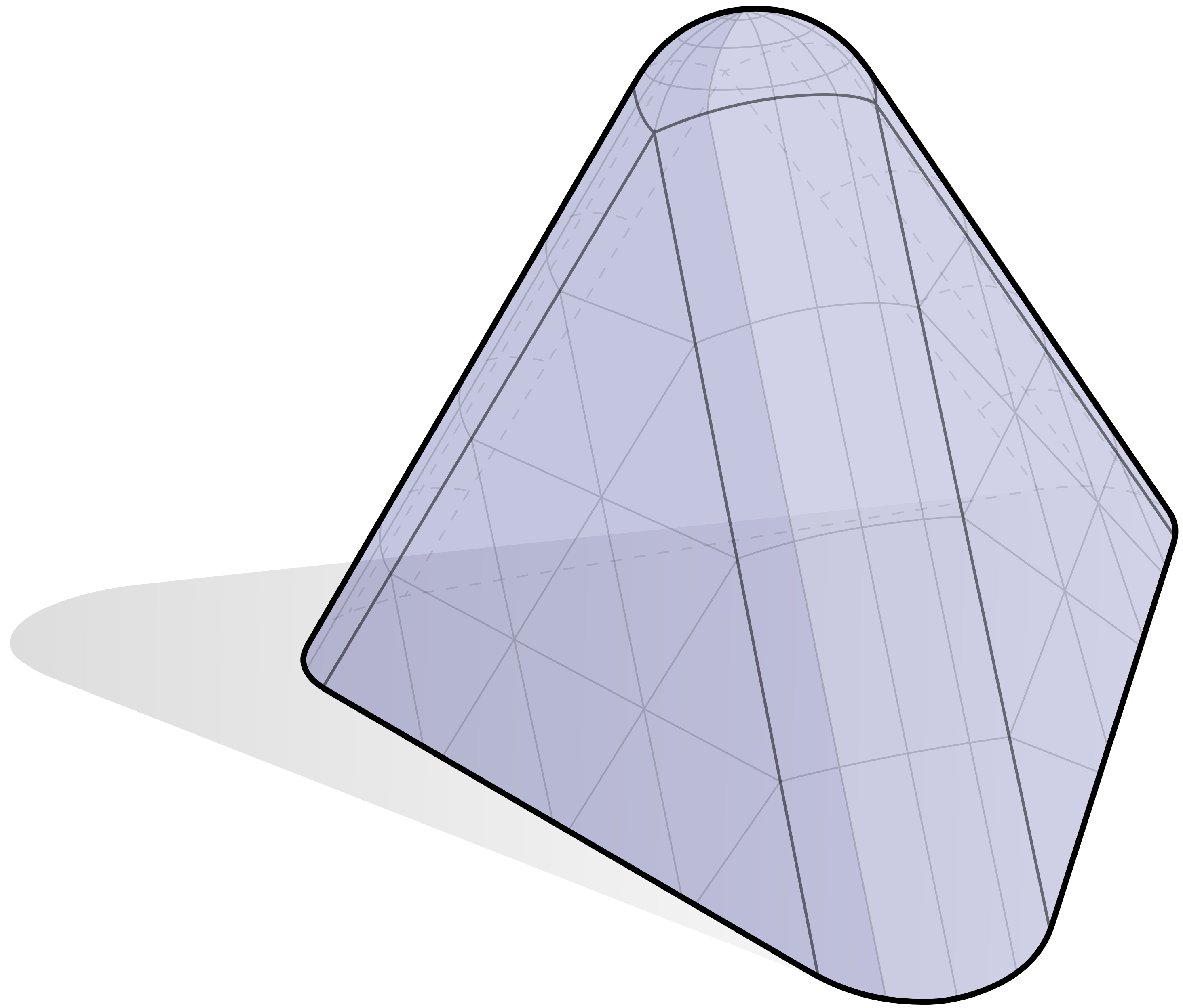
Figure 1: Cayley diagram of the Klein 4-group

The design space



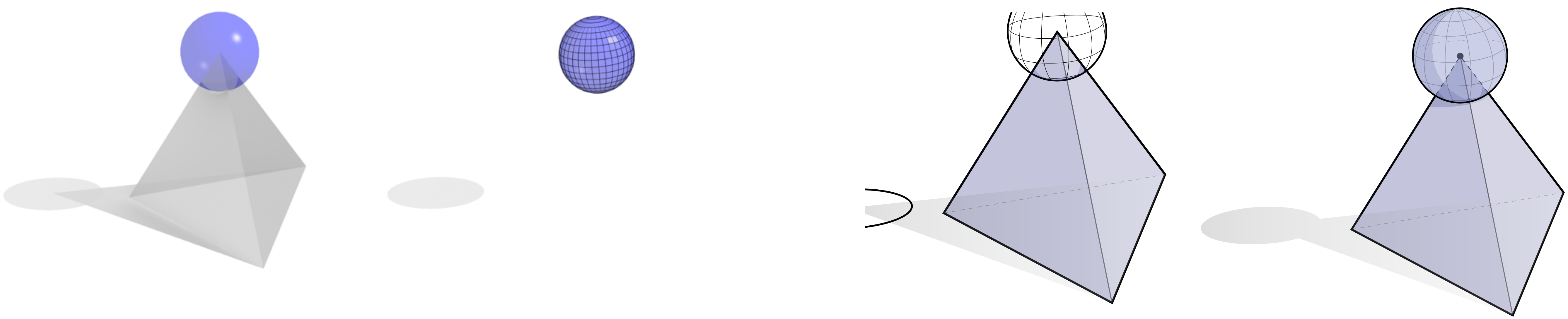
Creating a diagram via a GUI

curvature of a polyhedron

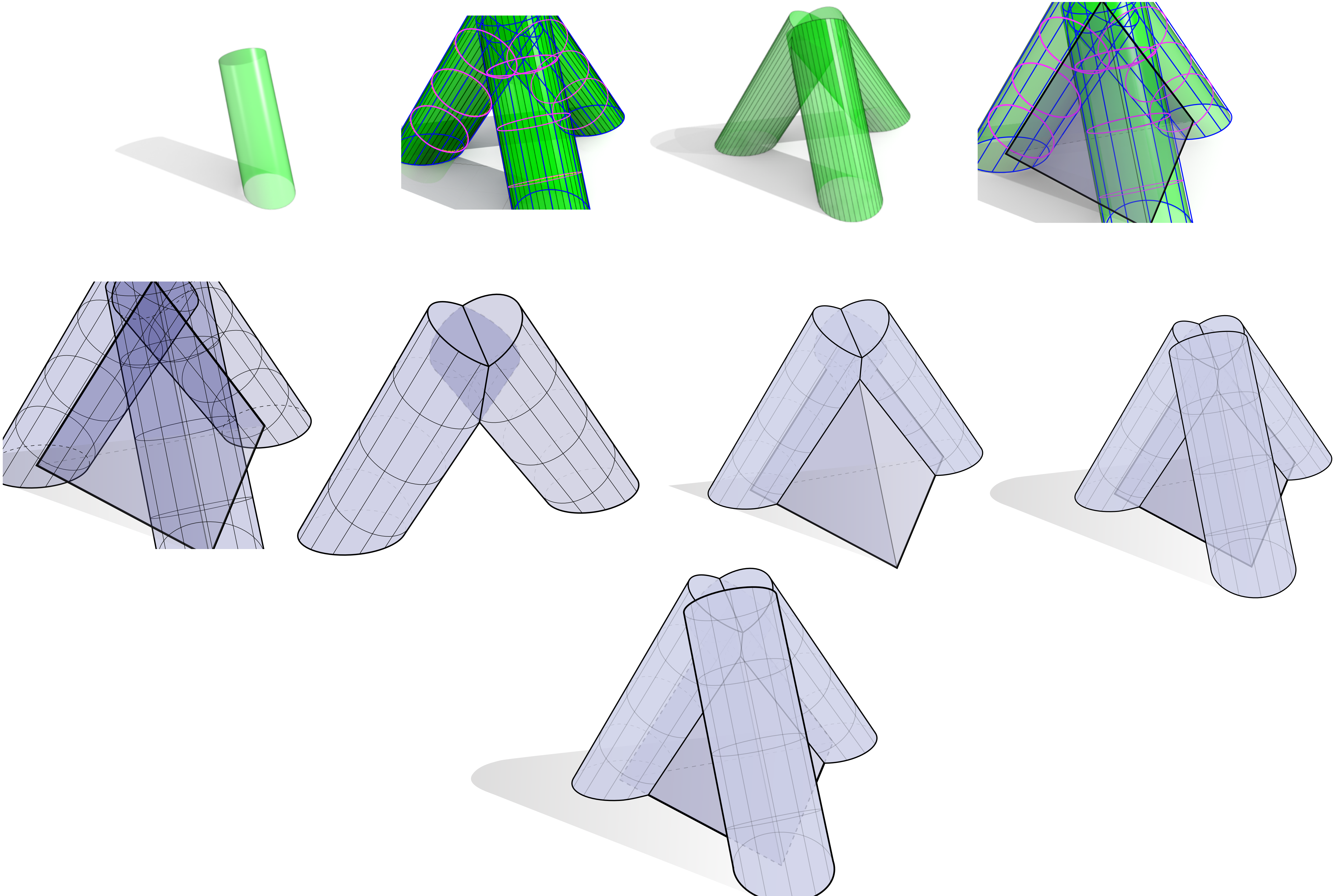


Source: Keenan Crane

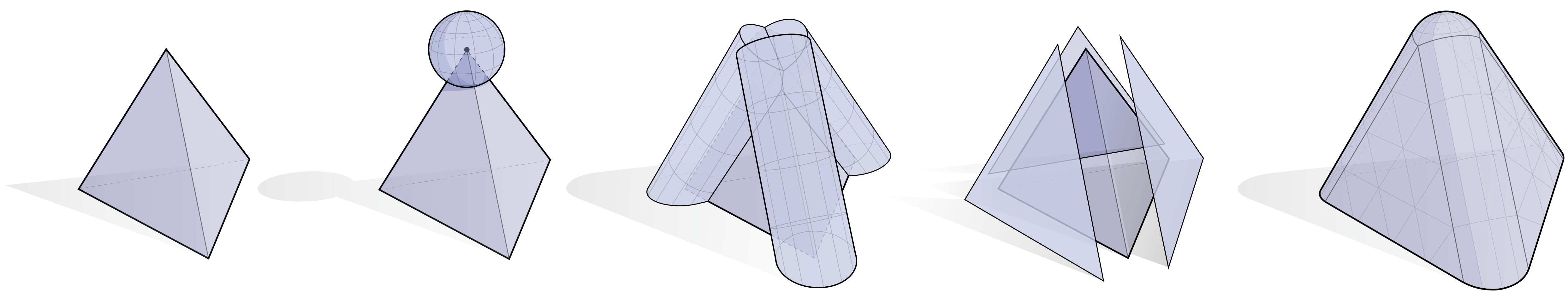
curvature of a polyhedron



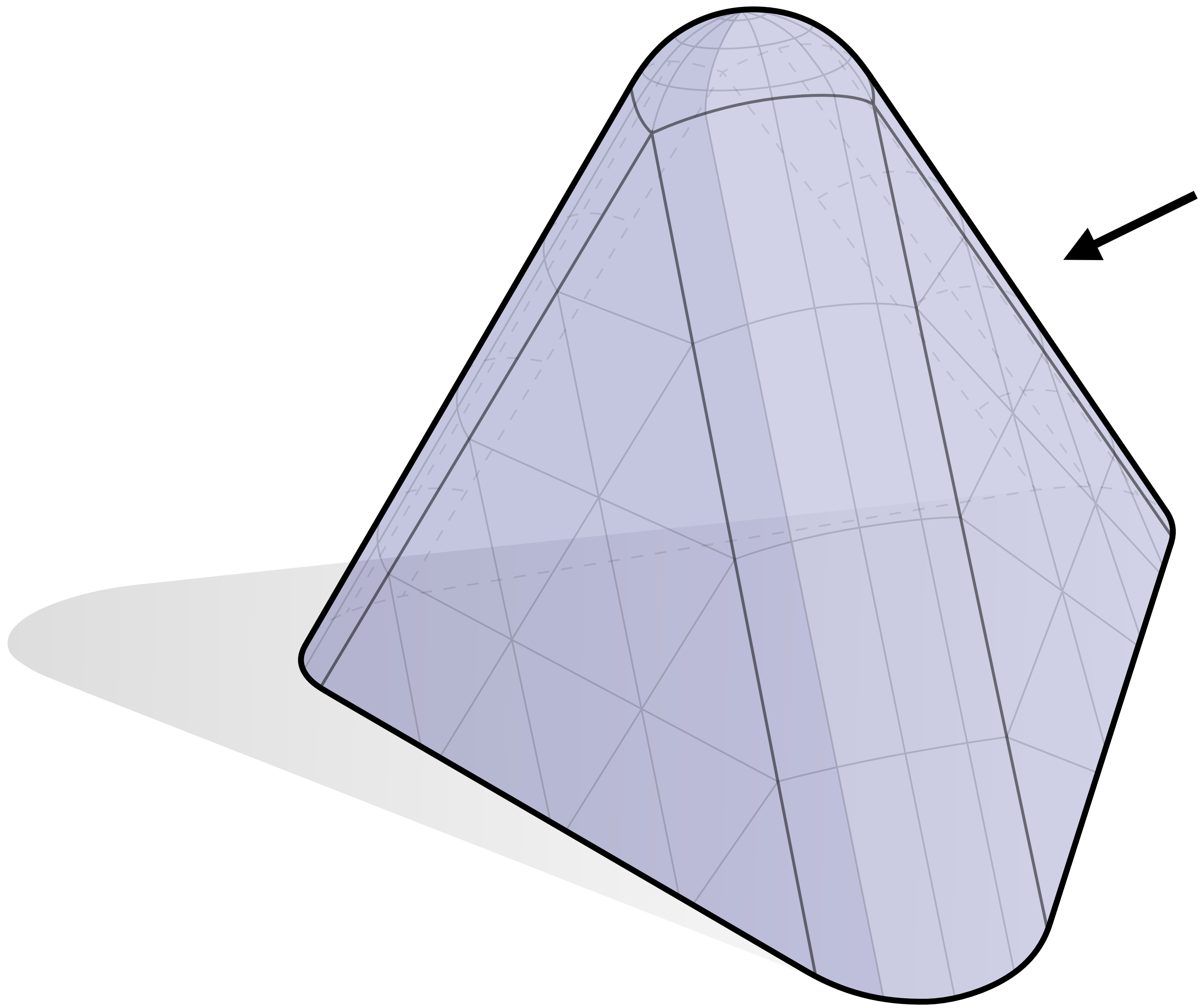
curvature of a polyhedron



curvature of a polyhedron

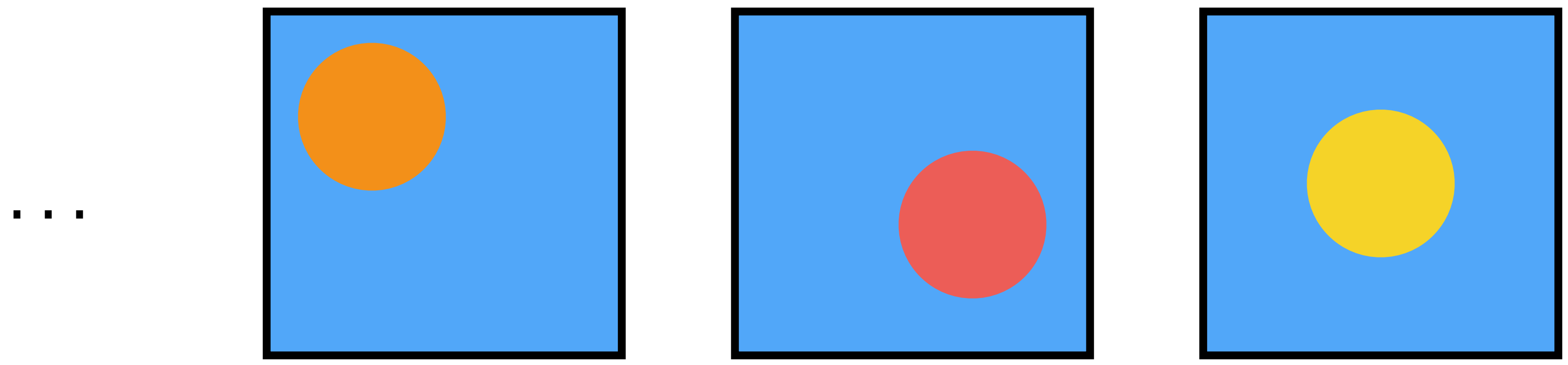


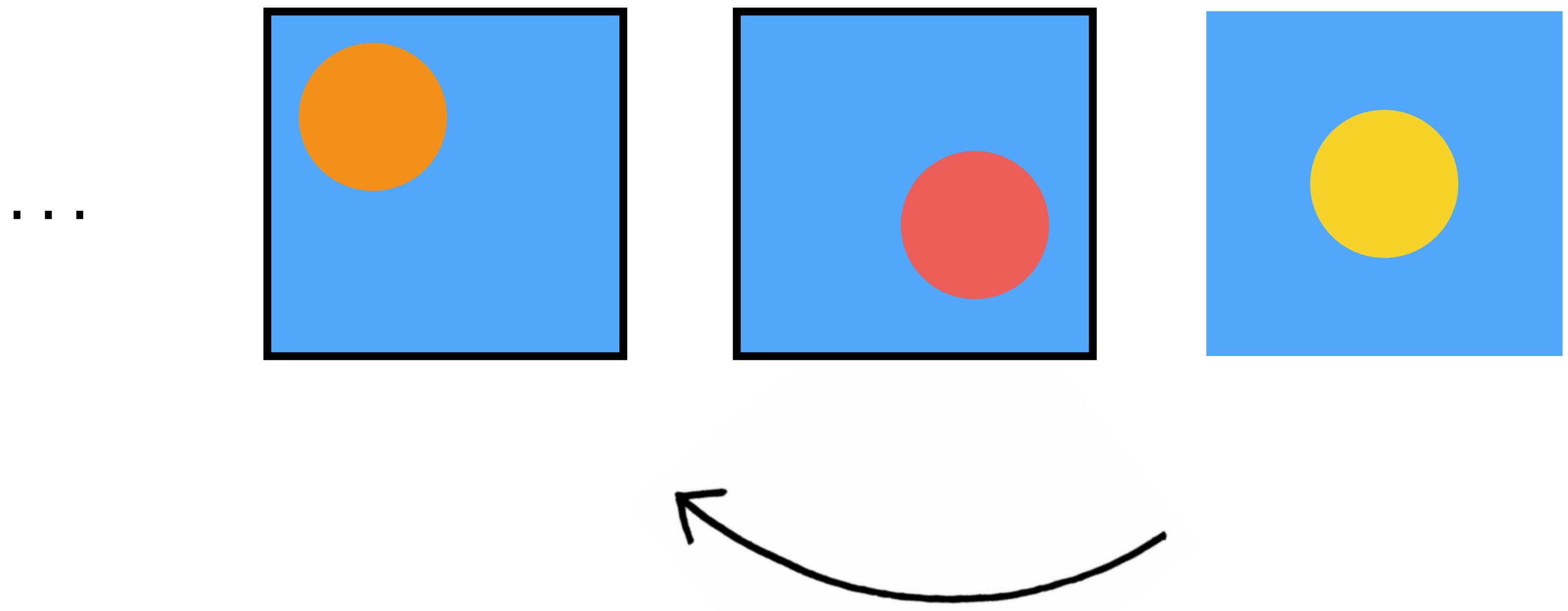
curvature of a polyhedron



8 hours of painstaking clicking and dragging

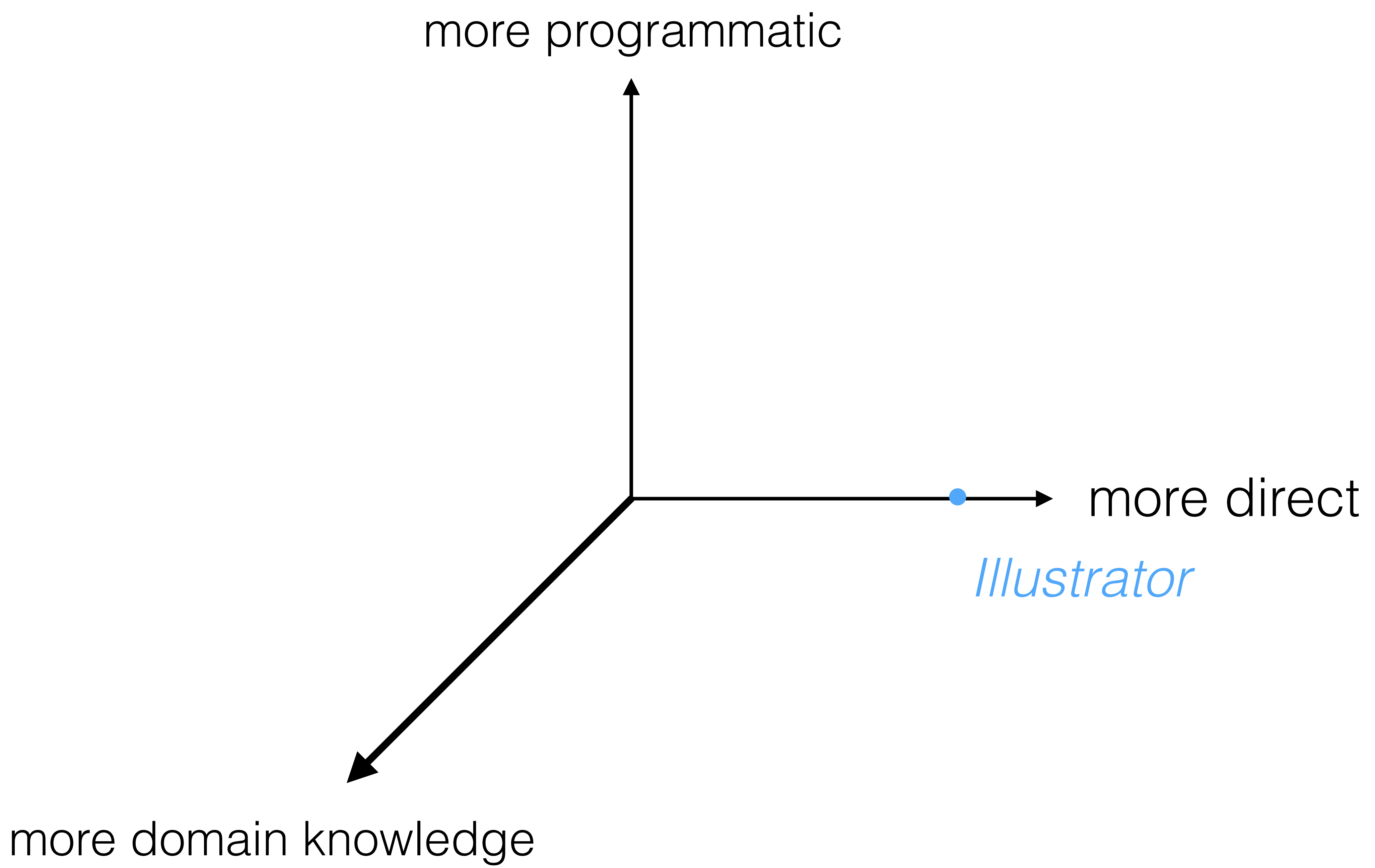
Source: Keenan Crane



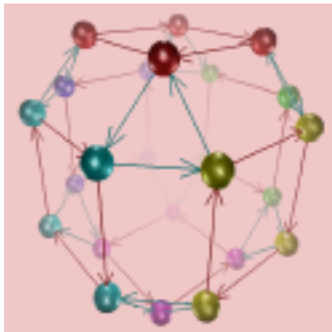


no style reuse!

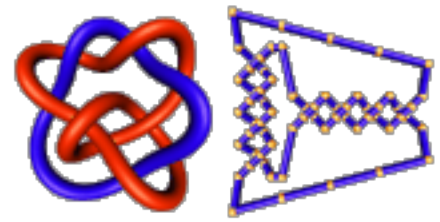
The design space



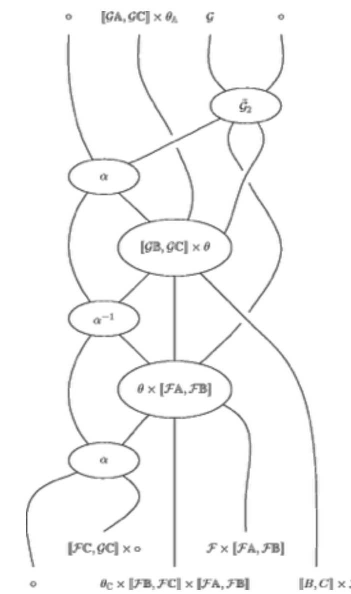
Burn it all down
and write your own software
or DSL?



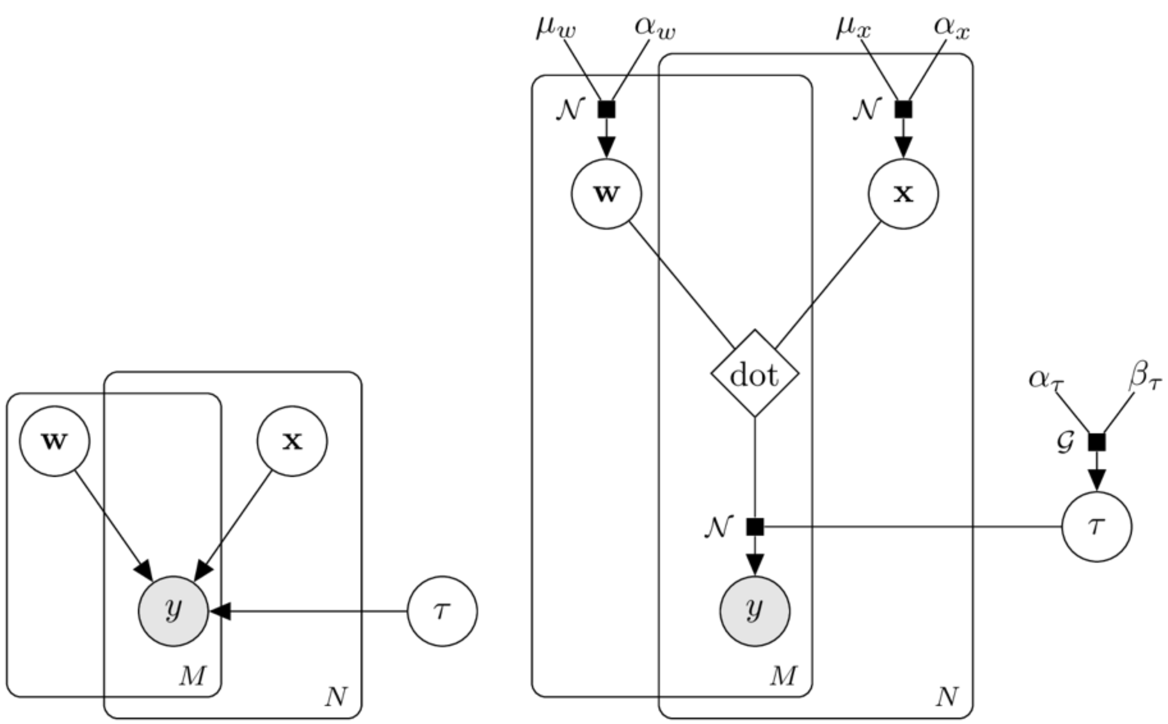
Group Explorer



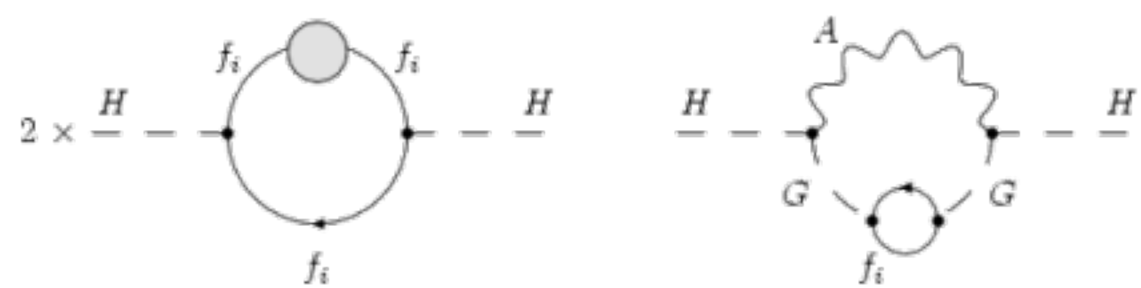
KnotPlot



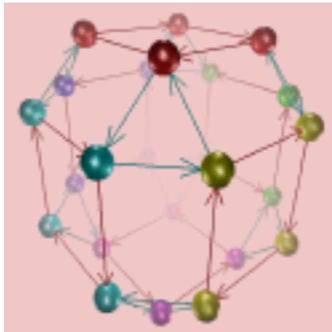
strid



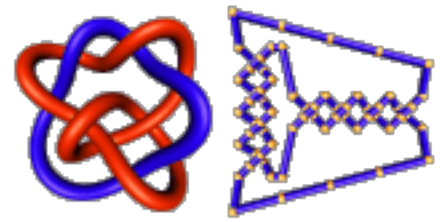
BayesNet



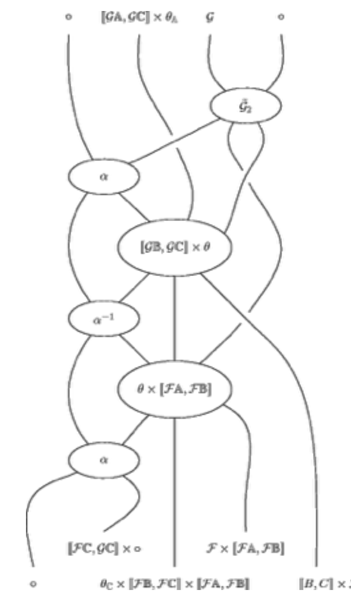
JaxoDraw



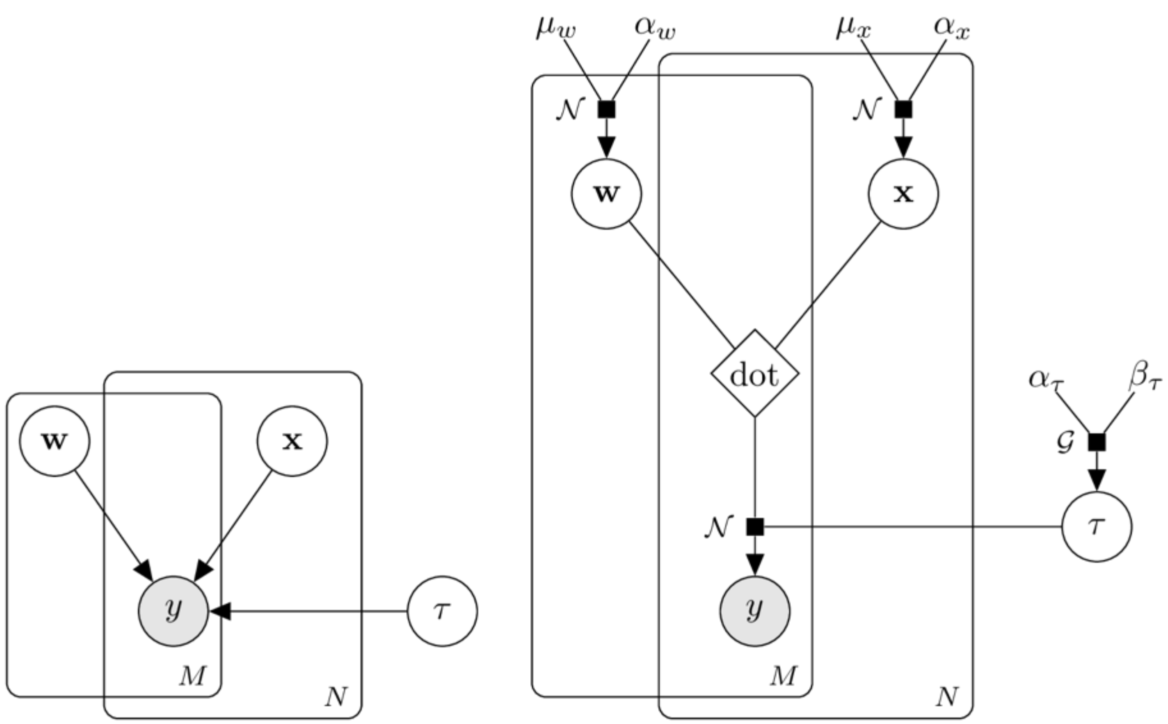
Group Explorer



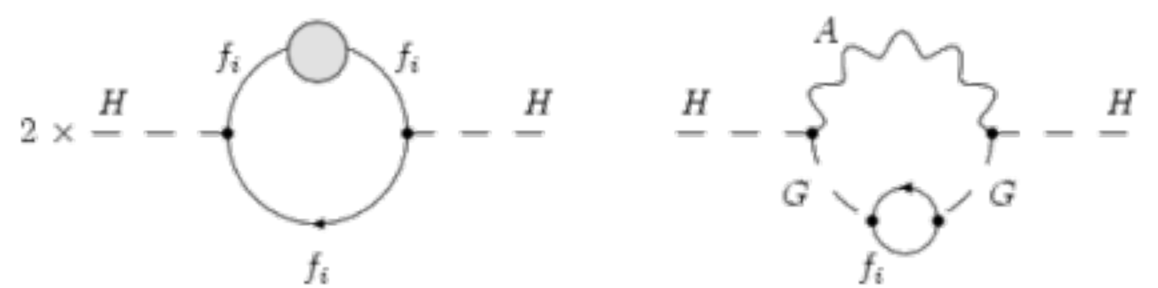
KnotPlot



strid

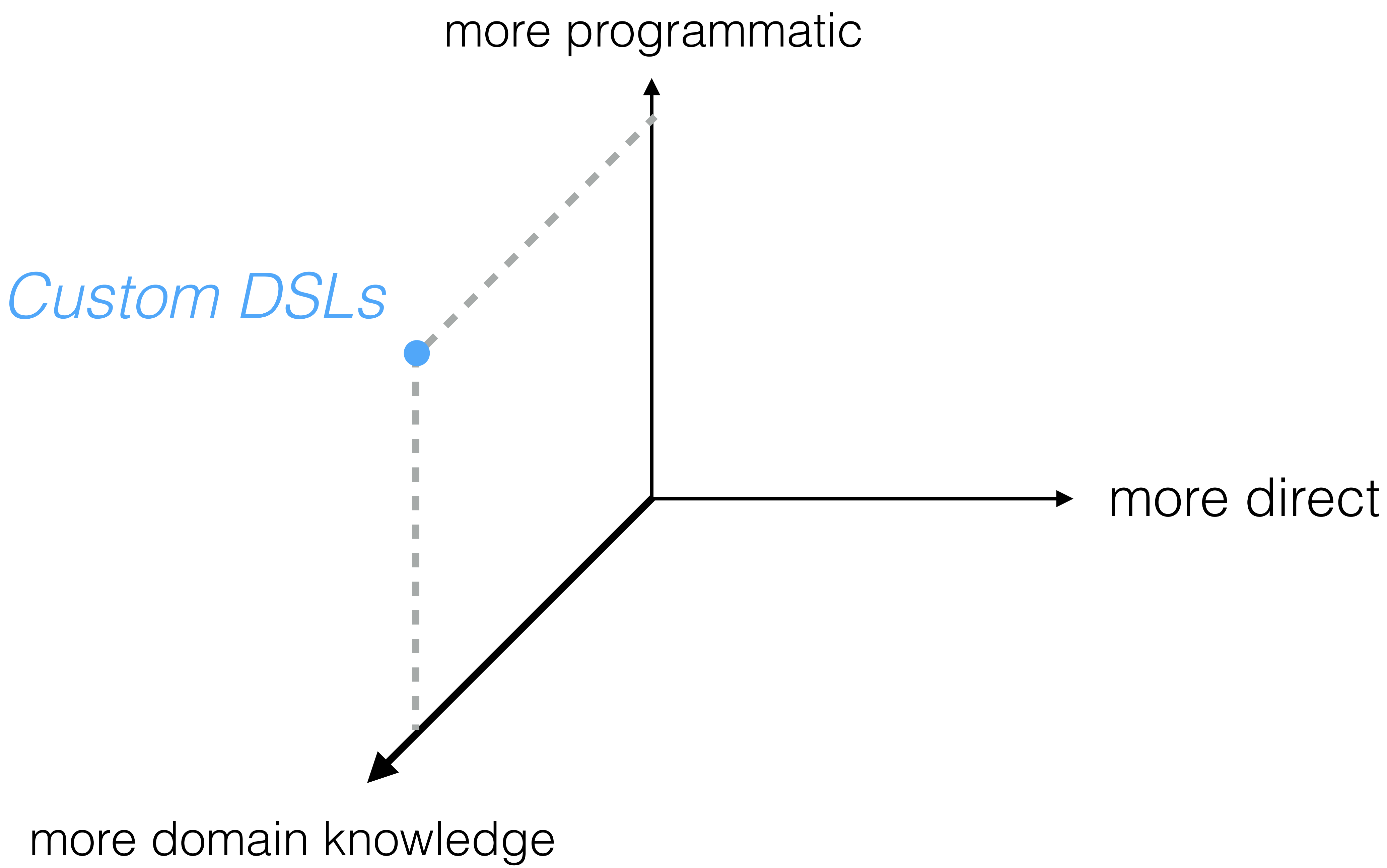


BayesNet



JaxoDraw

The design space



Penrose

Source: "An Algorithm to Generate Impossible Art?", Danny Tarlow

“Let A and B be sets, and p be a point in A.”

“Let A and B be sets, and p be a point in A.”

Notation

$$Set\ A$$
$$Set\ B$$
$$Point\ p$$
$$p \in A$$

Penrose architecture

“Let A and B be sets, and p be a point in A.”

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance *View* *Style*

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance *View* *Style*

```
Set A
Set B
Point p
In p A
```

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance

```
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Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

Penrose architecture

"Let A and B be sets, and p be a point in A."

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↘ ↓ ↙

Intermediate representations

↓

Sampling and optimization

↓

Penrose architecture

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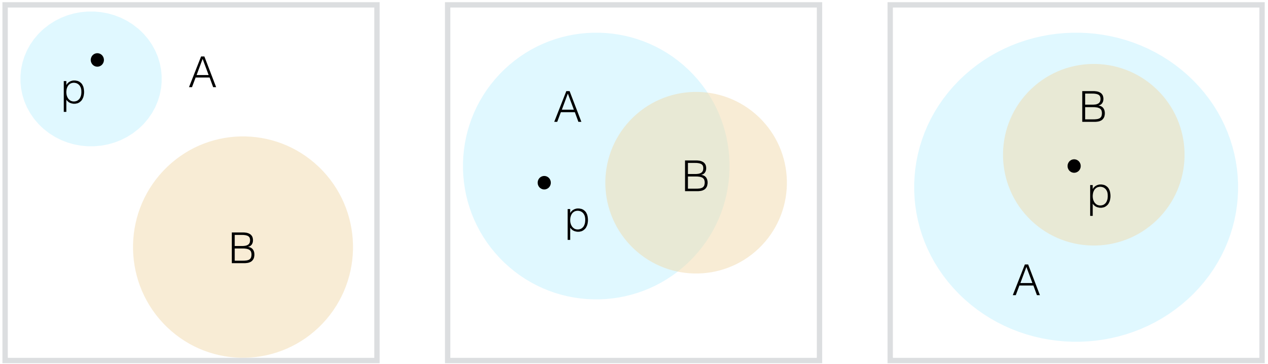
. . .



. . .

Penrose architecture

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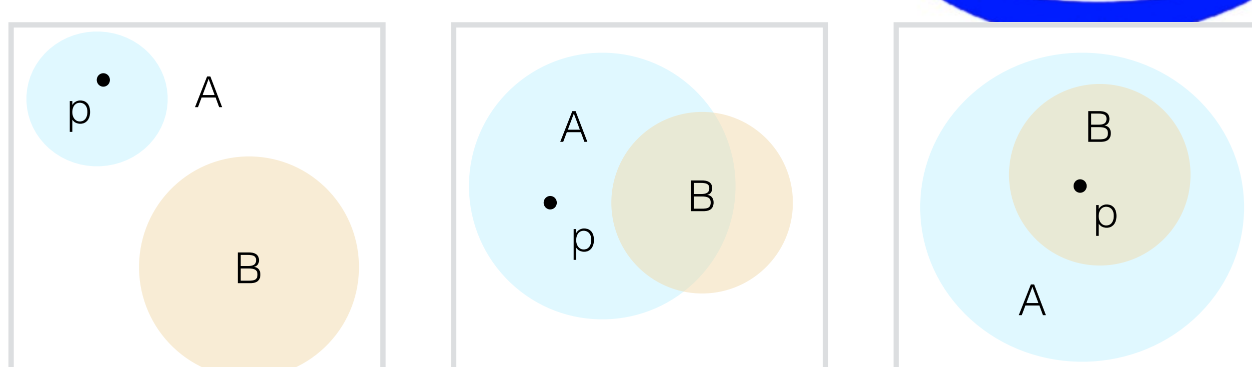

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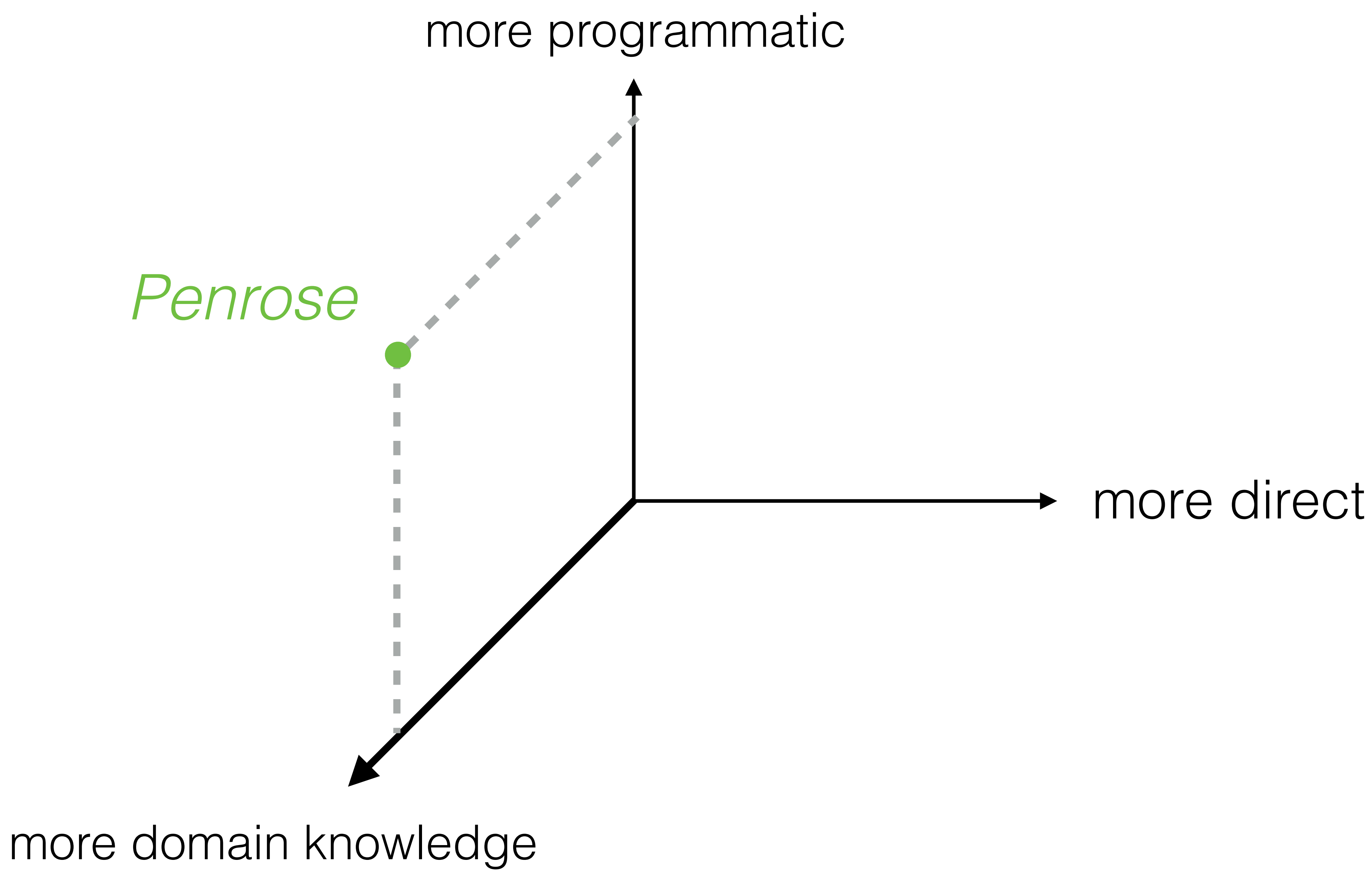
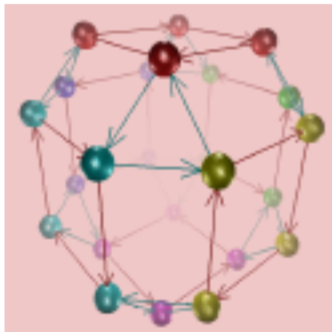
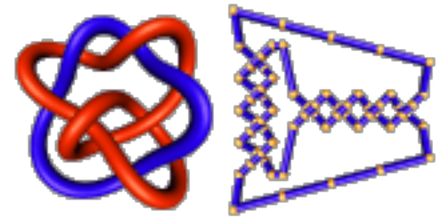
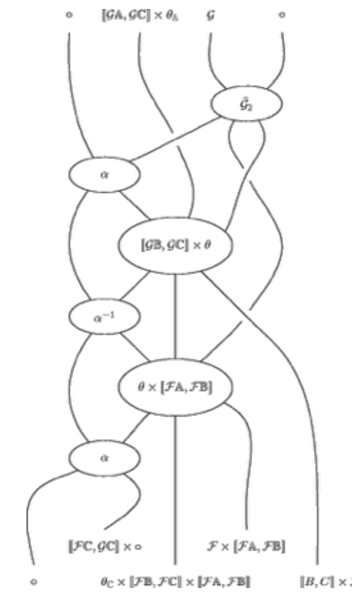
The design space

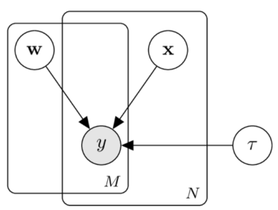


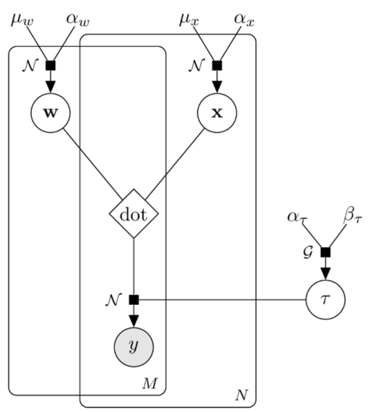
…plus extensibility!

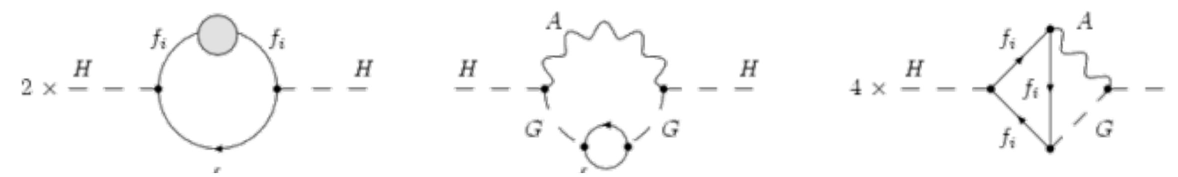




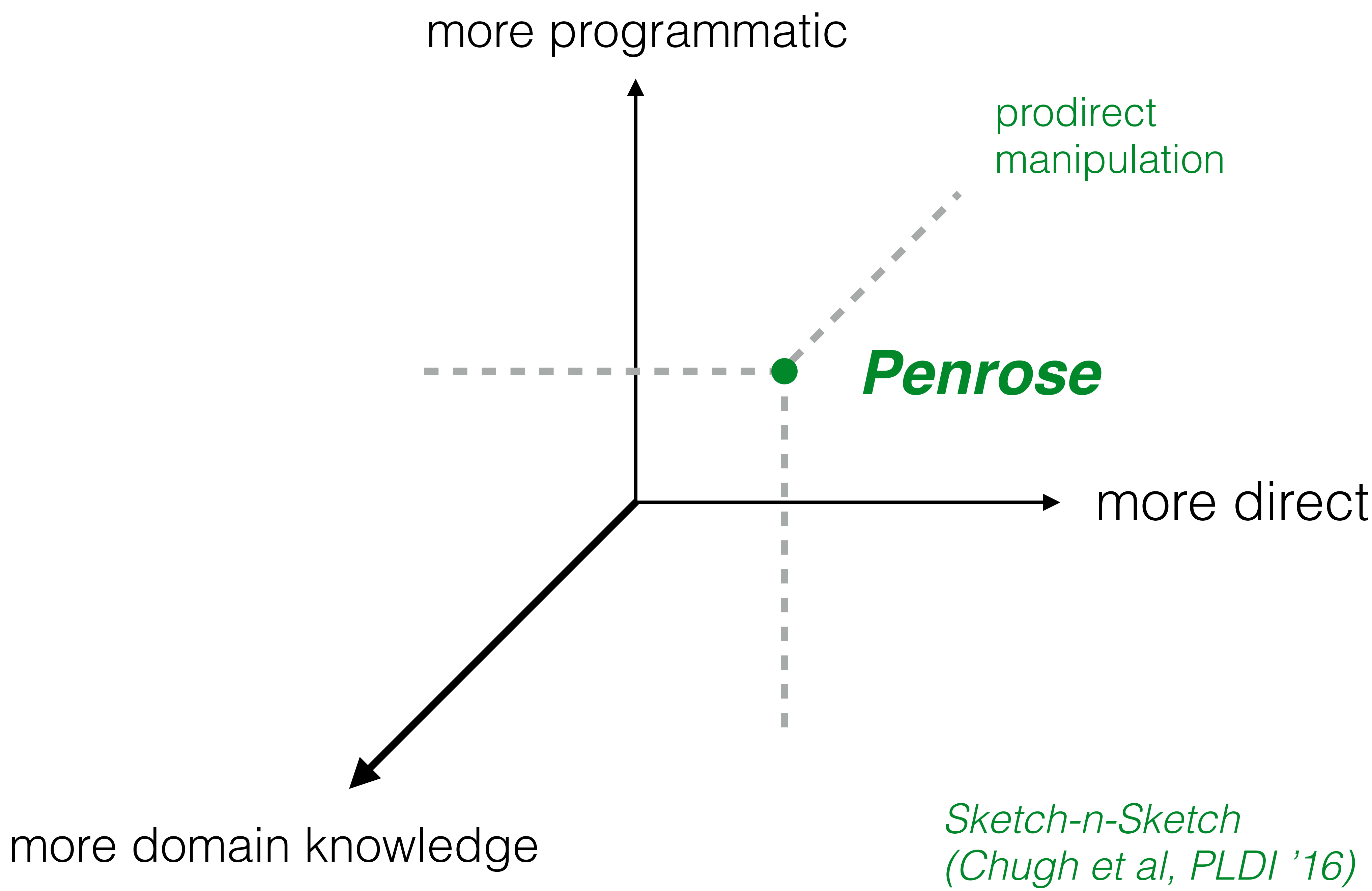








The design space



An example of our vision

I’m demonstrating a **mockup** of a workflow that I hope to demo live in a few months.

We have a very simple prototype for set theory, which I won't demo.

Domain:
Set theory →
Point-set topology

Phases:
Implementing the DSL →
using the DSL

Implementing the DSL

Implementing a set theory DSL

Declarations

Constraints

Implementing a set theory DSL

Declarations

```
data SubDecl = Decl Object
```

Constraints

Implementing a set theory DSL

Declarations

```
data SubDecl = Decl Object
data Object = OS Set | OP Point
data Set = Set’ String SetType

data Pt = Pt’ String
```

Constraints

Implementing a set theory DSL

Declarations

```
data SubDecl = Decl Object
data Object = OS Set | OP Point
data Set = Set’ String SetType
data SetType = Open | Closed | Clopen | Unspecified
data Pt = Pt’ String
```

Constraints

Implementing a set theory DSL

Declarations

```
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Constraints

Implementing a set theory DSL

Declarations

```
data SubDecl = Decl Object
data Object = OS Set | OP Point
data Set = Set’ String SetType
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data Pt = Pt’ String
```

Constraints

```
data Constraint = Subset String String
```

Implementing a set theory DSL

Declarations

```
data SubDecl = Decl Object
data Object = OS Set | OP Point
data Set = Set' String SetType
data SetType = Open | Closed | Clopen | Unspecified
data Pt = Pt' String
```

Constraints

```
data Constraint = Subset String String
                | PointIn String String
```

Set theory DSL

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

Extending the set theory DSL

Declarations

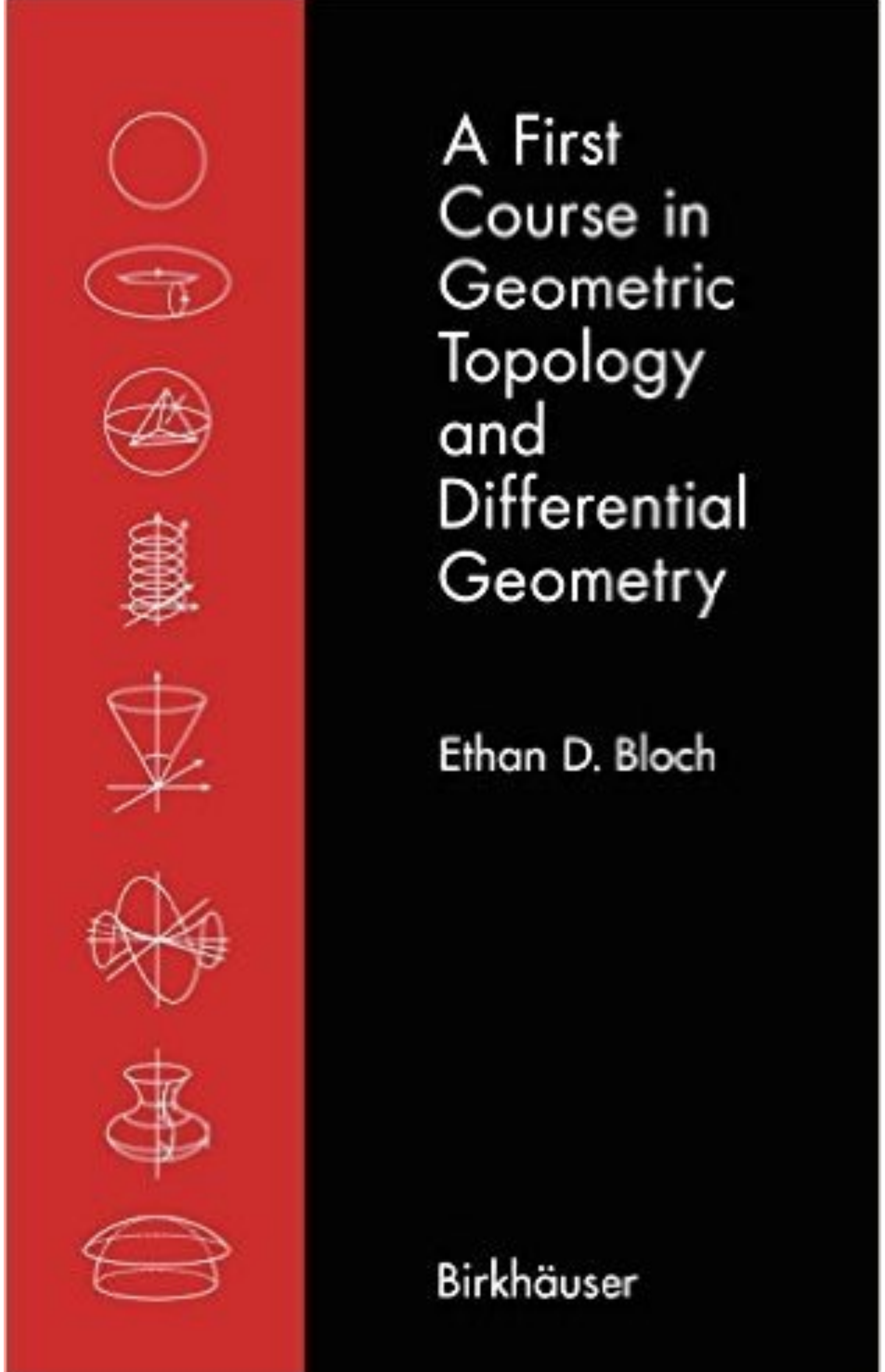
```
data SubDecl = Decl Object
data Object = OS Set | OP Point | OM Map
data Set = Set' String SetType
data SetType = Open | Closed | Clopen | Unspecified
data Pt = Pt' String
data Map = Map' String String String
```

Constraints

```
data Constraint = Subset String String
                | PointIn String String
```

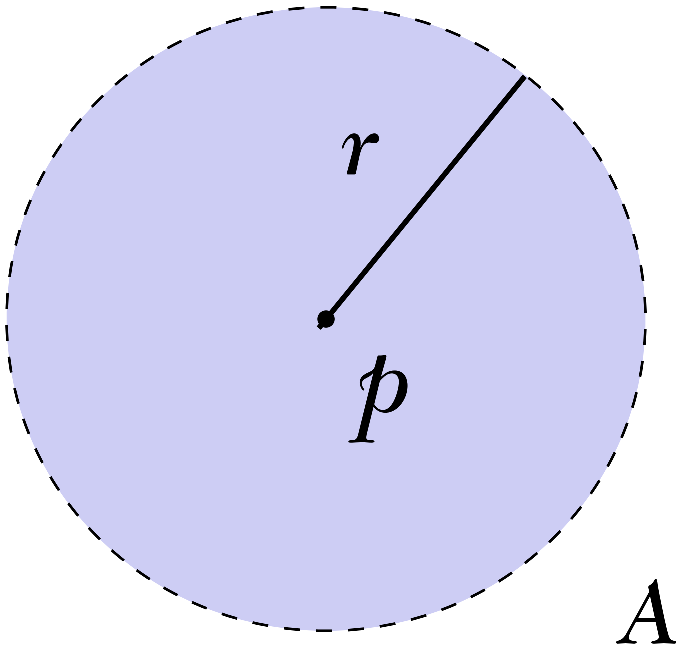
Using the DSL

Open sets, closed sets, continuous maps


A First Course in Geometric Topology and Differential Geometry
Ethan D. Bloch
Birkhäuser

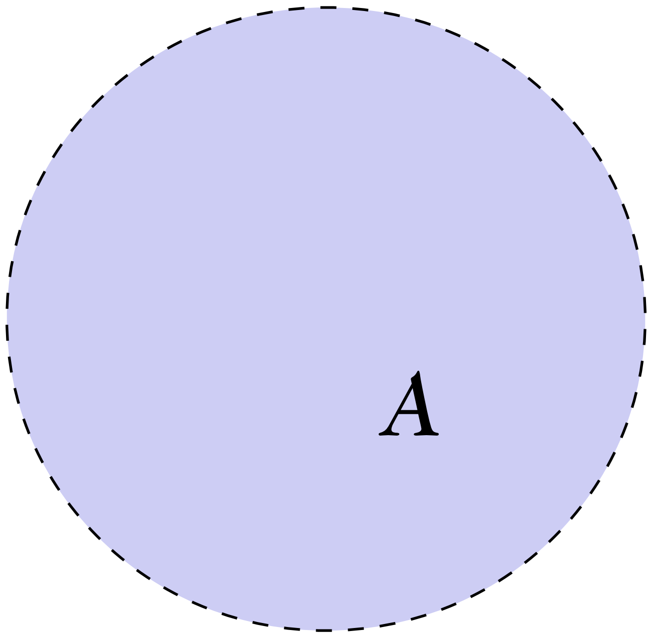
An open ball in R^2

```
A = Or(p, Rn)
```



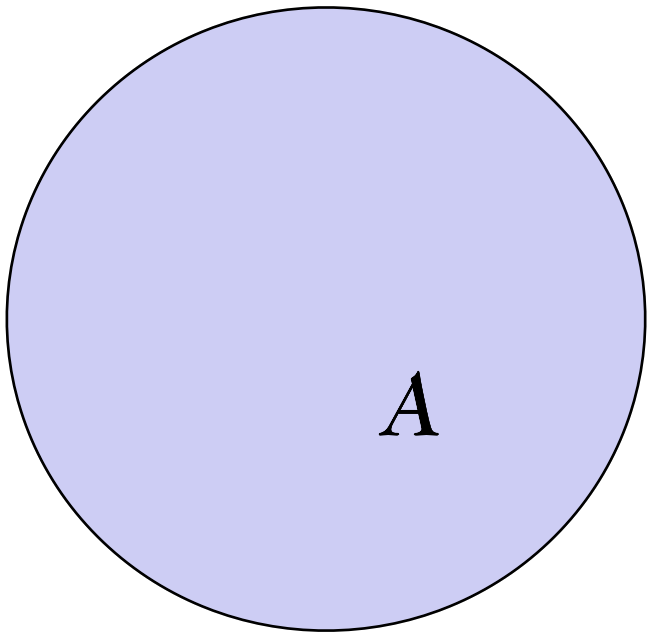
An open ball in R^2

```
OpenBall A
```



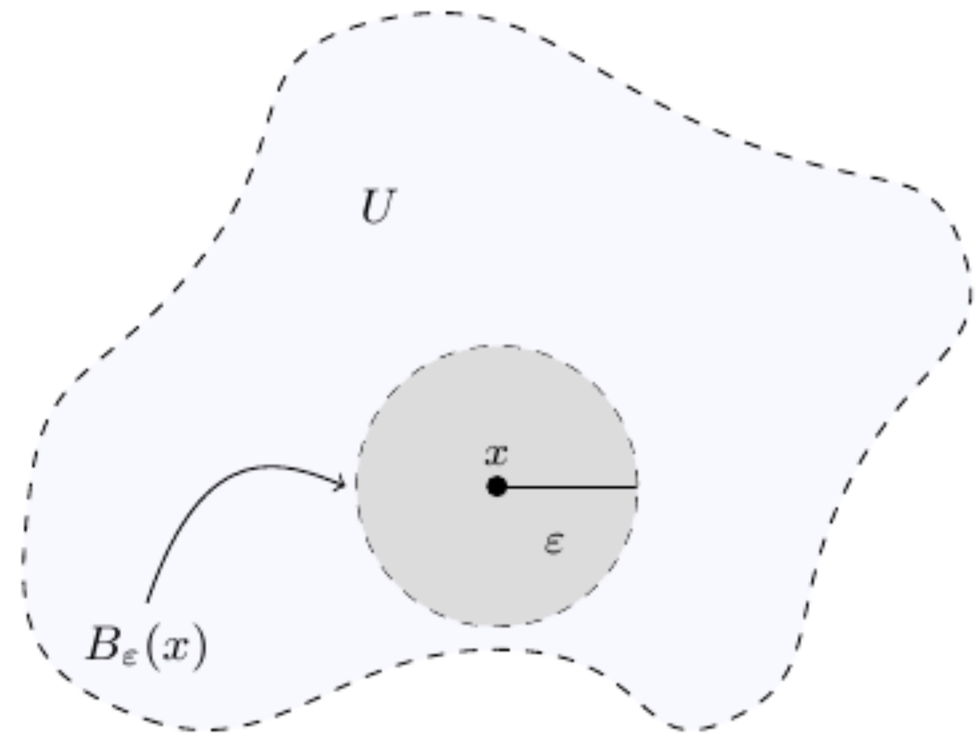
An closed ball in R^2

```
ClosedBall A
```



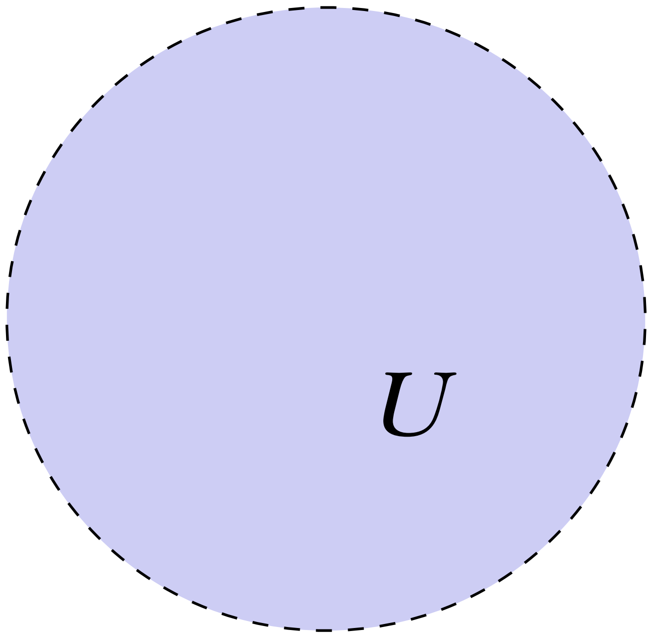
An open set in R^2

```
OpenSet U
```



An open set in R^2, simplified

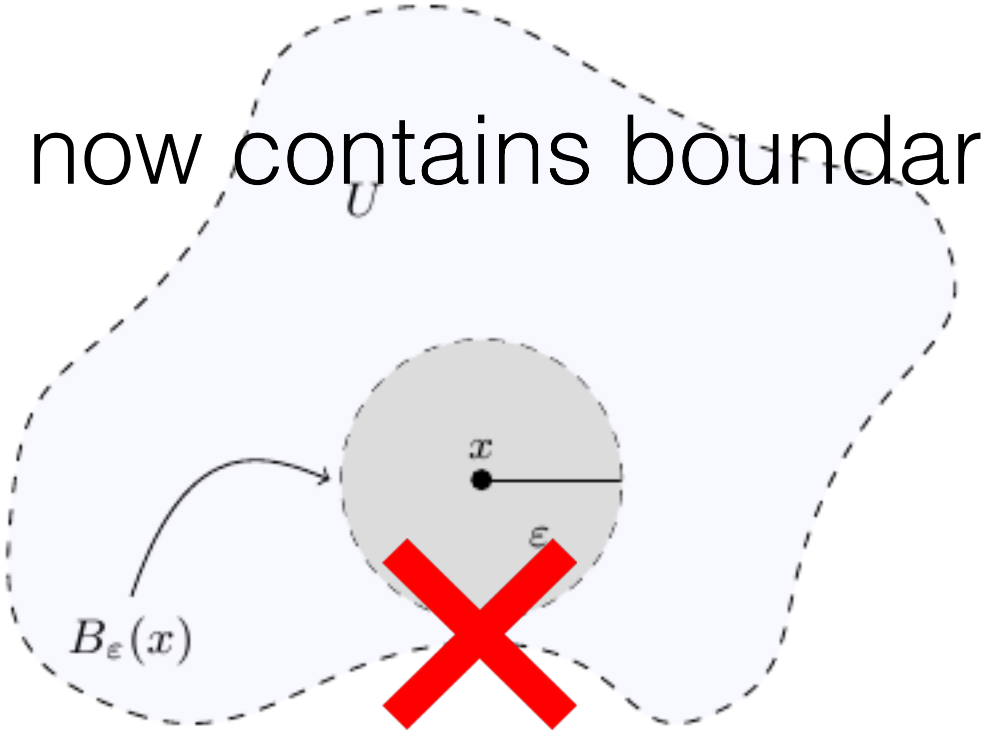
```
OpenSet U
```



An closed set in R²

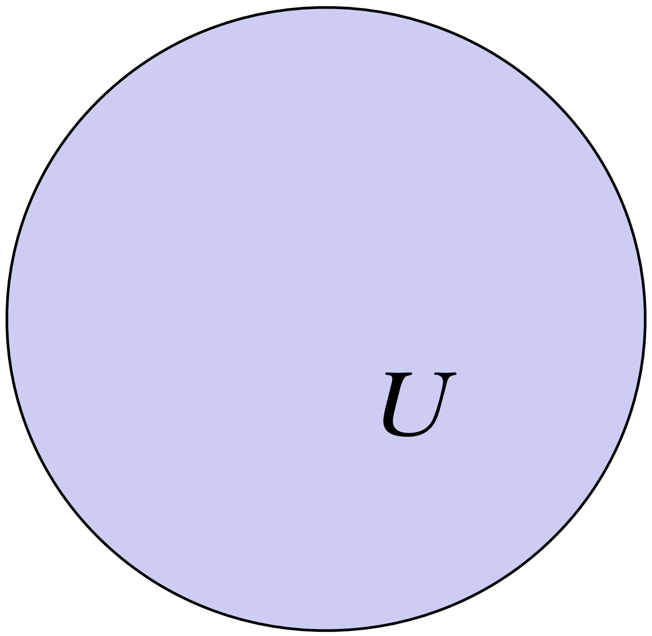
```
ClosedSet U
```

now contains boundary



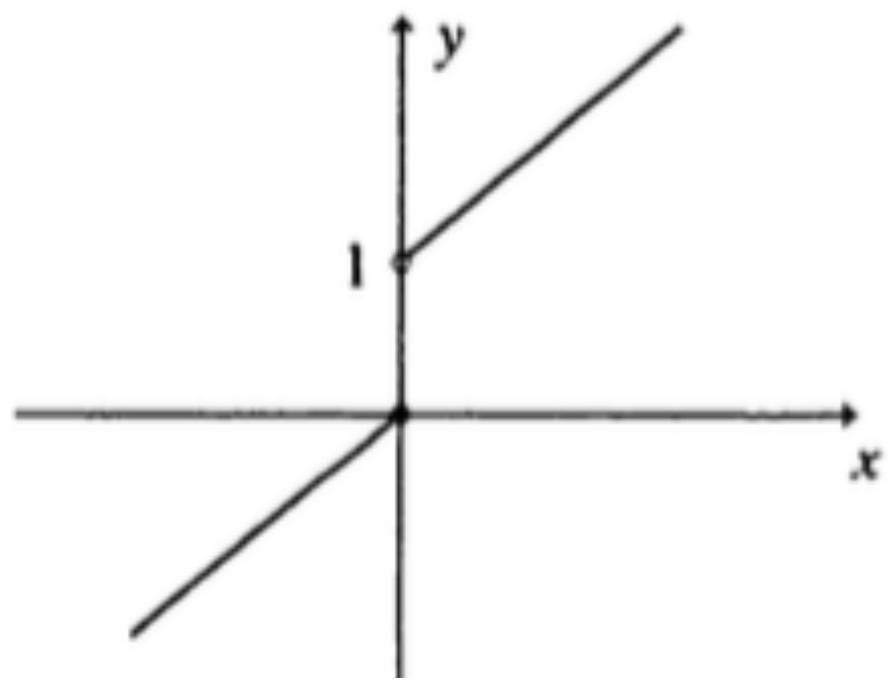
An closed set in R^2, simplified

```
ClosedSet U
```

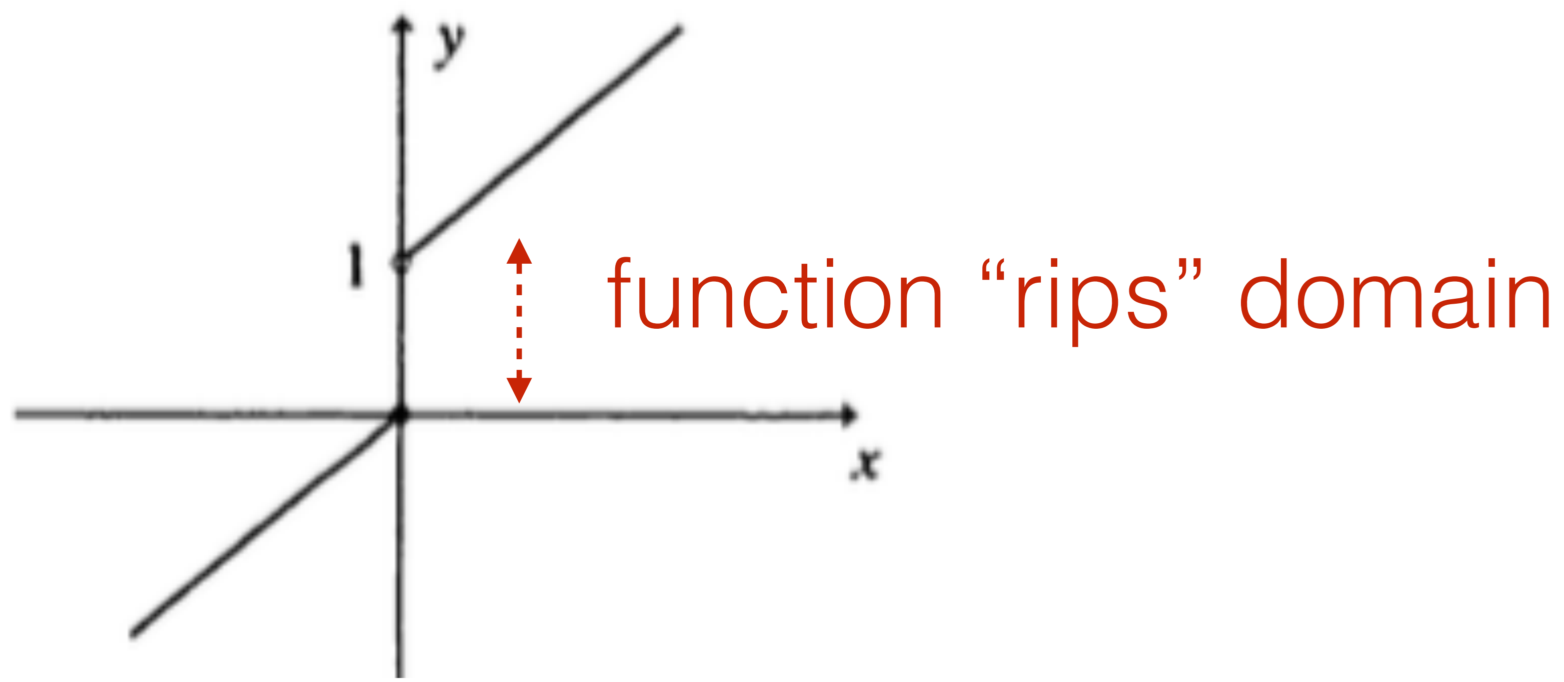


Continuous maps

Concrete anti-example: discontinuous map



Concrete anti-example: discontinuous map



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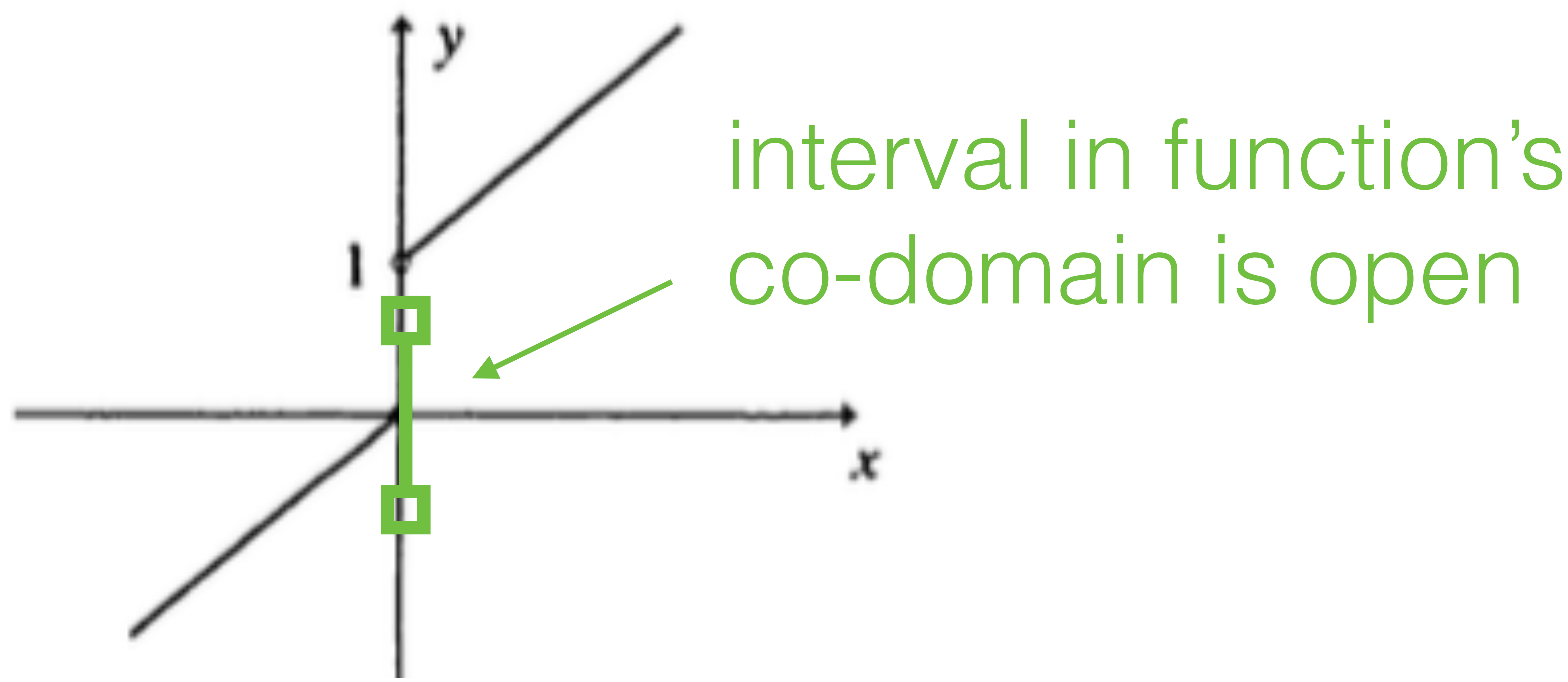

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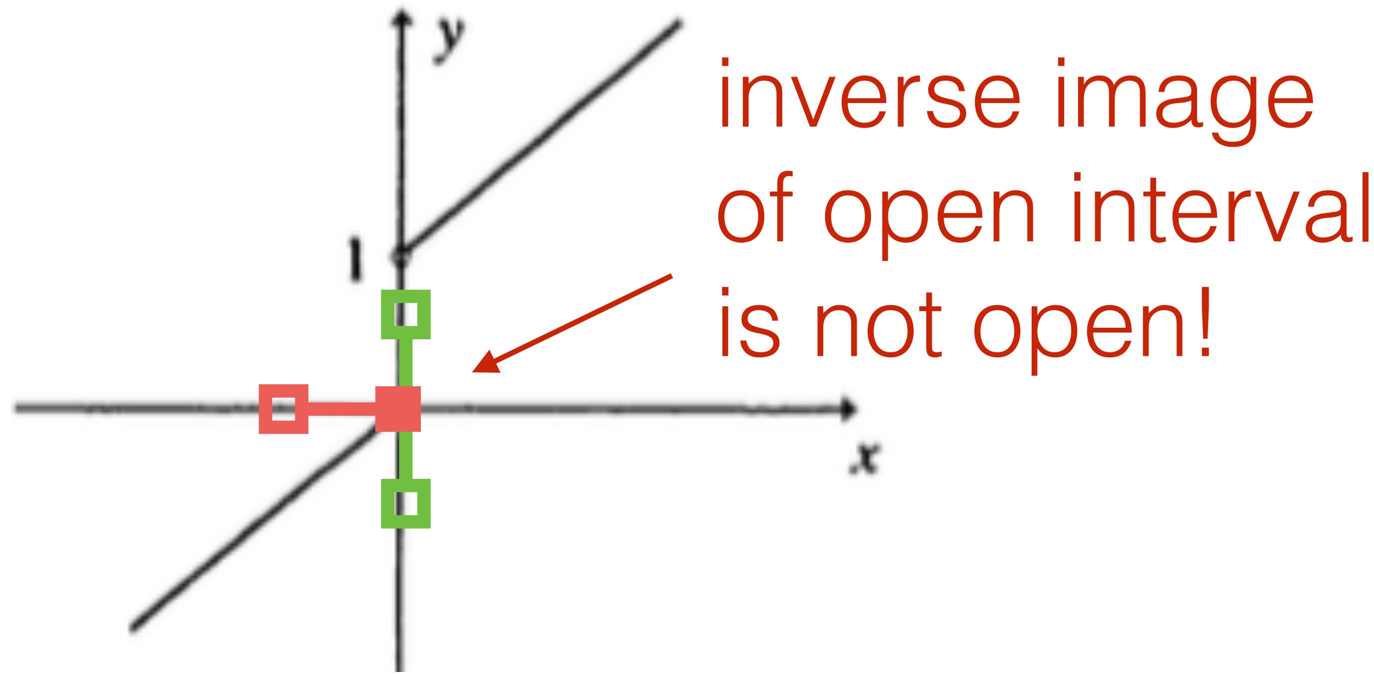

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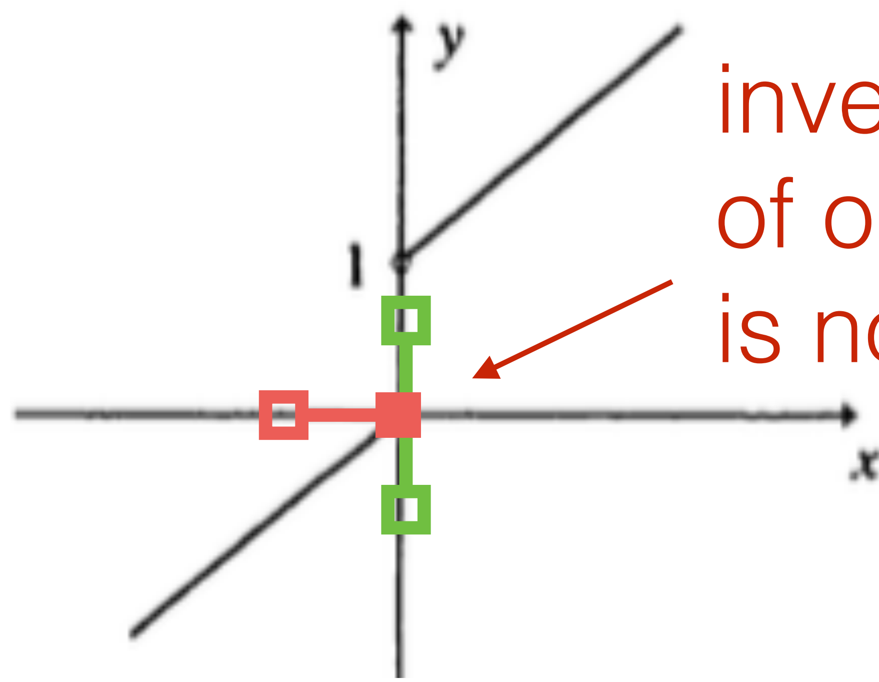

inverse image
of open interval
is not open!

Generalizing the intuition

Generalizing the intuition

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊

Generalizing the intuition

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Picture this!

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊

Substance

```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
Set f^{-1}(U)
Subset f^{-1}(U) A
```

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊

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Set B
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Set R^m
Subset A R^n
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Map f A B
OpenSet U
Subset U B
Set f^{-1}(U)
Subset f^{-1}(U) A
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declarations

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f : A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊

Substance

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Set A
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Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
Set f^{-1}(U)
Subset f^{-1}(U) A
```

constraints

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊

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Set R^m
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Subset B R^m
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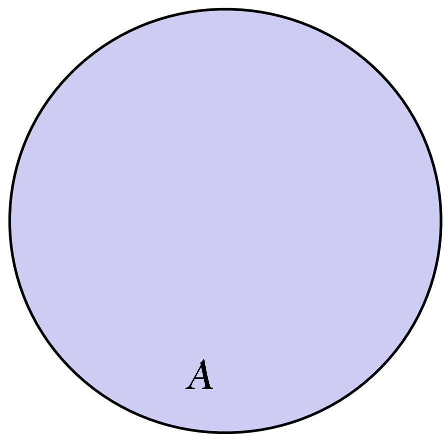
OpenSet U
Subset U B
Set f^{-1}(U)
Subset f^{-1}(U) A
```

Style

```
Style All Auto
```

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f\colon A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊



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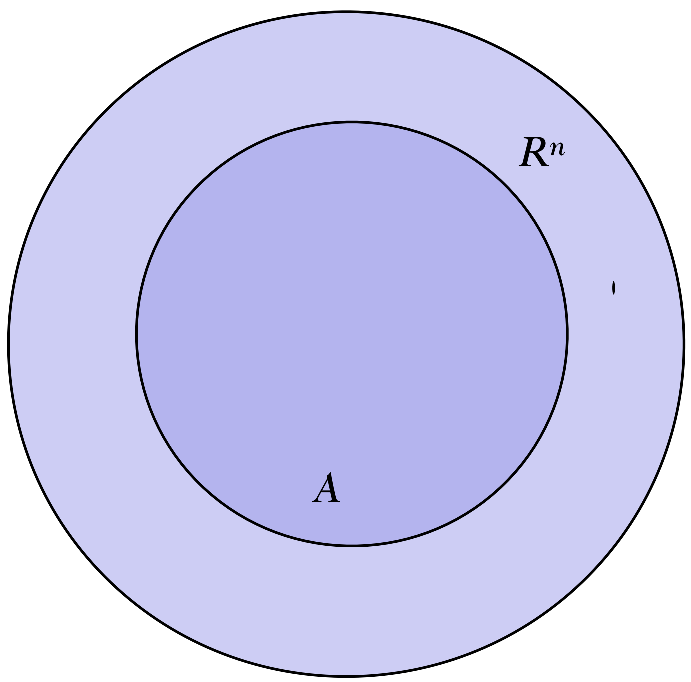

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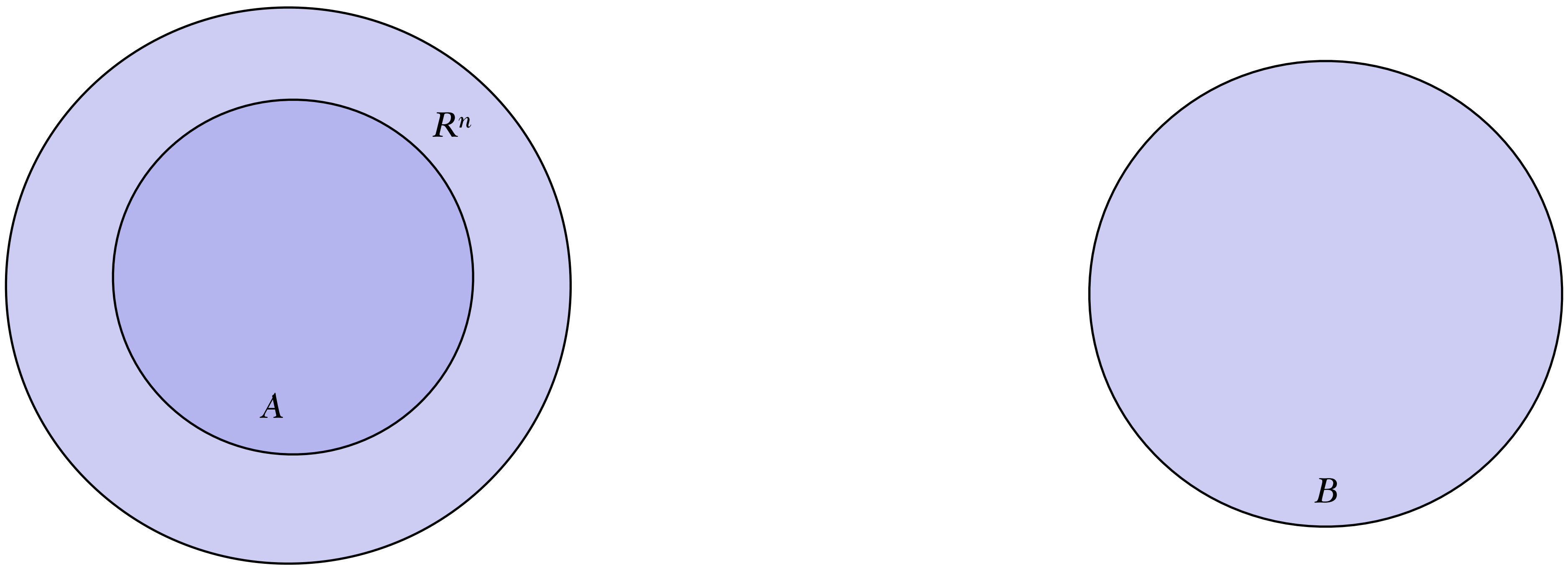

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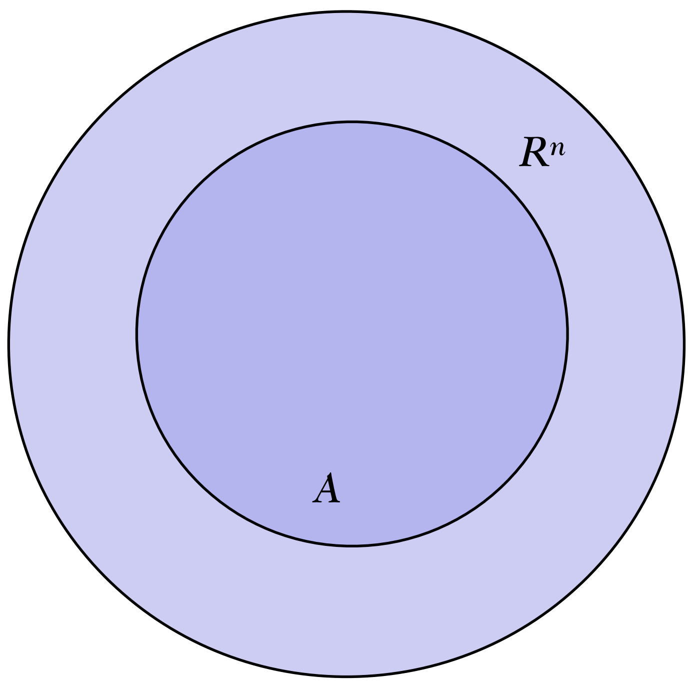

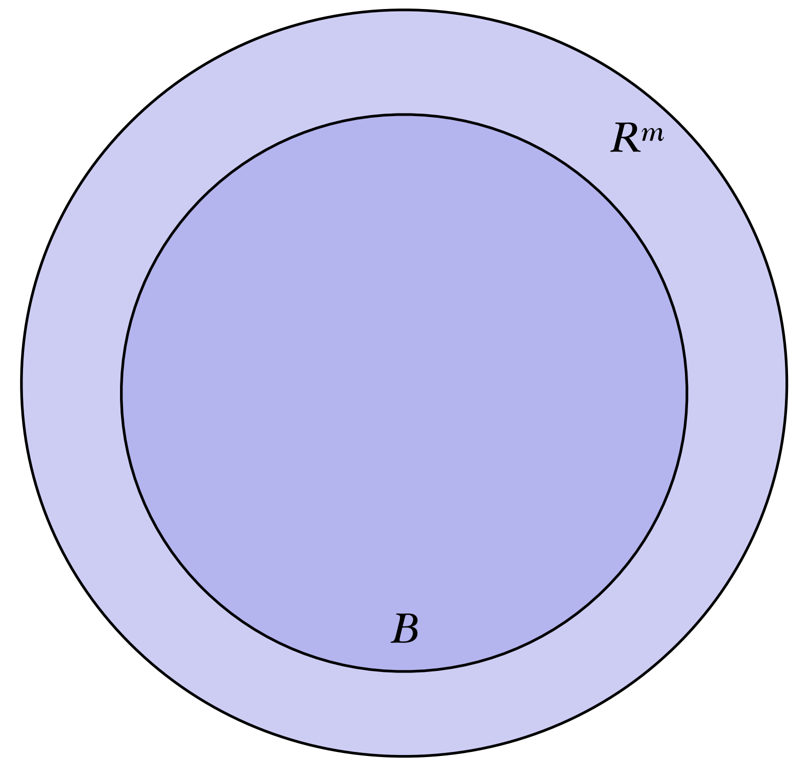

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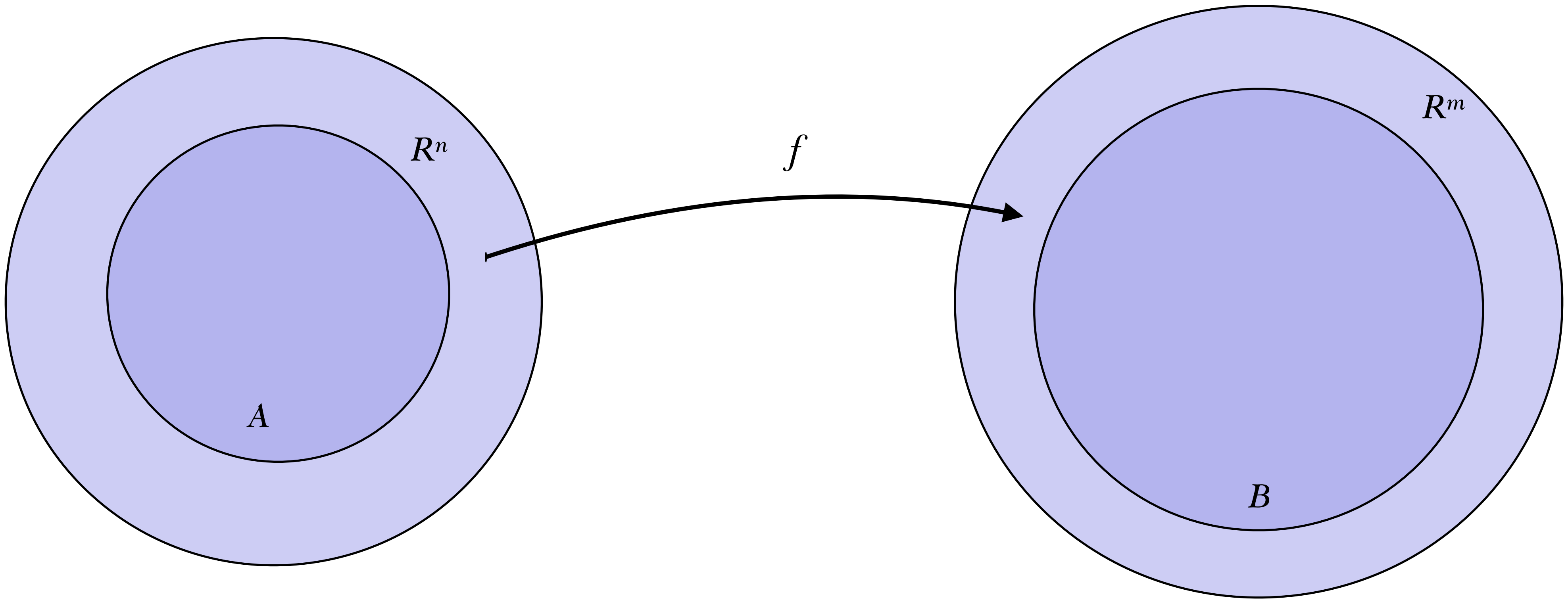

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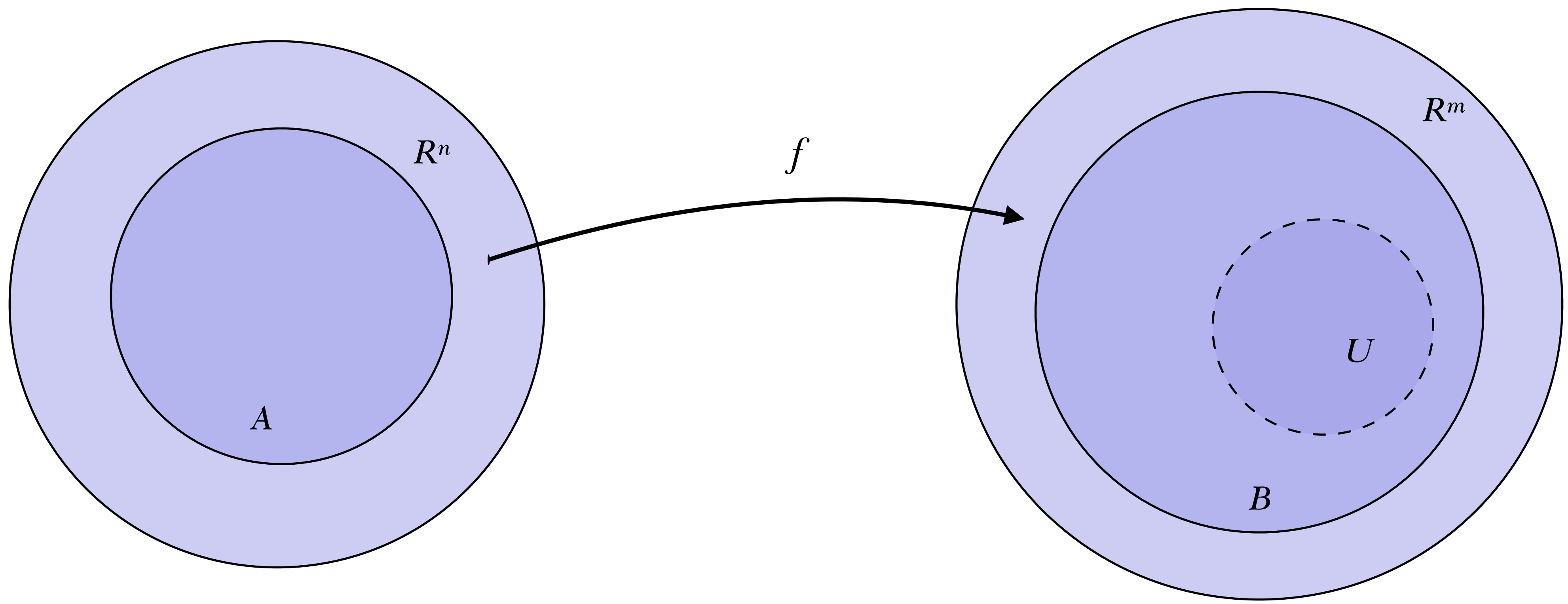

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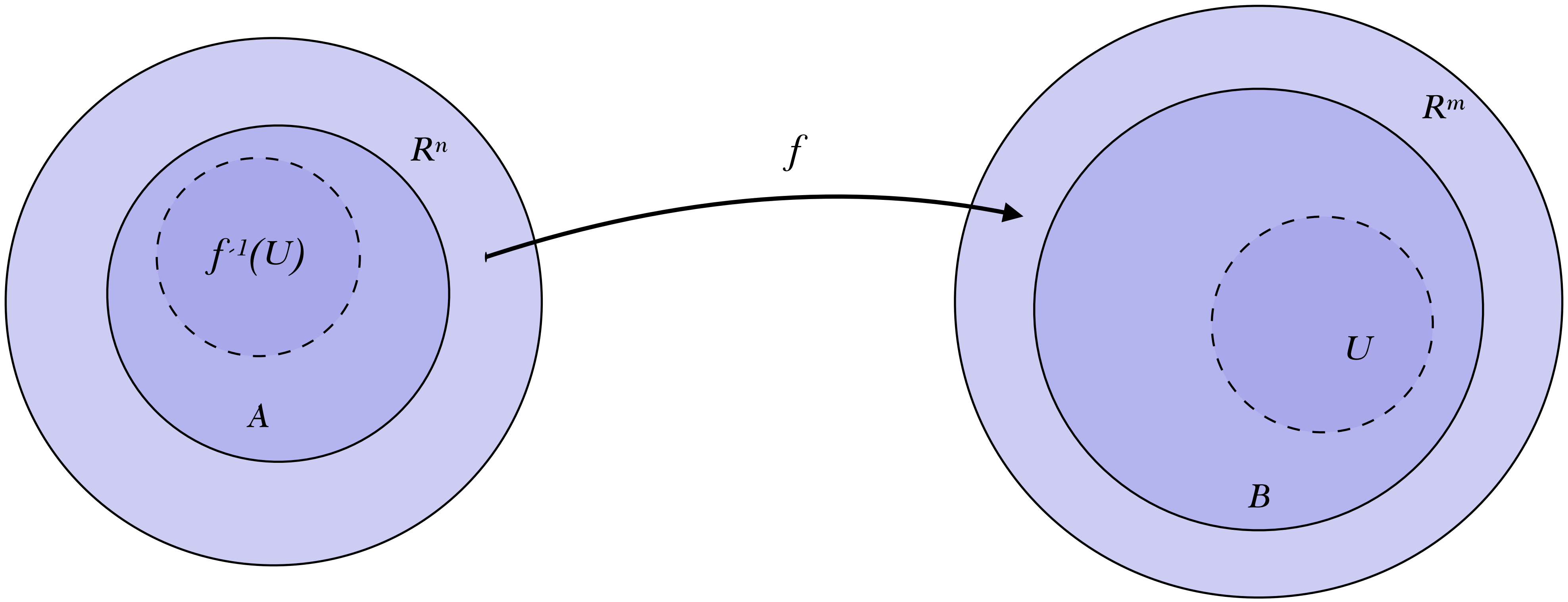

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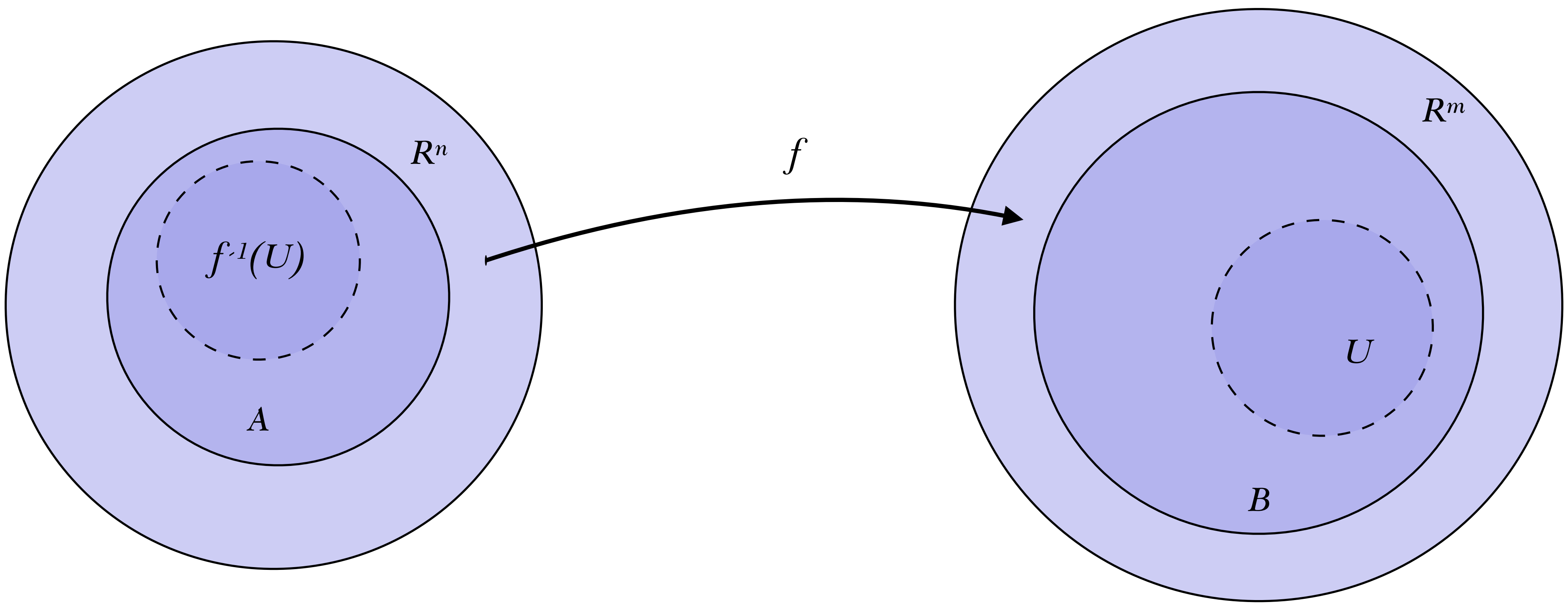

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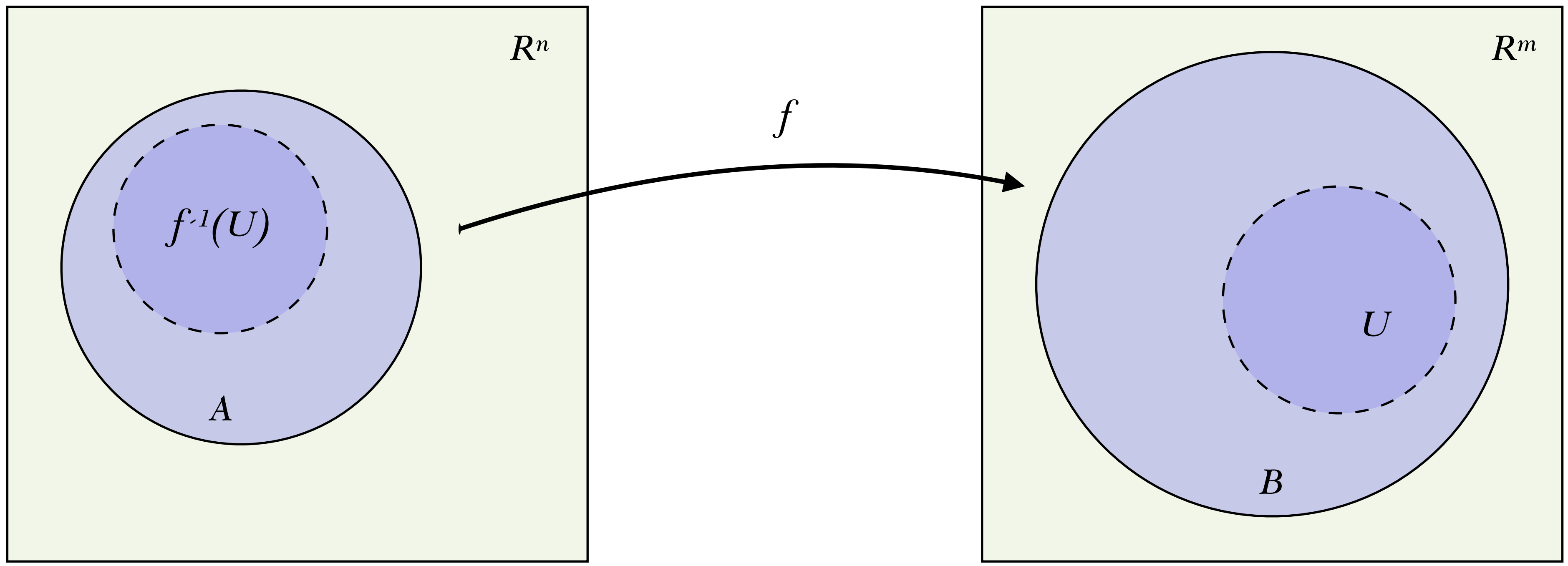
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
Set f^{-1}(U)
Subset f^{-1}(U) A
```

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊



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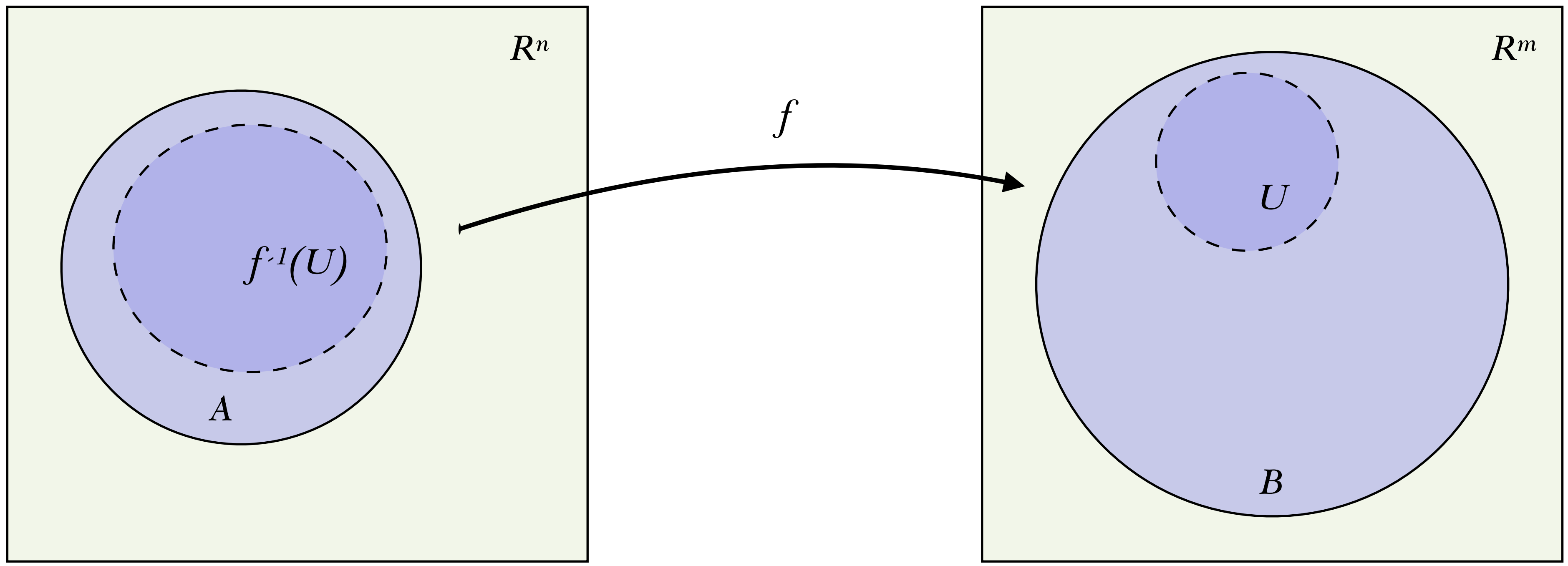
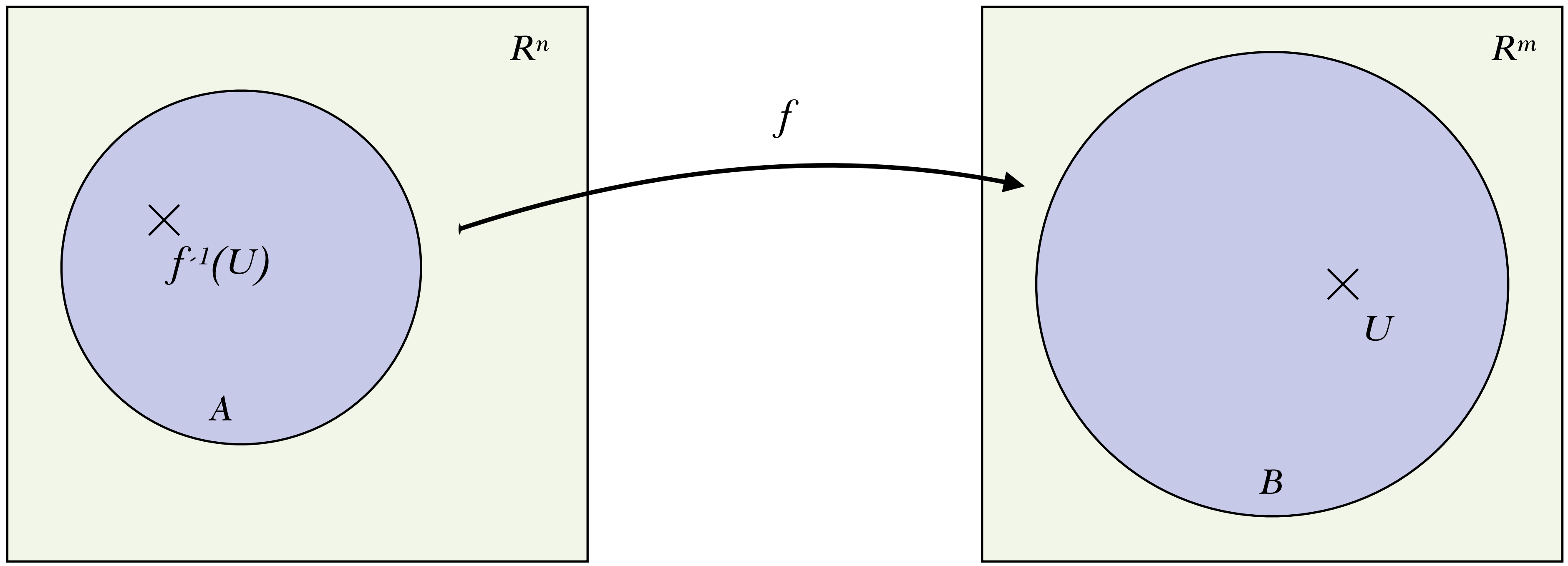


animation, interactivity, fuzzing

Definition. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The map f is **continuous** if for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A. ◊



definition holds for the "edge case" of empty sets

animation, interactivity, fuzzing

A harder definition…

Proposition 1.3.3. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The following statements are equivalent.*

(1) *The map f is continuous.*

(2) *For every point $p \in A$, and every open subset $U \subset B$ containing $f(p)$, there is an open subset $V \subset A$ containing p such that $f(V) \subset U$.*

(3) *For every point $p \in A$ and every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $x \in A$ and $\|x - p\| < \delta$ then $\|f(x) - f(p)\| < \epsilon$.*

Proposition 1.3.3. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The following statements are equivalent.*

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Picture this!

Proposition 1.3.3. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The following statements are equivalent.*

(1) for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A.

(2) *For every point $p \in A$, and every open subset $U \subset B$ containing $f(p)$, there is an open subset $V \subset A$ containing p such that $f(V) \subset U$.*

(3) *For every point $p \in A$ and every number $\epsilon > 0$, there is a number $\delta > 0$ such that if $x \in A$ and $\|x - p\| < \delta$ then $\|f(x) - f(p)\| < \epsilon$.*

Picture this!

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Substance

```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
```

```
Point p
In p A
Point f(p)
In f(p) U
OpenSet V
Subset V A
In p V
Set f(V)
Subset f(V) U
In f(p) f(V)
```

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

underline denotes implicit in notation. f(V) might not be open but is depicted as such

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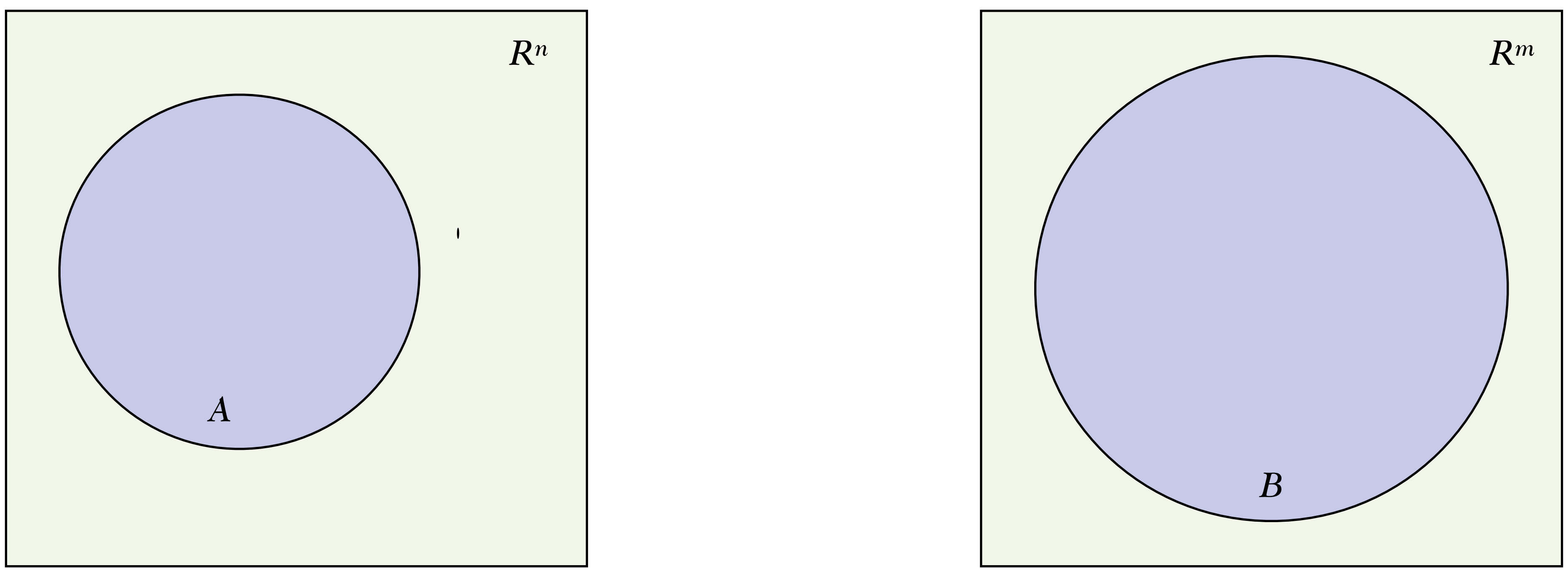

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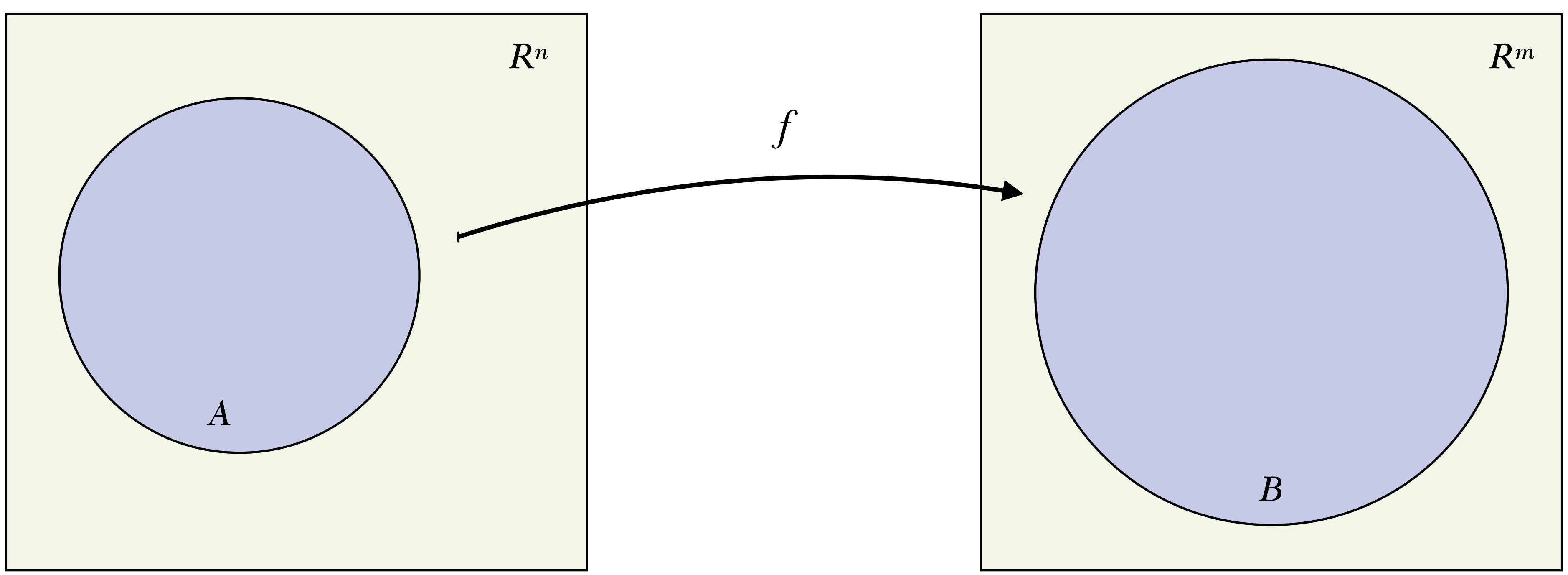

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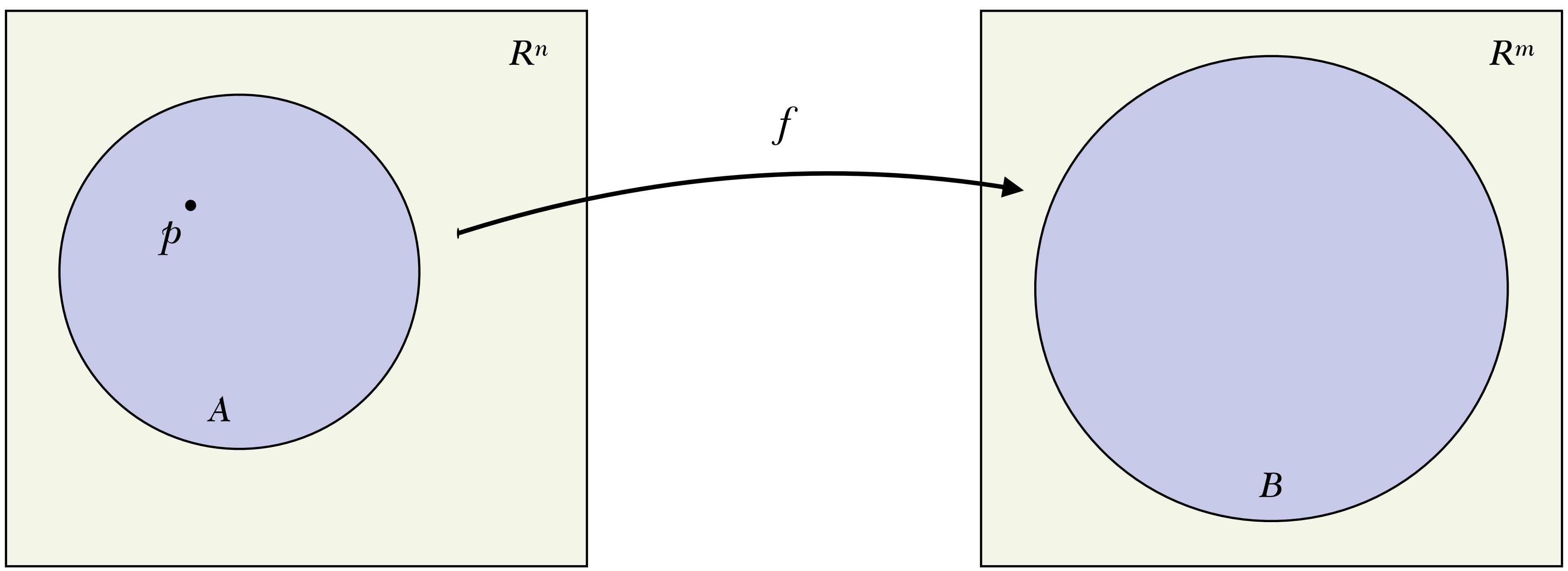

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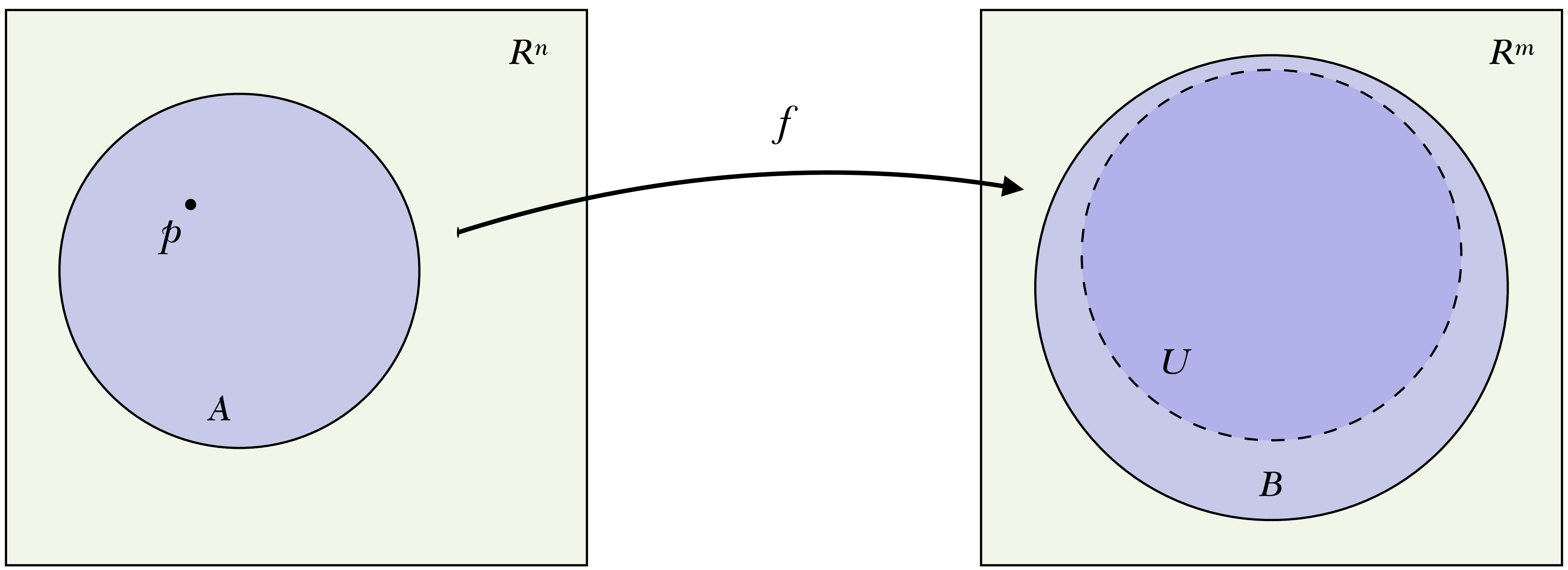

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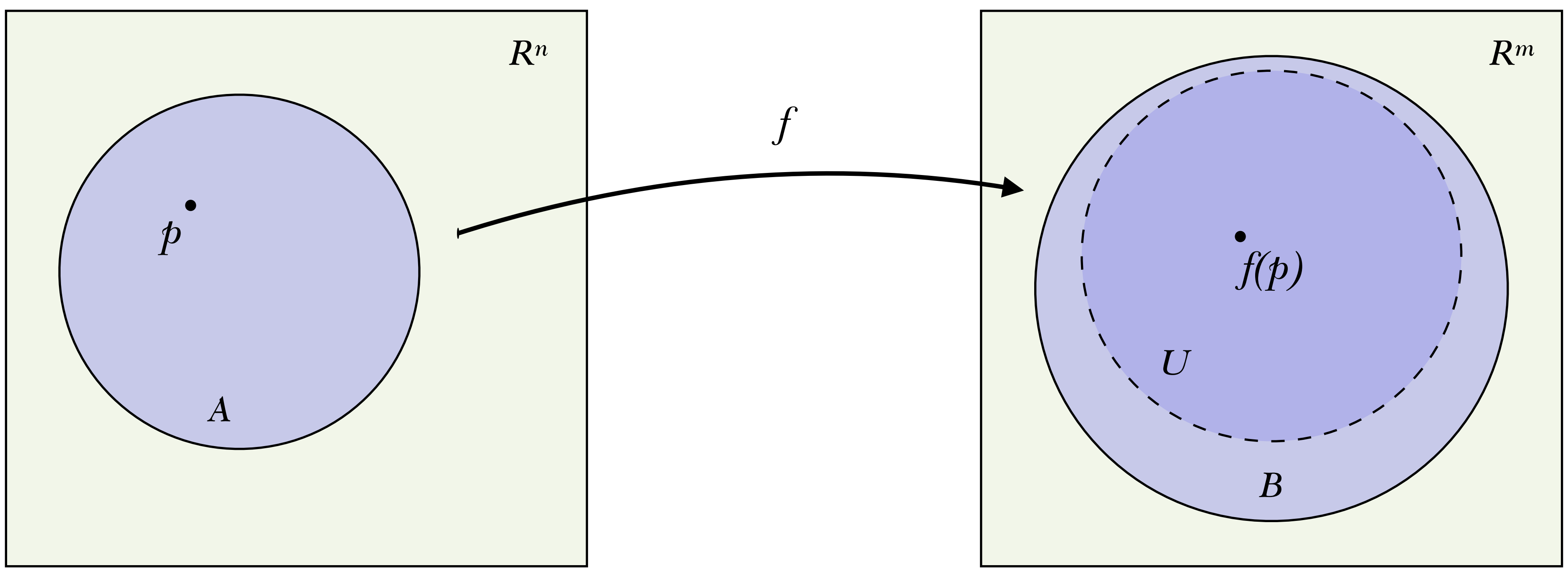

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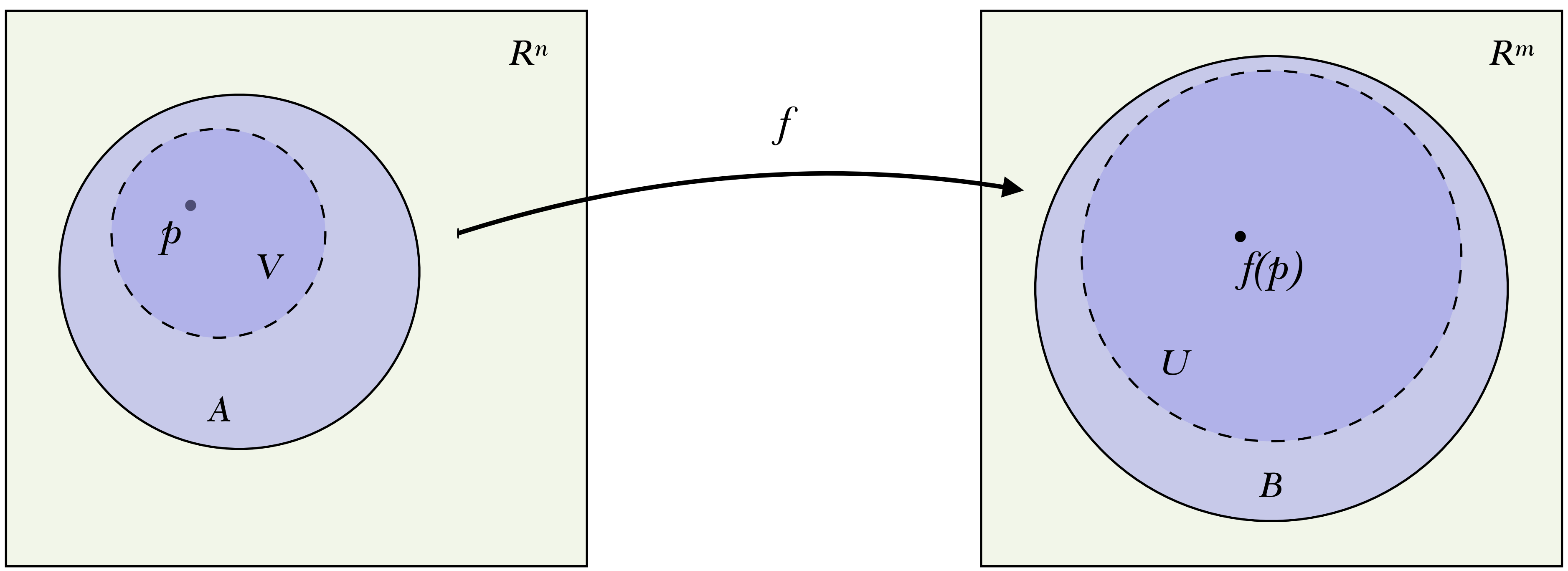

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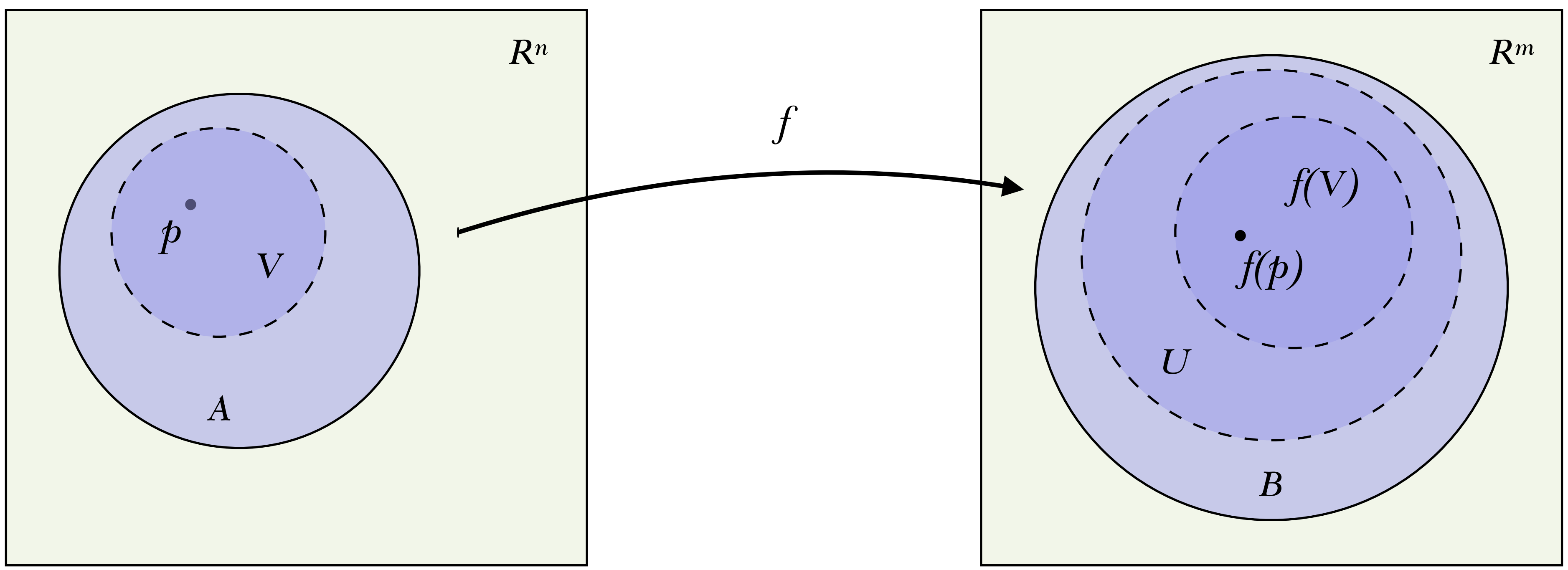

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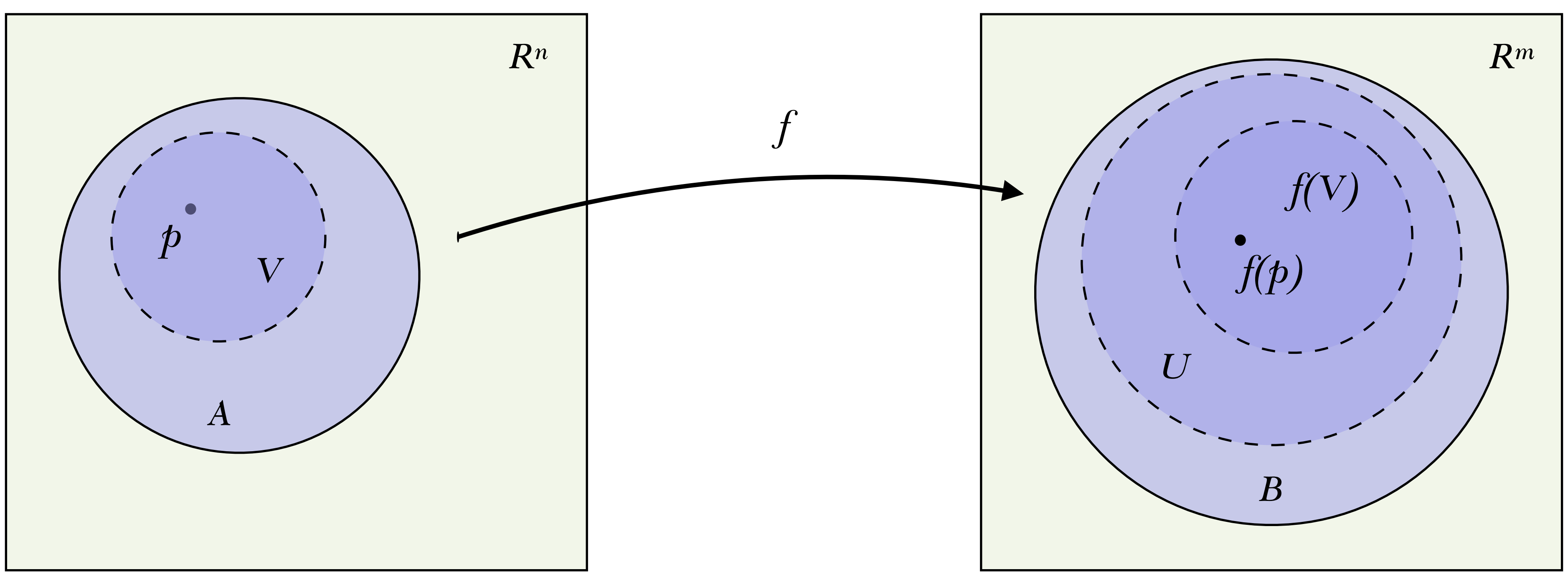

How do the two definitions relate?

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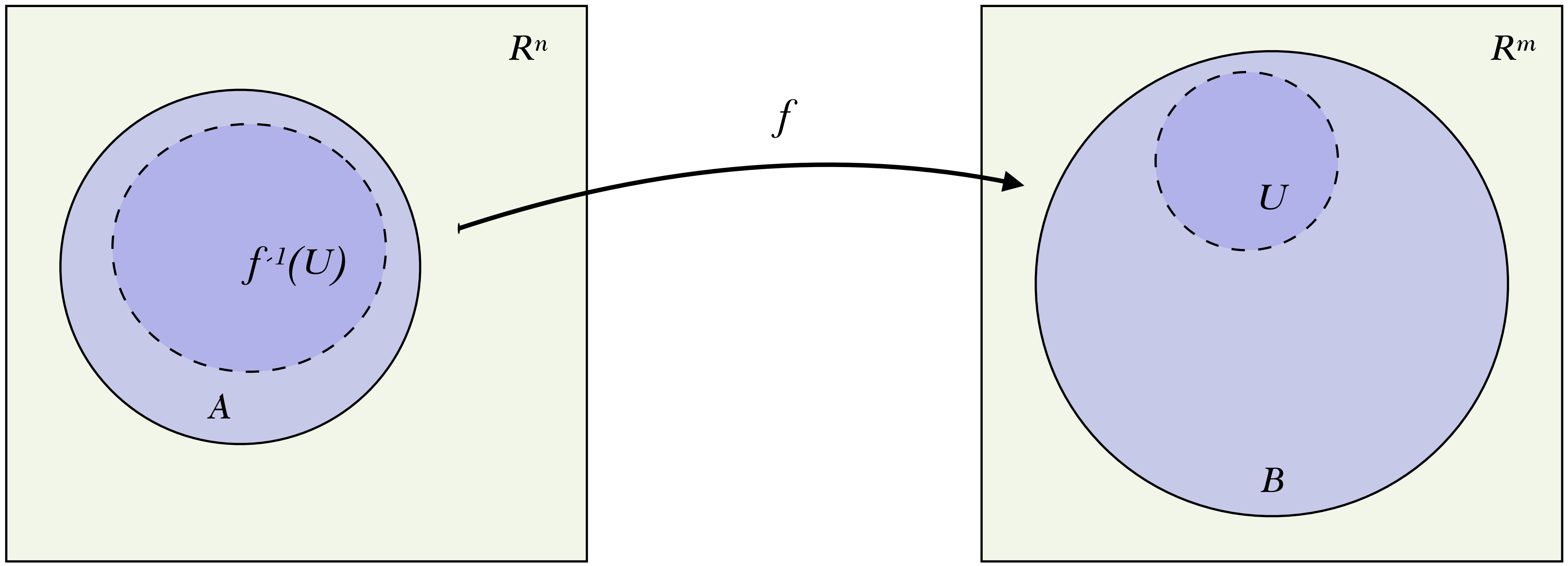

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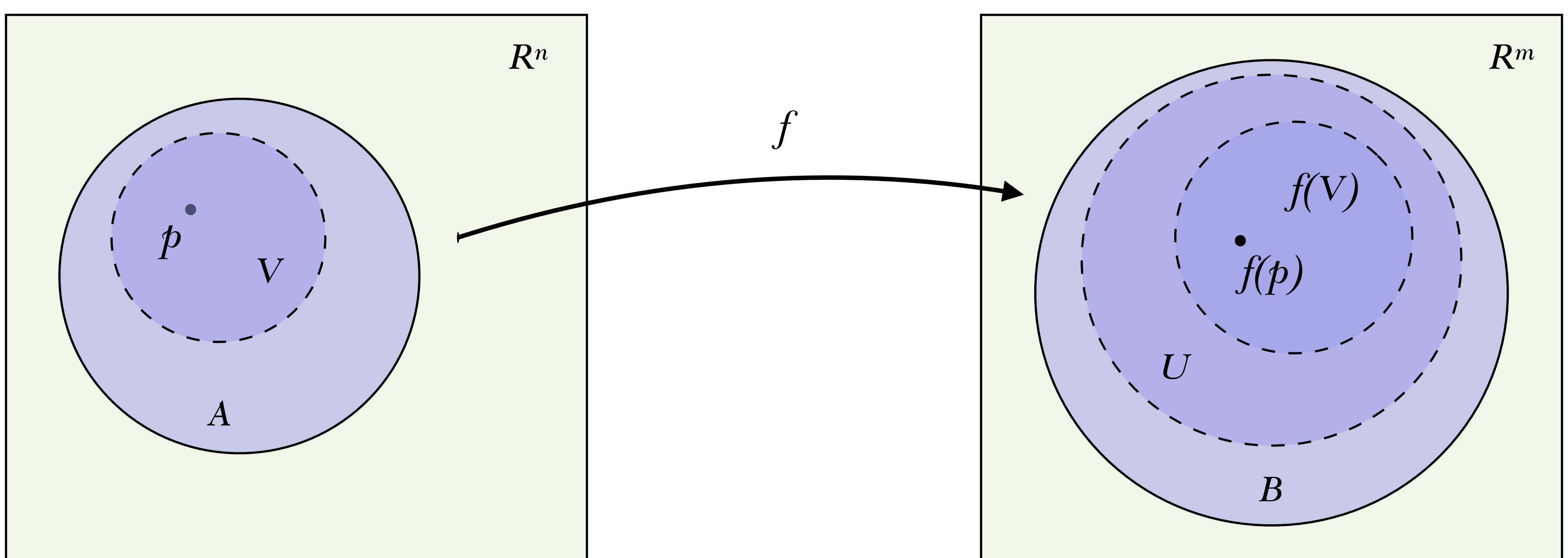

Can prove (1) → (2) via diagram!

We now prove (1) ⇒ (2) ⇒ (3′) ⇒ (1).

(1) ⇒ (2). Let $p \in A$ and $U \subset B$ containing $f(p)$ be given. By assumption, the map f is continuous, so that $f^{-1}(U)$ is an open subset of A. Observe that $p \in f^{-1}(U)$. By the definition of openness there is thus some open ball of the form $O_\delta(p, A)$ contained in $f^{-1}(U)$. It follows that $f(O_\delta(p, A)) \subset U$. By Exercise 1.2.5 the open ball $O_\delta(p, A)$ is an open subset of A, so let $V = O_\delta(p, A)$.

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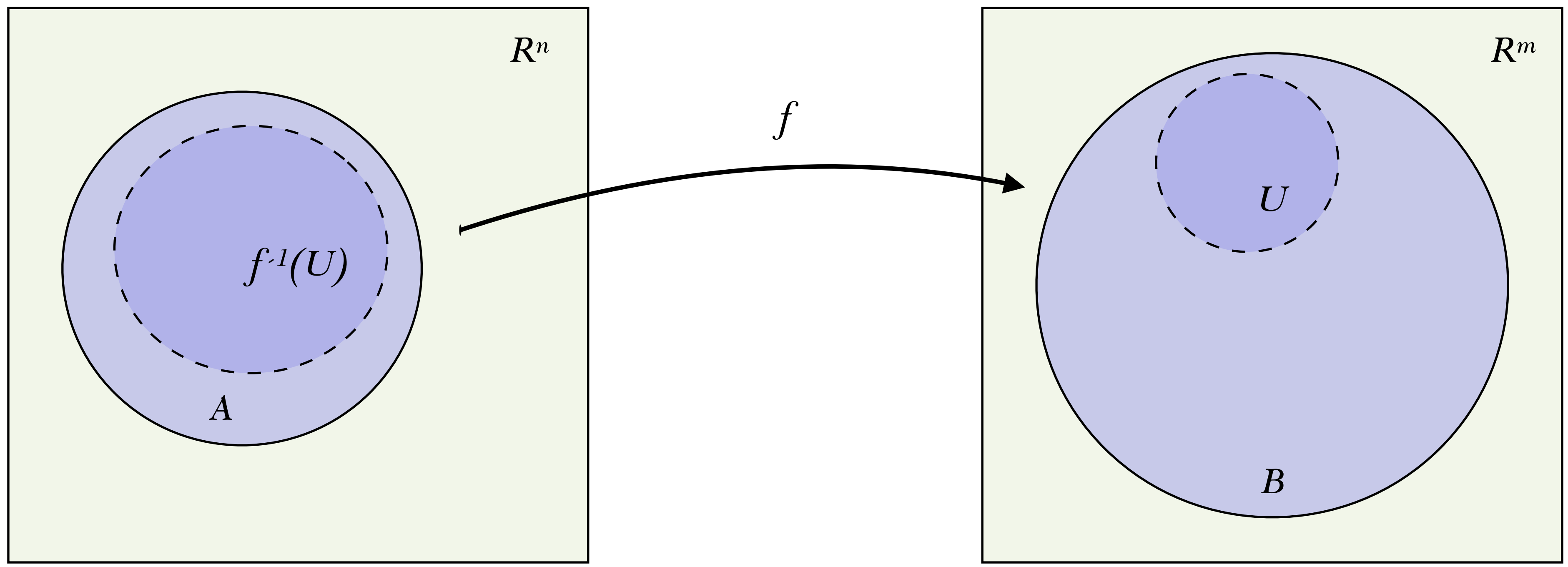

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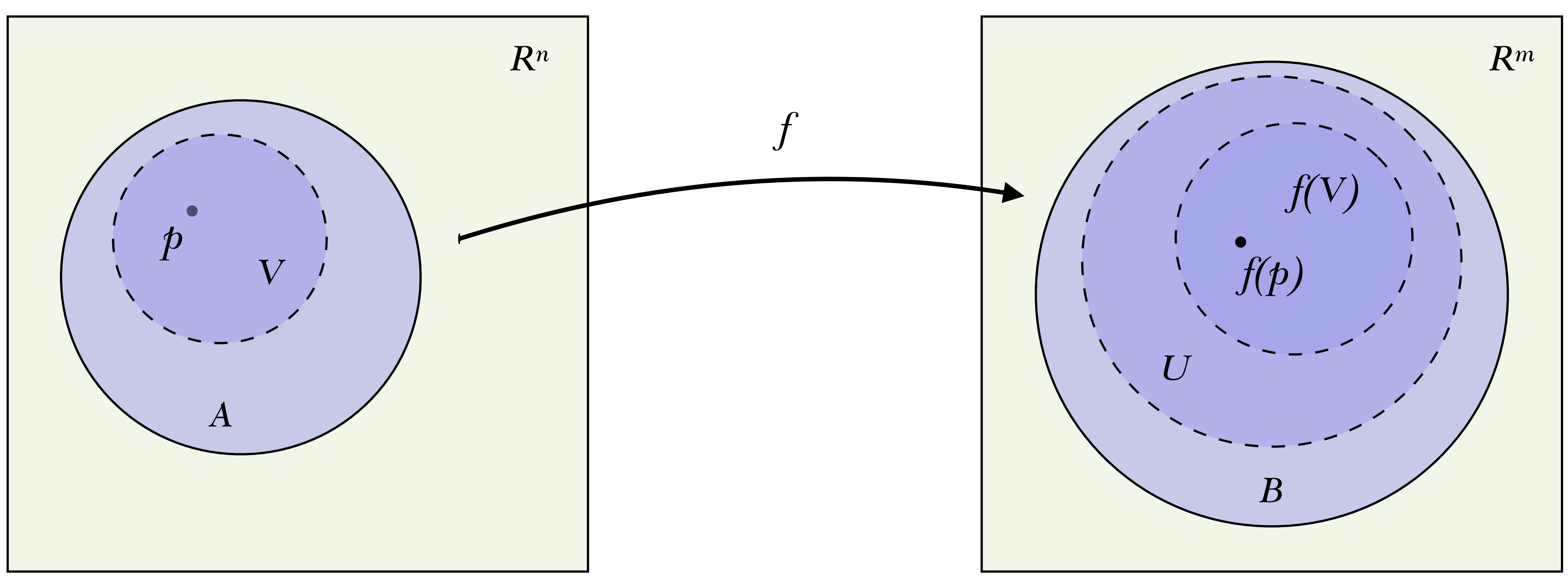
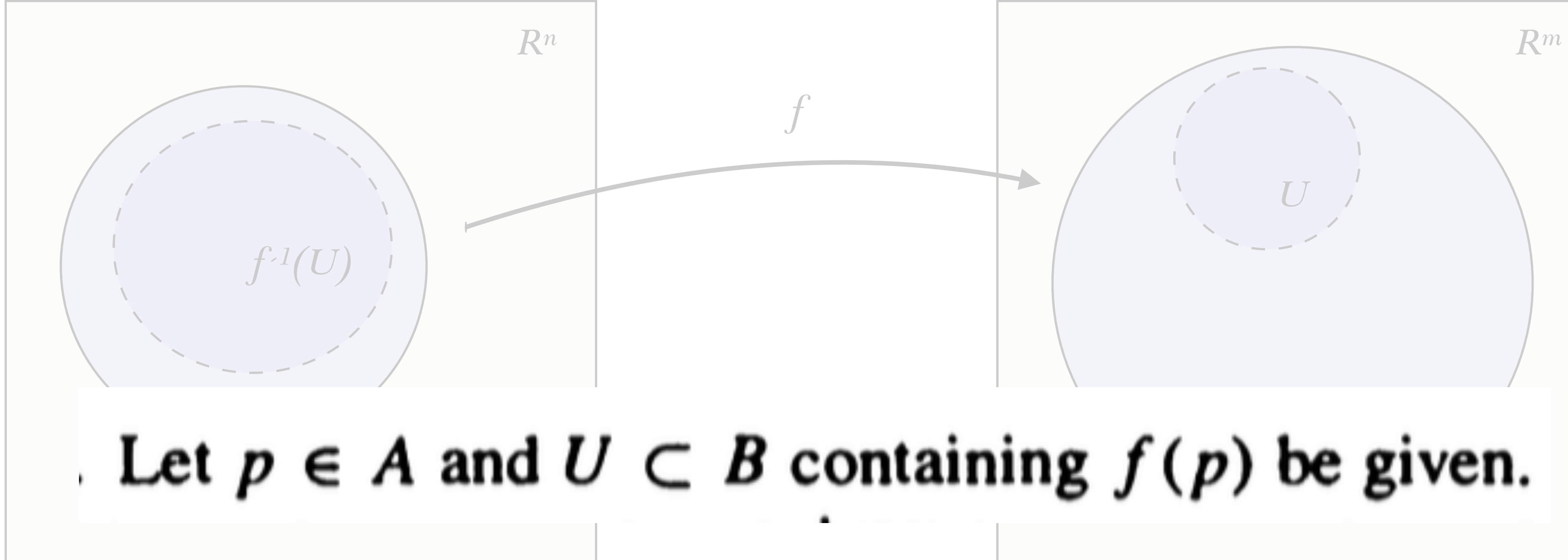
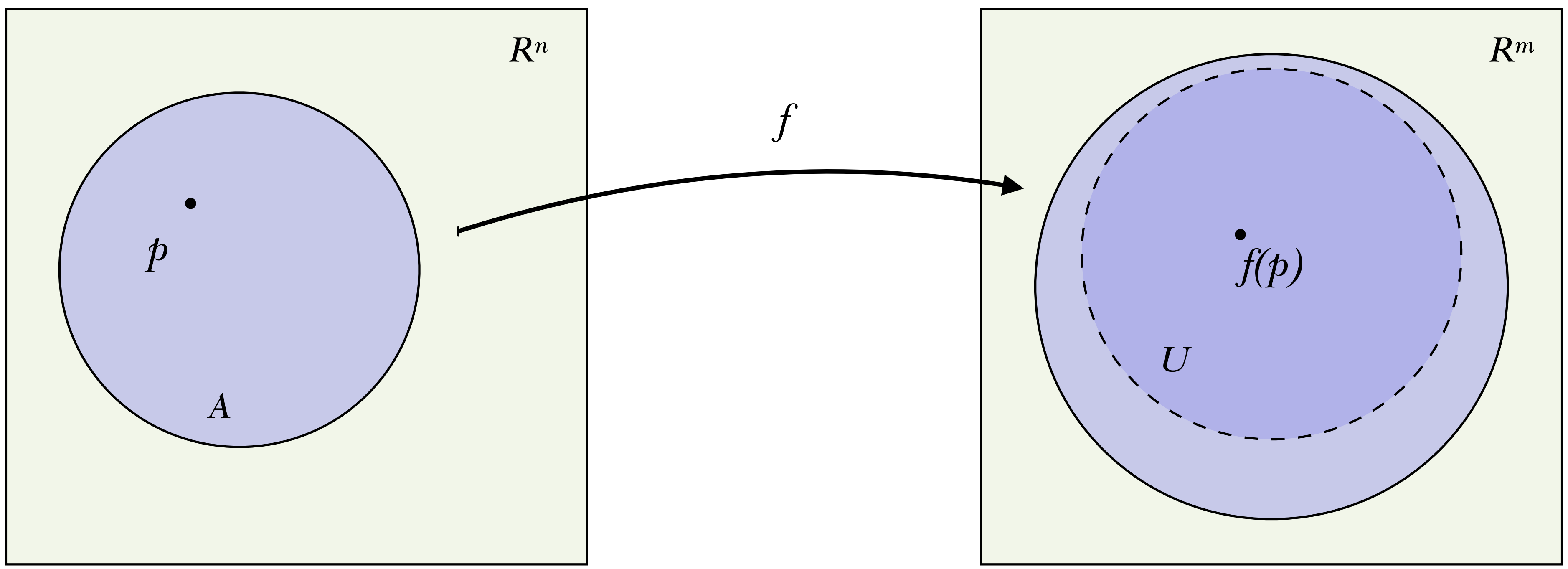


(1)



Let $p \in A$ and $U \subset B$ containing $f(p)$ be given.

(1) -> (2)

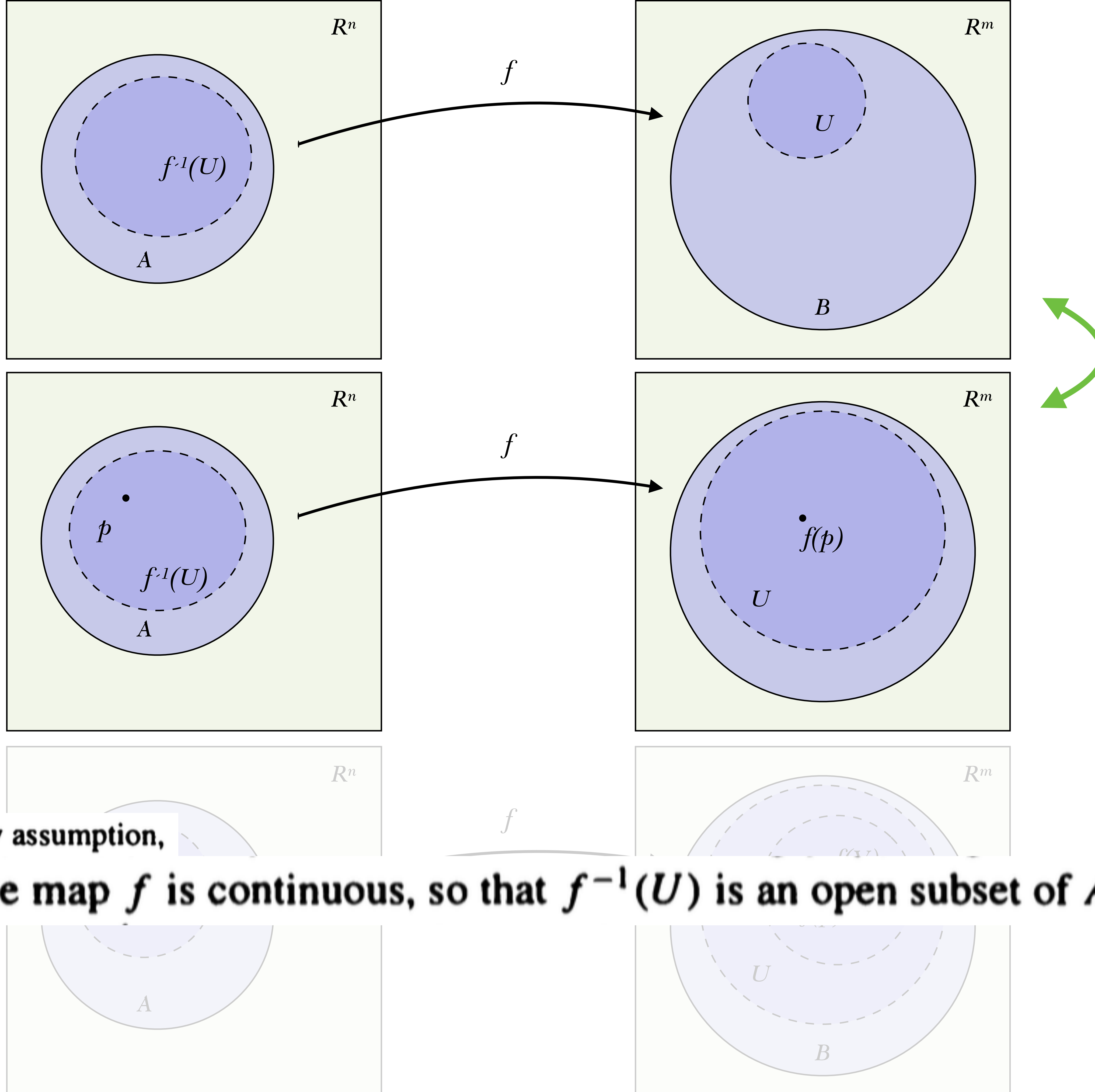


(2)

(1)

(1) -> (2)

(2)



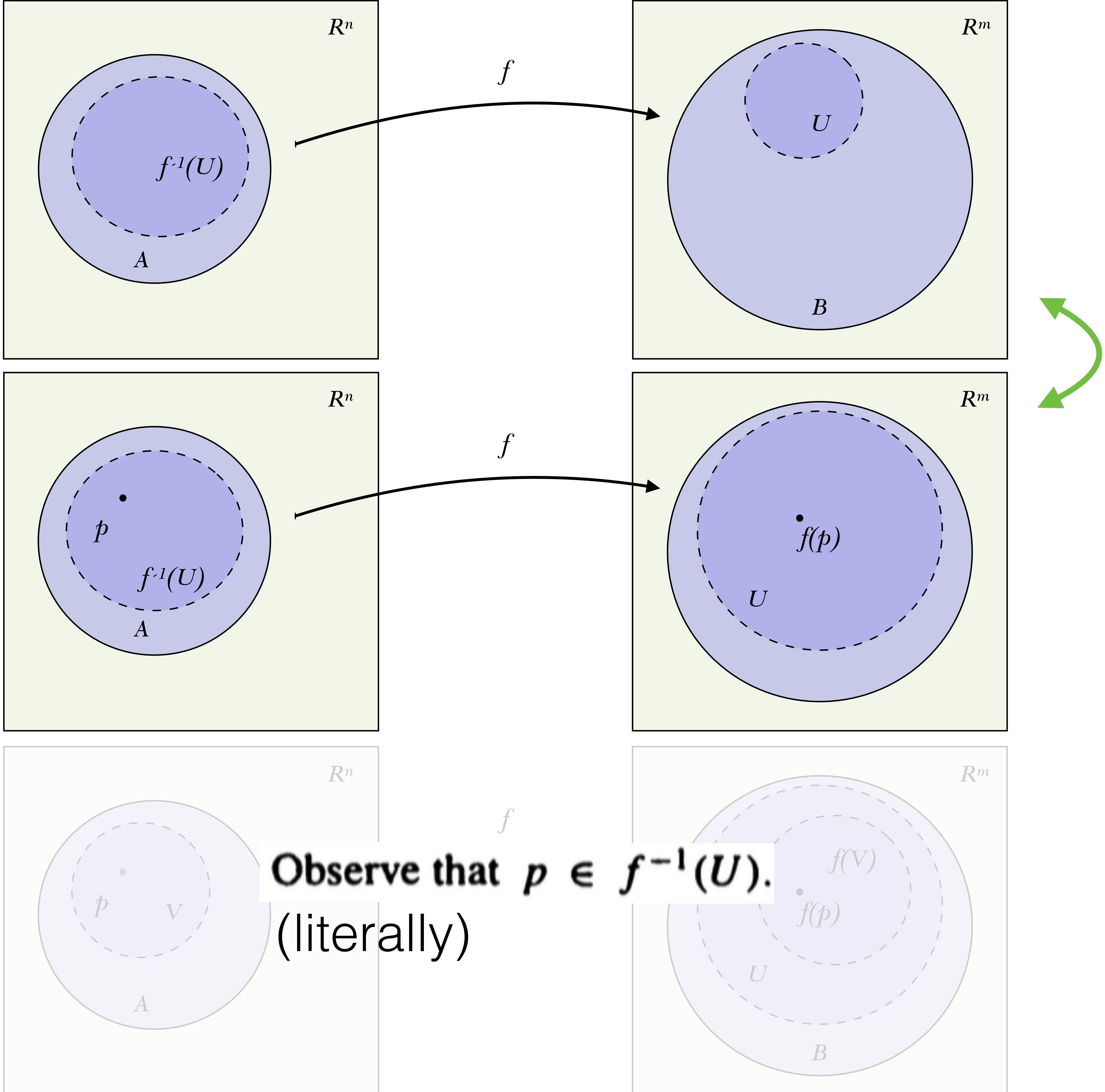
By assumption,

the map f is continuous, so that $f^{-1}(U)$ is an open subset of A.

(1)

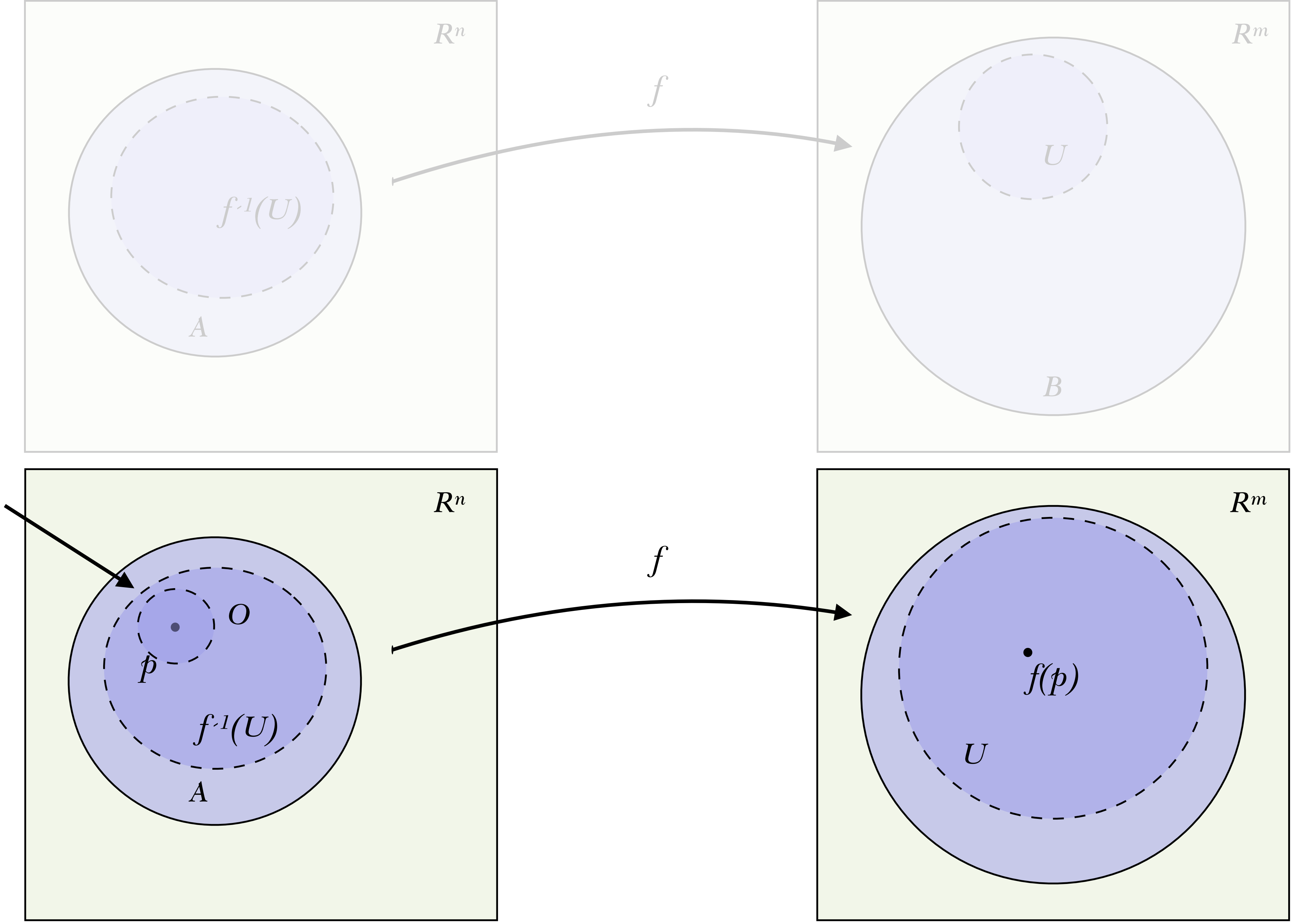
(1) -> (2)

(2)



Observe that $p \in f^{-1}(U)$.
(literally)

(1)

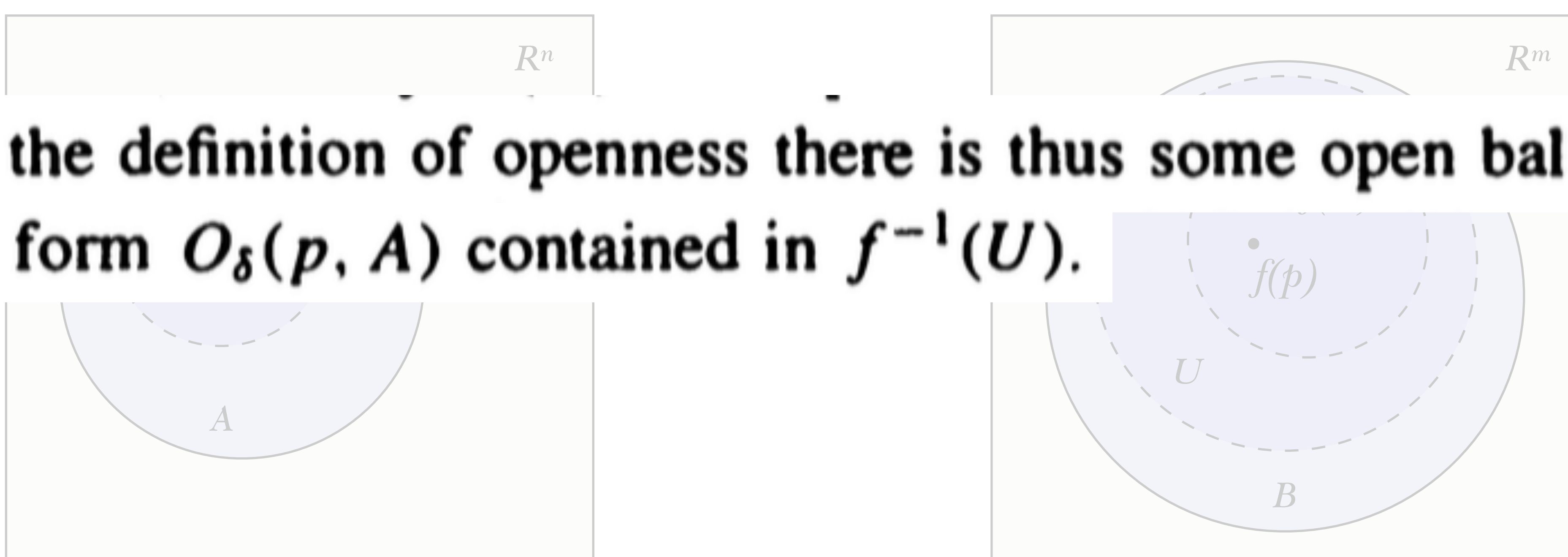


open ball
centered at p

(1) -> (2)

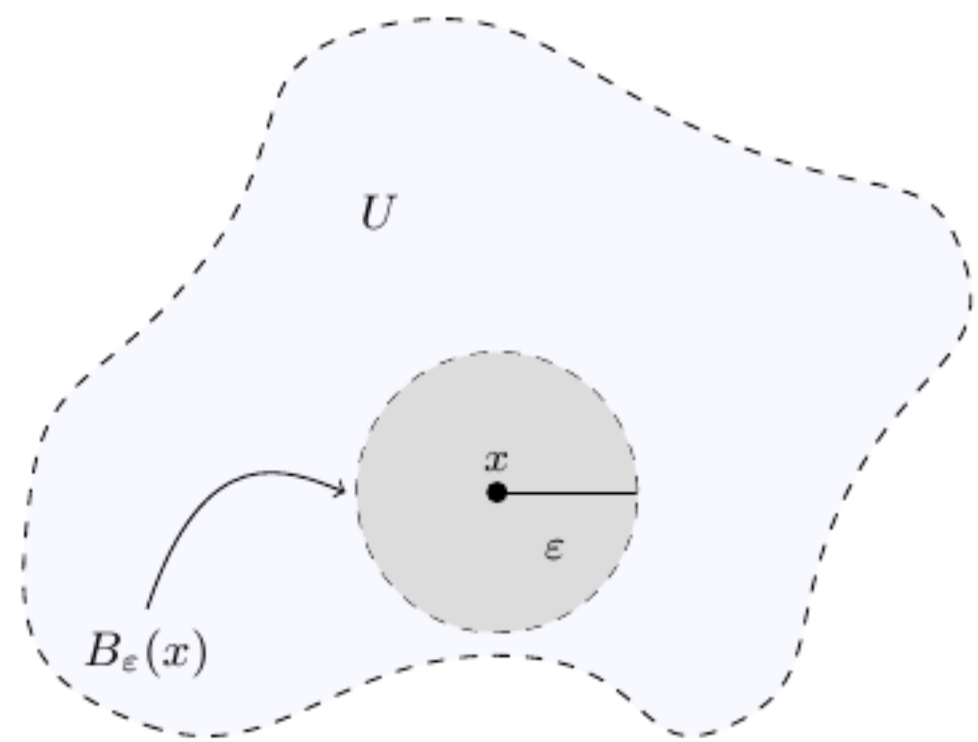
By the definition of openness there is thus some open ball of the form $O_\delta(p, A)$ contained in $f^{-1}(U)$.

(2)



recall: an open set in R^2

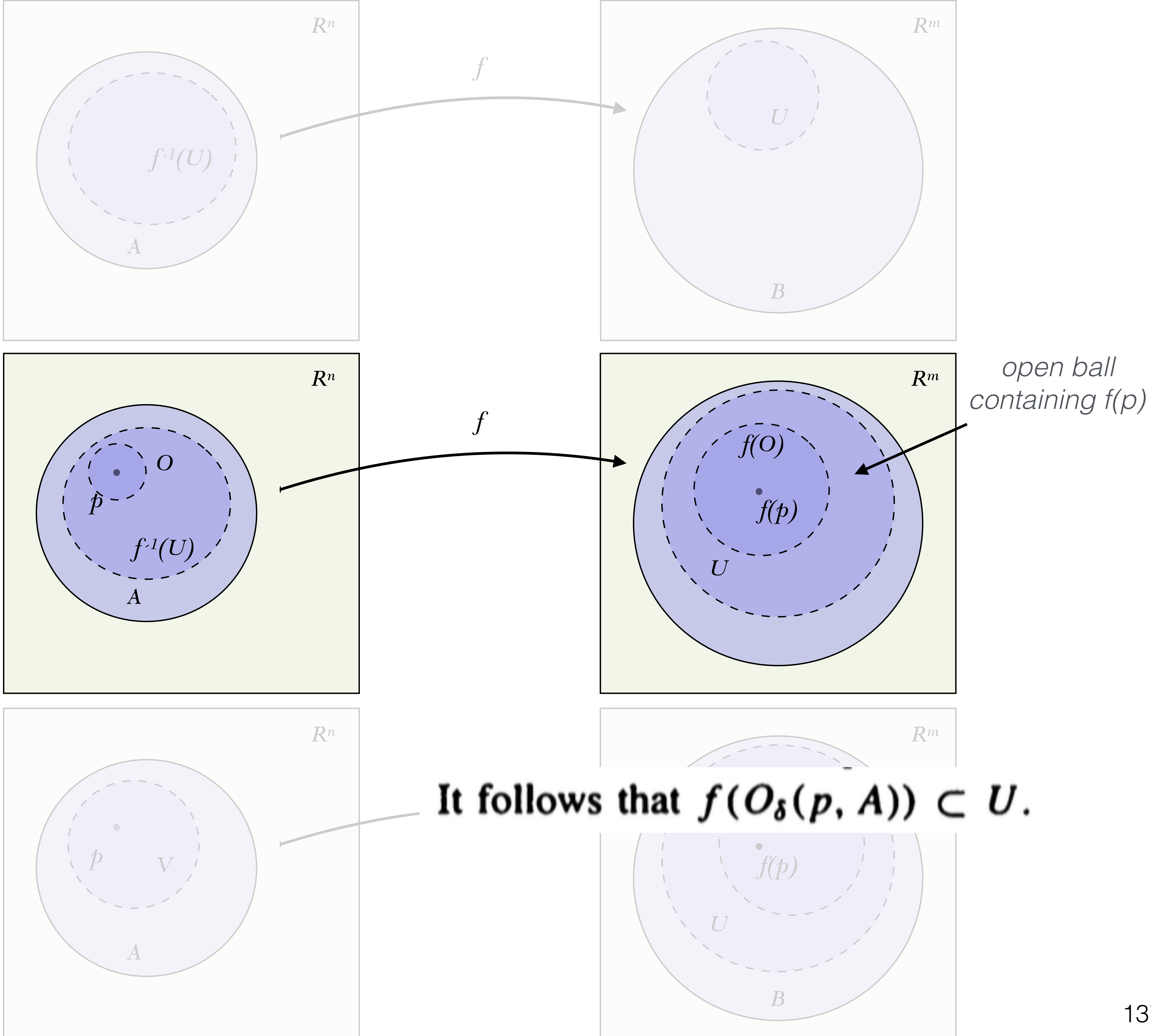
```
OpenSet U
```



(1)

(1) -> (2)

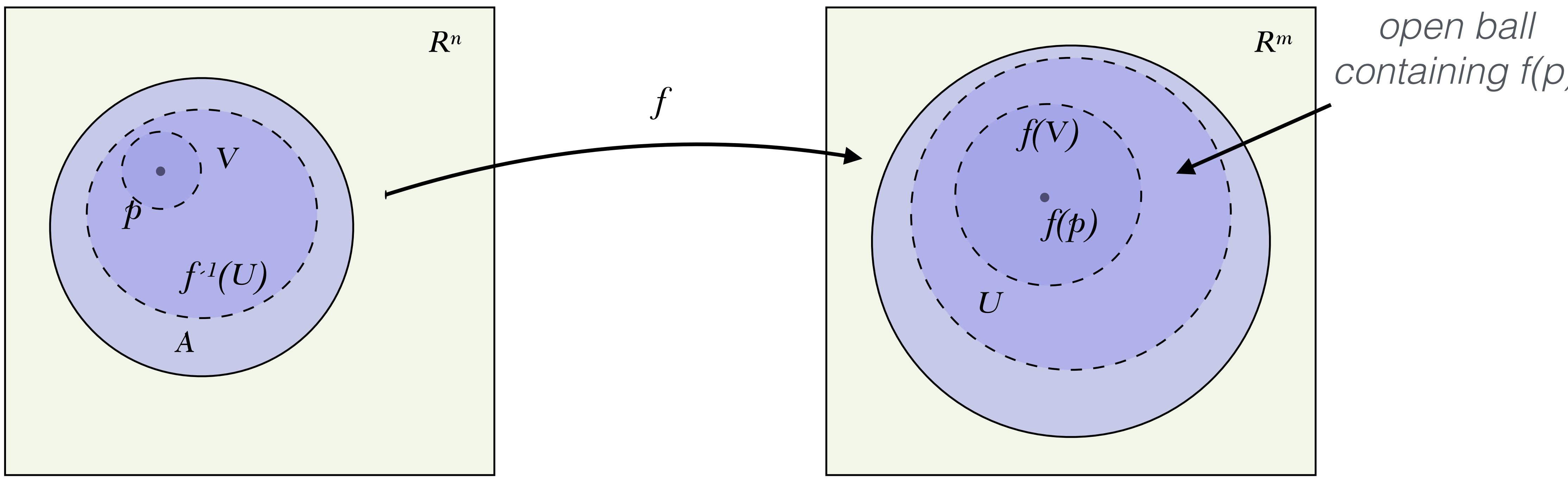
(2)



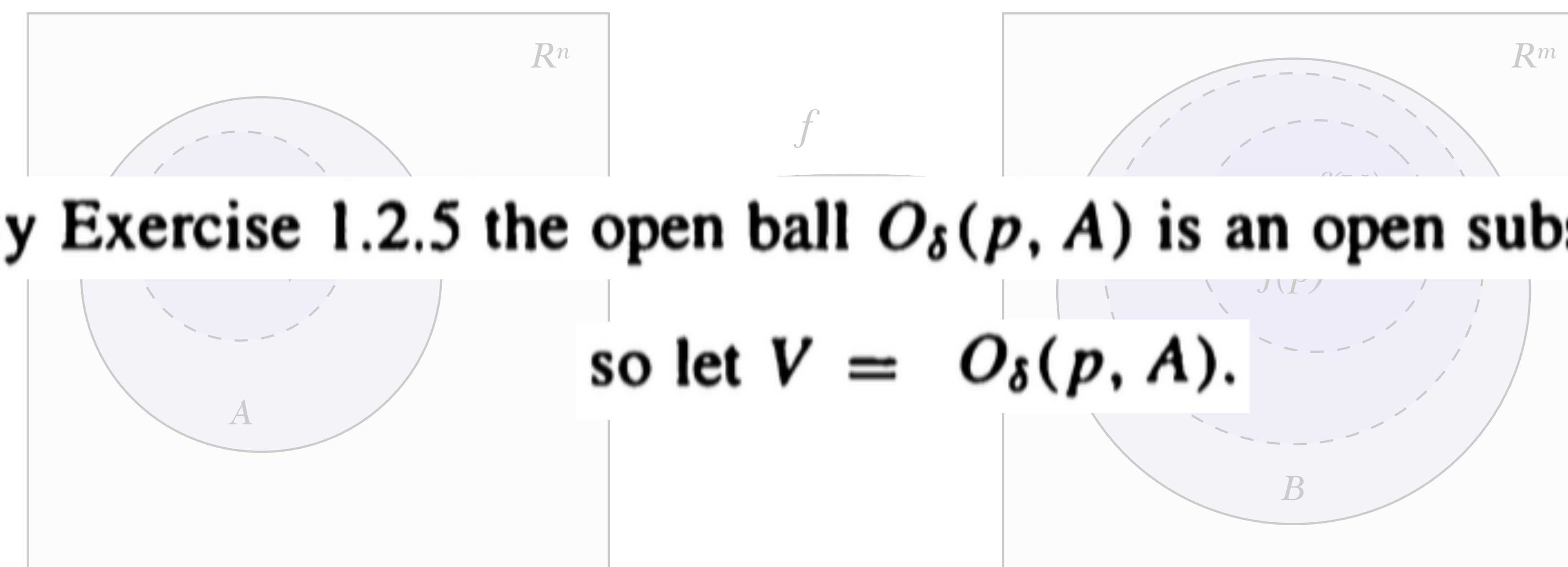
It follows that $f(O_\delta(p, A)) \subset U$.

(1)

(1) -> (2)



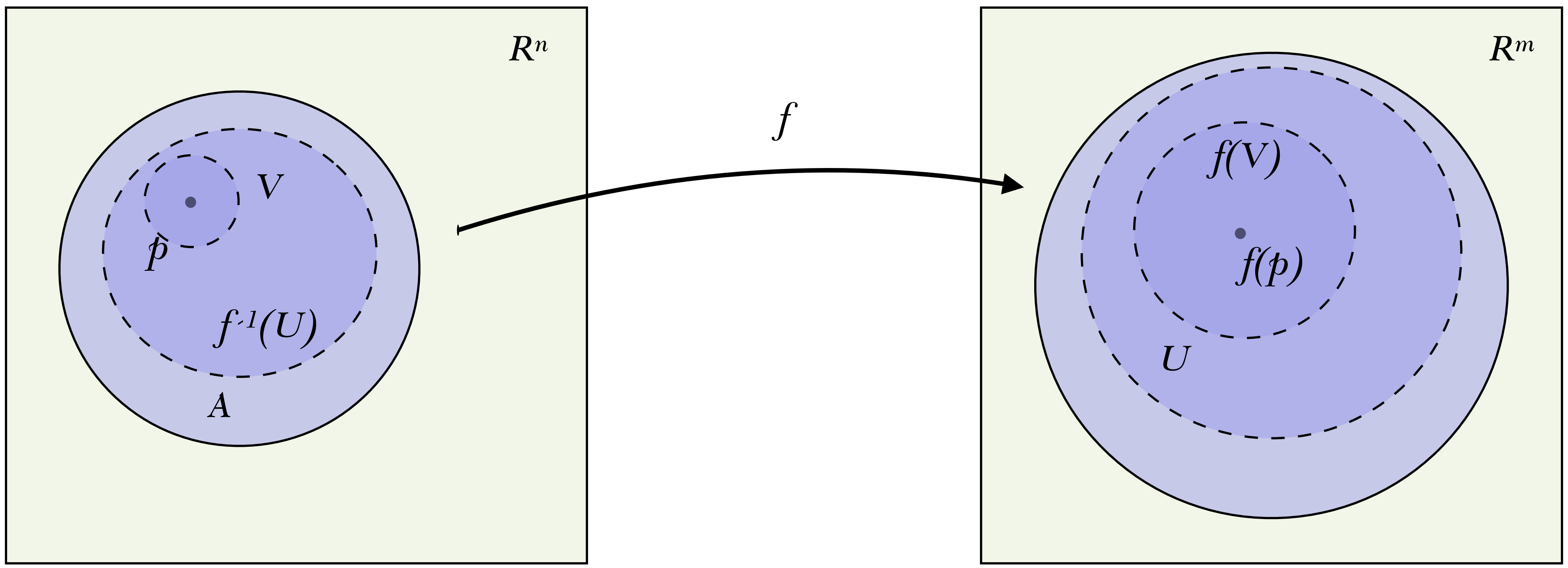
(2)



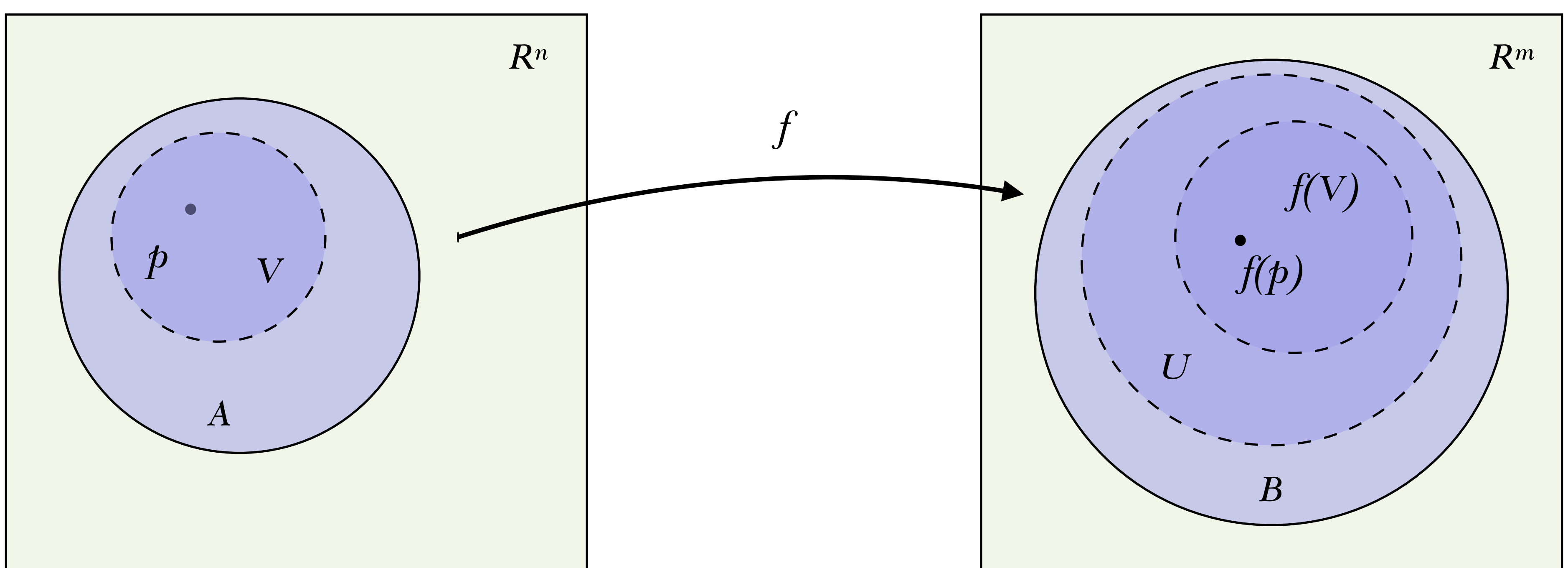
By Exercise 1.2.5 the open ball $O_\delta(p, A)$ is an open subset of A, so let $V = O_\delta(p, A)$.

(1)

(1) -> (2)



(2)



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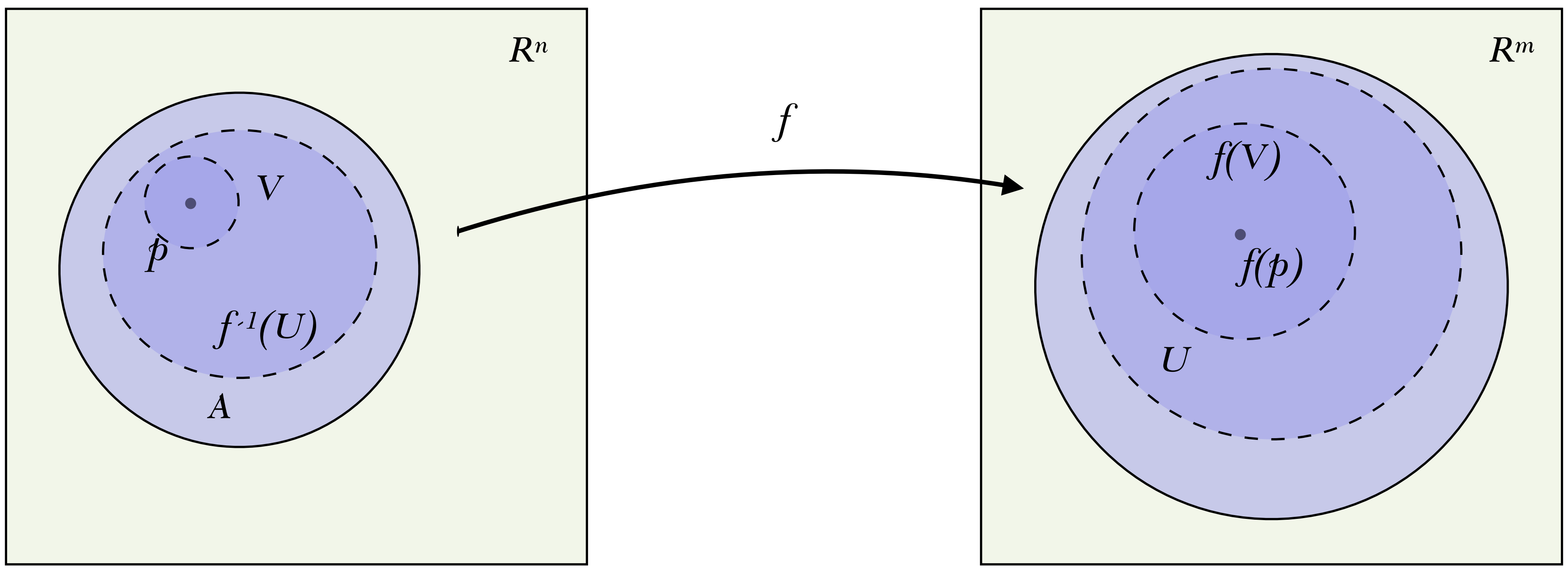
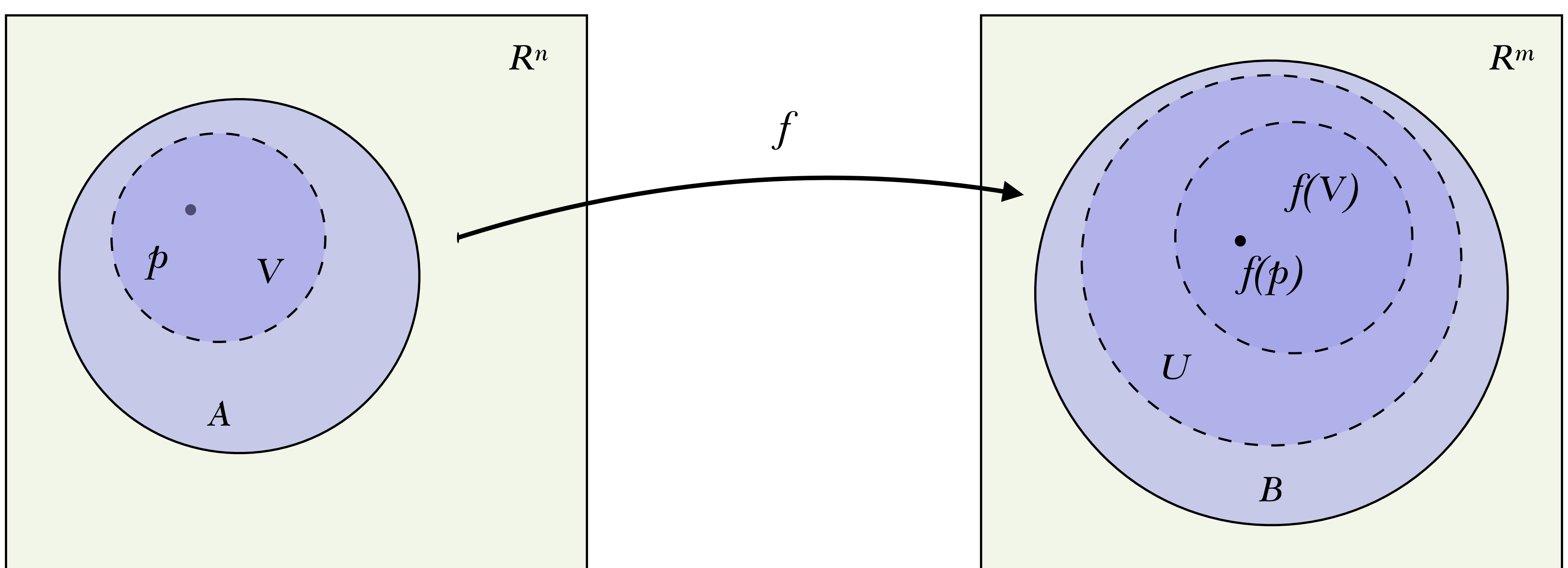
(1) -> (2)

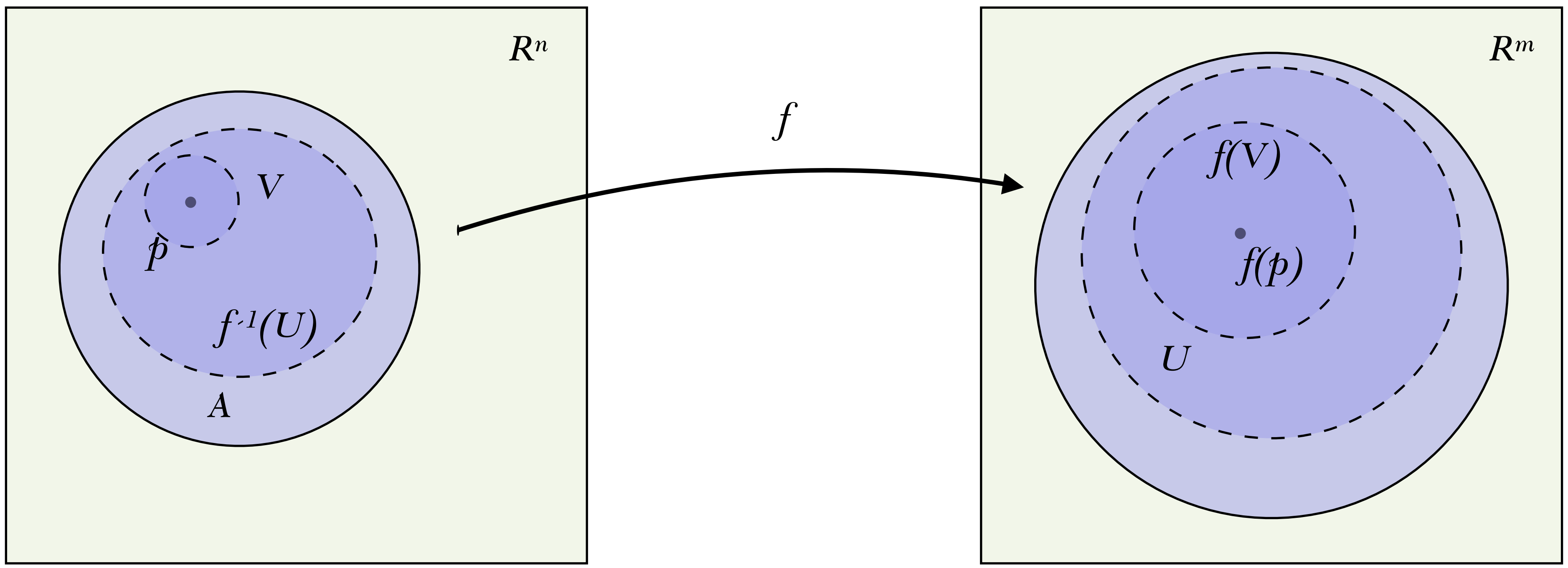


(2)



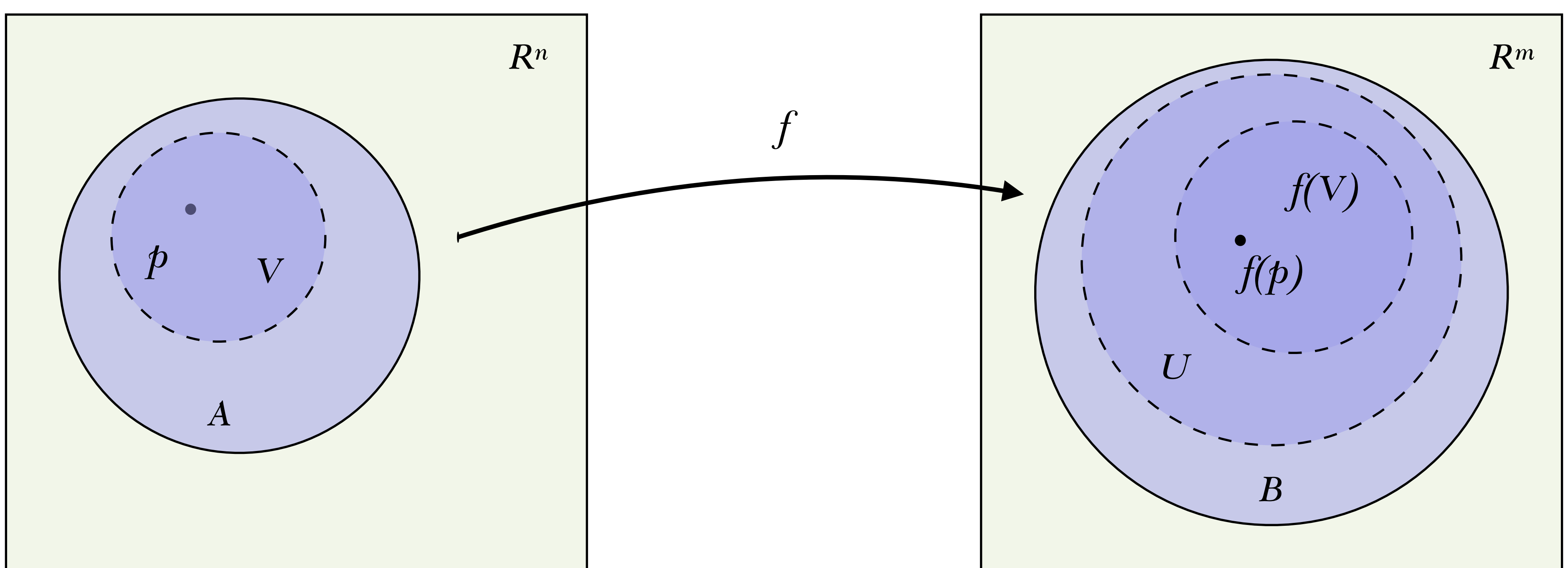
(1)

(1) -> (2)



not centered!

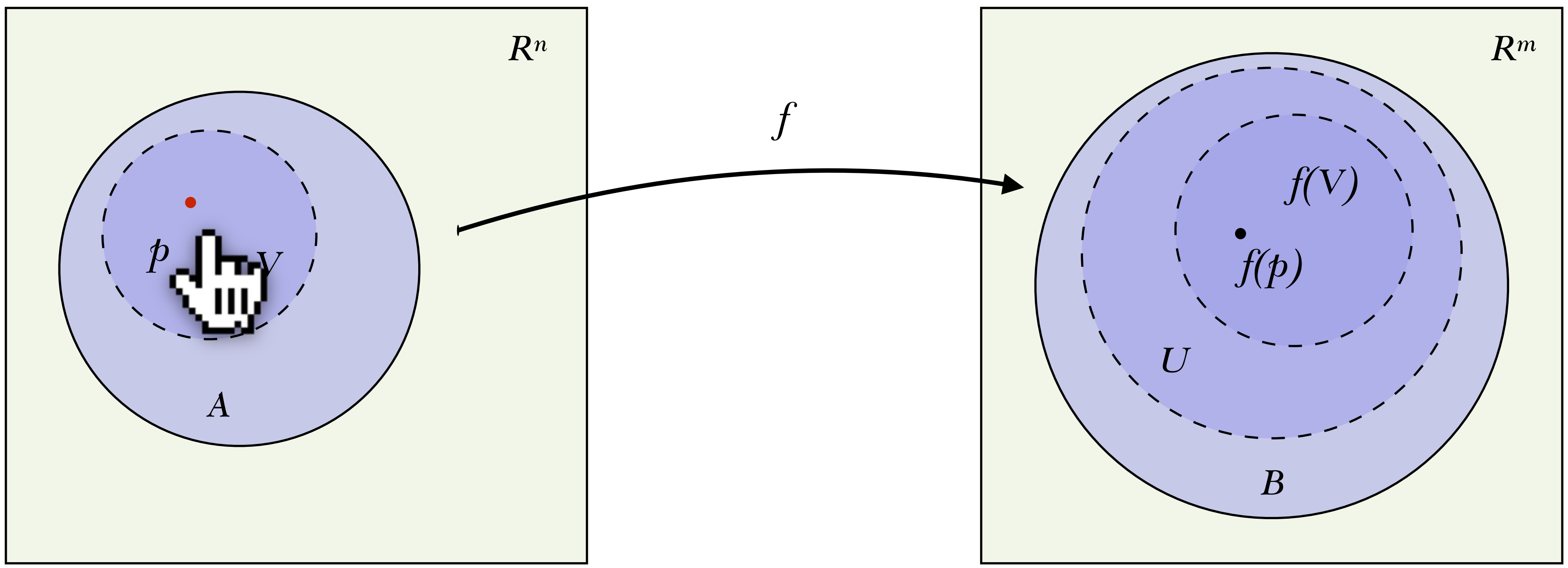
(2)



Prodirect manipulation: center the points

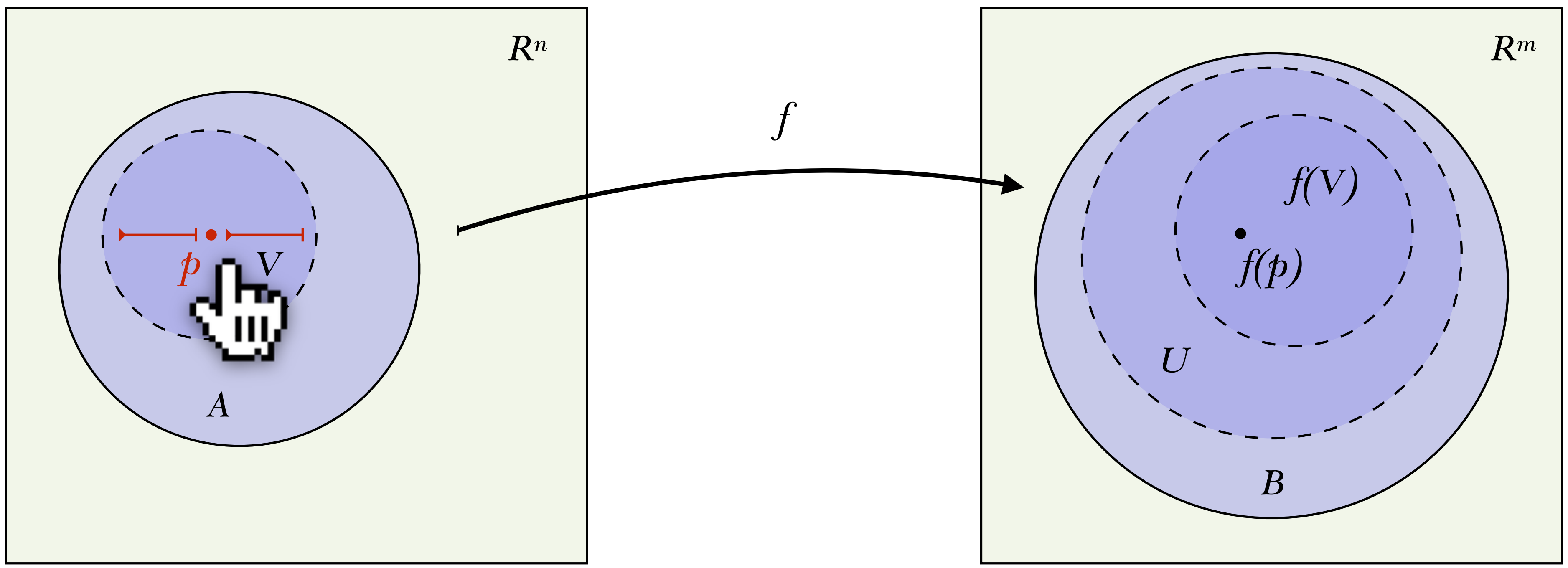
want to center this point

(2)



Prodirect manipulation: center the points

(2)



Substance

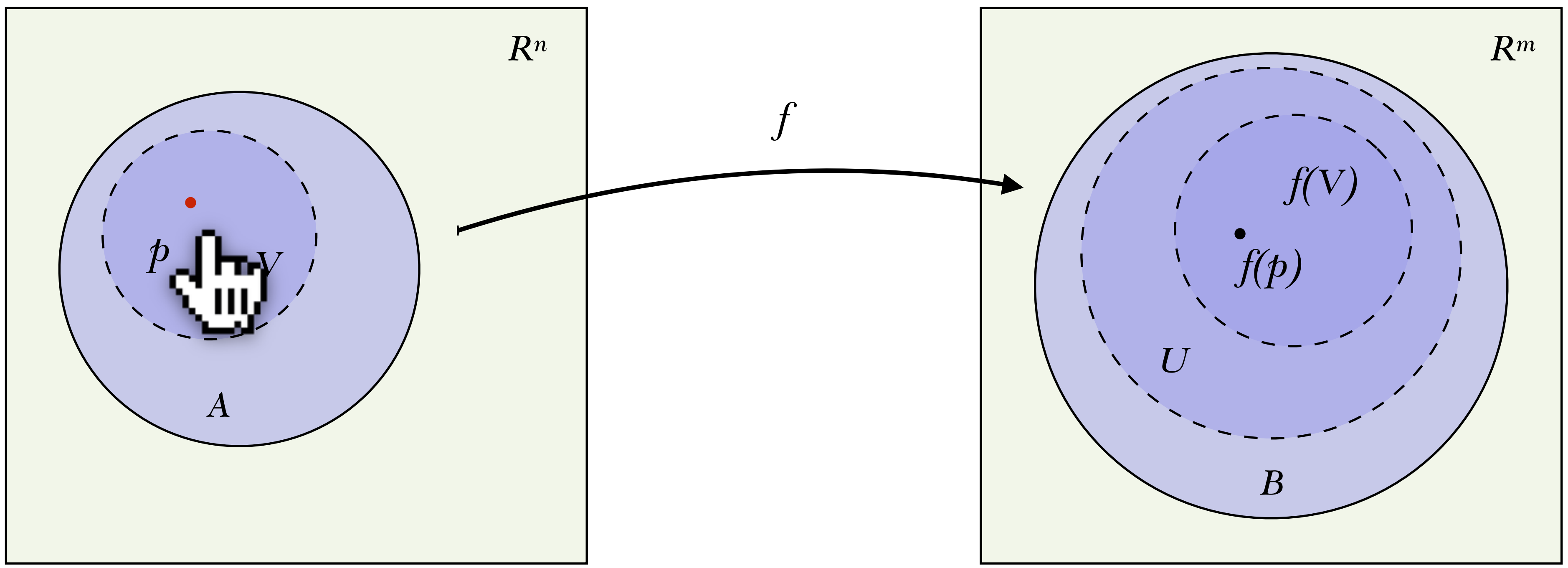
```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
```

```
Point p
In p A
Point f(p)
In f(p) U
OpenSet V
Subset V A
In p V
Set f(V)
Subset f(V) U
In f(p) f(V)
```

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

(2)



Substance

```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
```

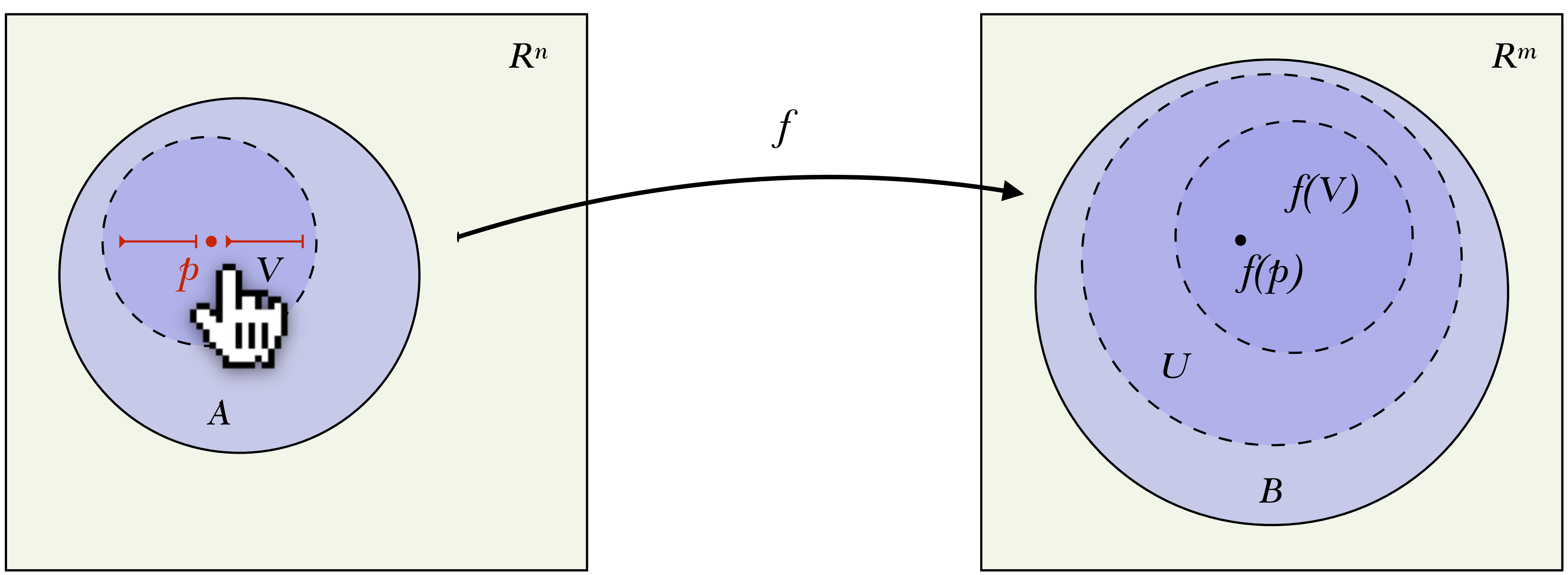
```
Point p
In p A
Point f(p)
In f(p) U
OpenSet V p
Subset V A
In p V
Set f(V)
Subset f(V) U
In f(p) f(V)
```

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

runtime infers code changes!

(2)



Substance

```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
```

```
Point p
In p A
Point f(p)
In f(p) U
OpenSet V p
Subset V A
In p V
Set f(V)
Subset f(V) U
In f(p) f(V)
```

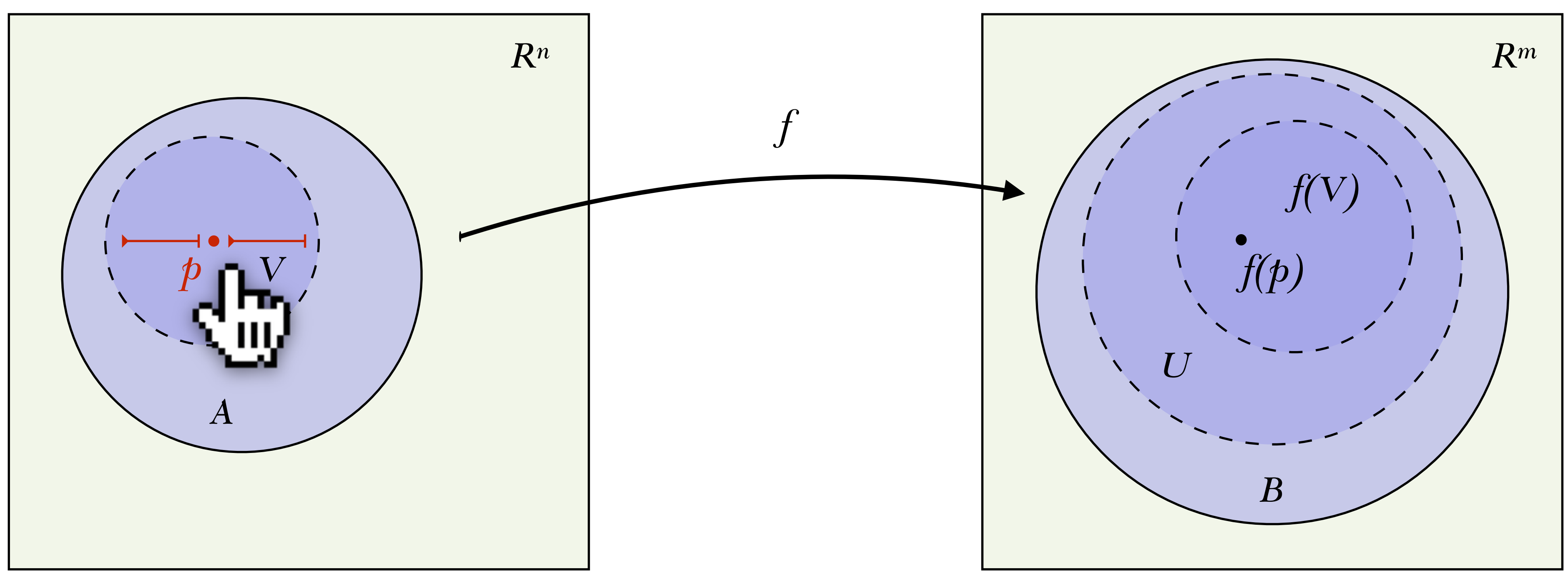
now centered at p

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

runtime infers code changes!

(2)



Penrose takeaways

- Easily implement and extend DSLs

Penrose takeaways

- Easily implement and extend DSLs
- Easily reuse styles

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- Easily implement and extend DSLs
- Easily reuse styles

Author's Guide to the
ACM SIGPLAN Class
(`sigplanconf.cls`)
v3.6

This guide describes Version 3.6 of the class file, which ends with a complete revision history that starts with the file's creation in September 2004.

Penrose takeaways

- Easily implement and extend DSLs
- Easily reuse styles
- Expertise encoded! Substantial benefits accrue with each new diagram

Penrose takeaways

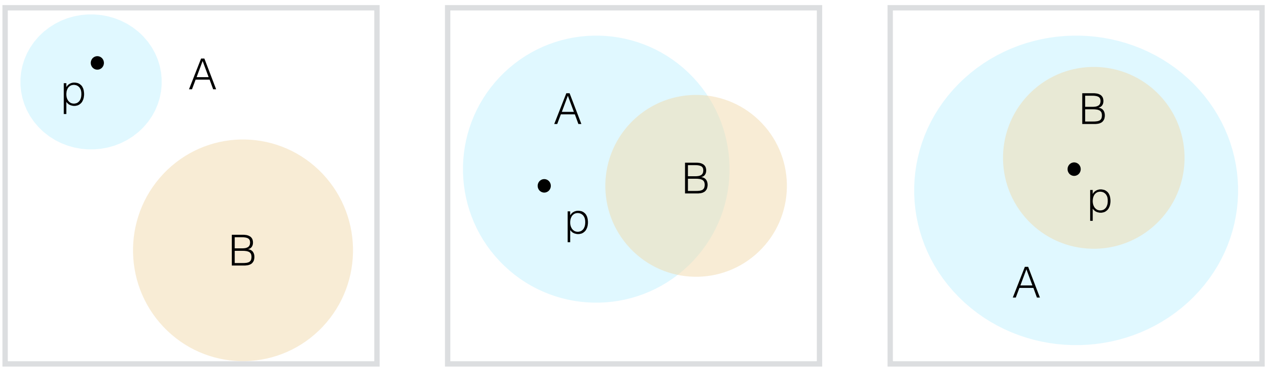
- Easily implement and extend DSLs
- Easily reuse styles
- Expertise encoded! Substantial benefits accrue with each new diagram
- DSL users can just throw in notation; get a useful illustration

Language design challenges

Penrose provides an extensible visual semantics for mathematical notation.

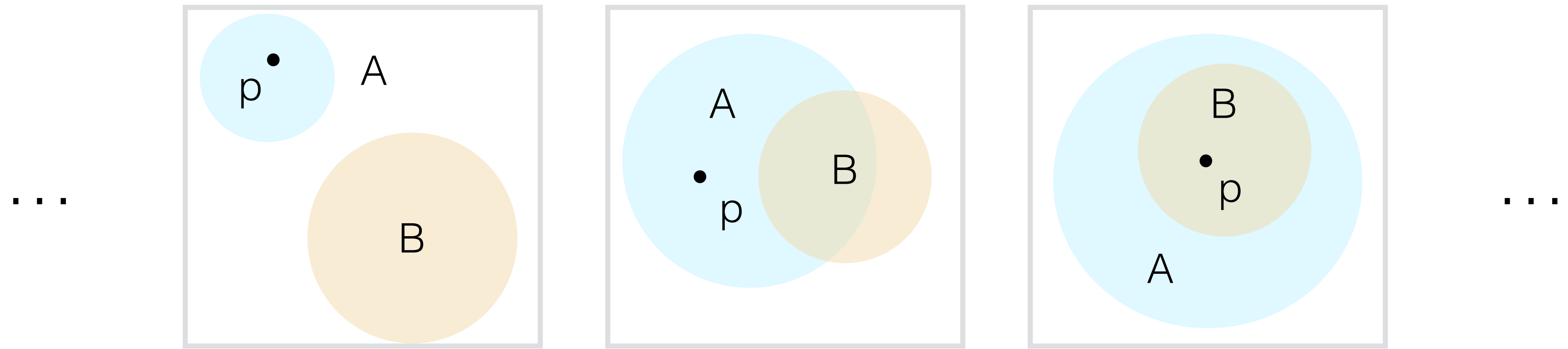
“Let A and B be sets, and p be a point in A.”

…



…

“Let A and B be sets, and p be a point in A.”



Where are the semantics?

“Let A and B be sets, and p be a point in A.”

Notation

$$Set\ A$$
$$Set\ B$$
$$Point\ p$$
$$p \in A$$

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

“Let A and B be sets, and p be a point in A.”

Substance

```
Set A
Set B
Point p
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View

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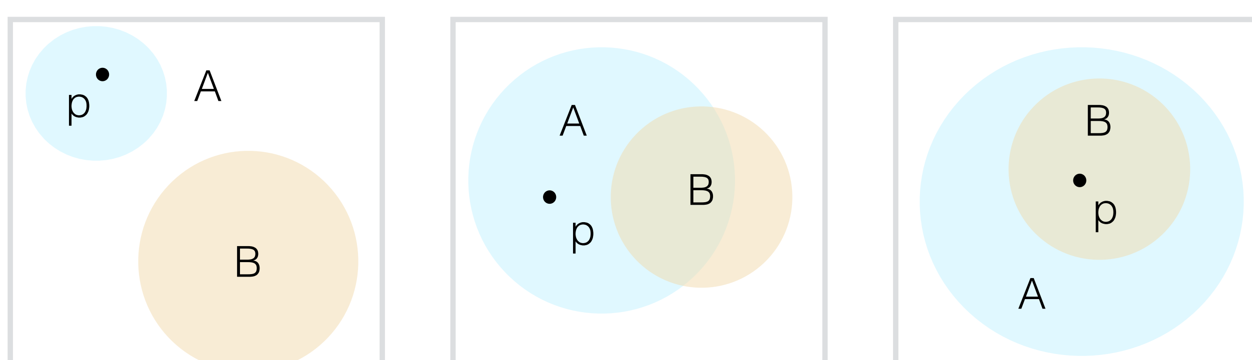


. . .

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

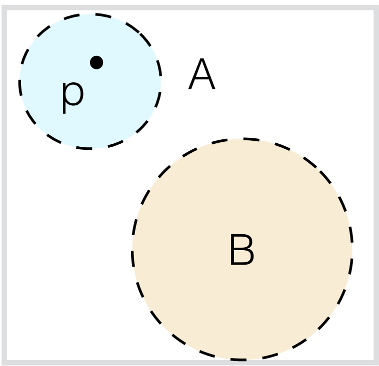
```
Dimension 2
Coordinates None
```

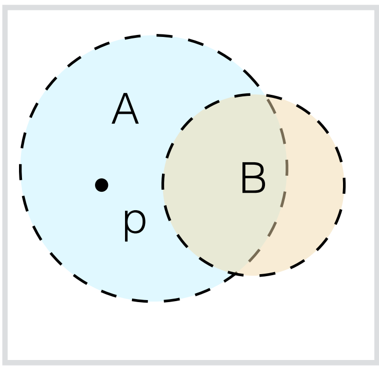
Style

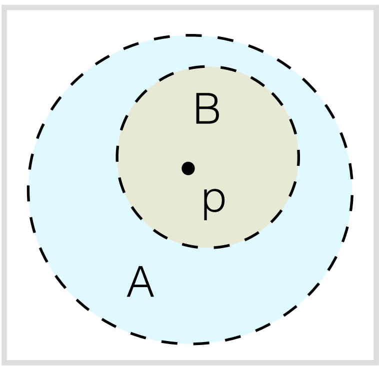
```
Shape Set DottedCircle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

incorrect semantics?

…







…

Another way to give semantics to mathematical notation…

Another way to give semantics to mathematical notation…

The derivative of the coordinate path Dq is the function that maps time to velocity components:

$$Dq(t) = (Dx(t), Dy(t), Dz(t)).$$

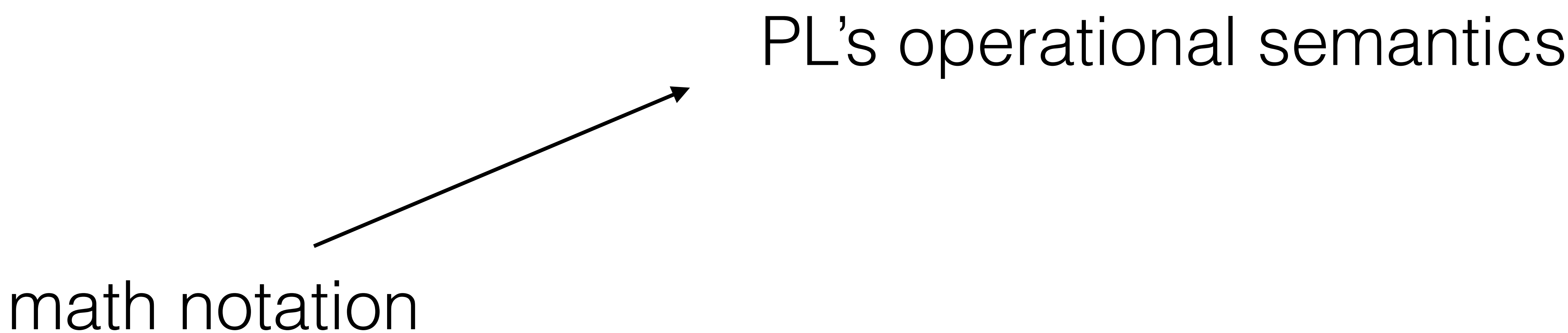
We can make and use the derivative of a function.[31] For example, we can write:

```
(print-expression ((D q) 't))
(up ((D x) t) ((D y) t) ((D z) t))
```

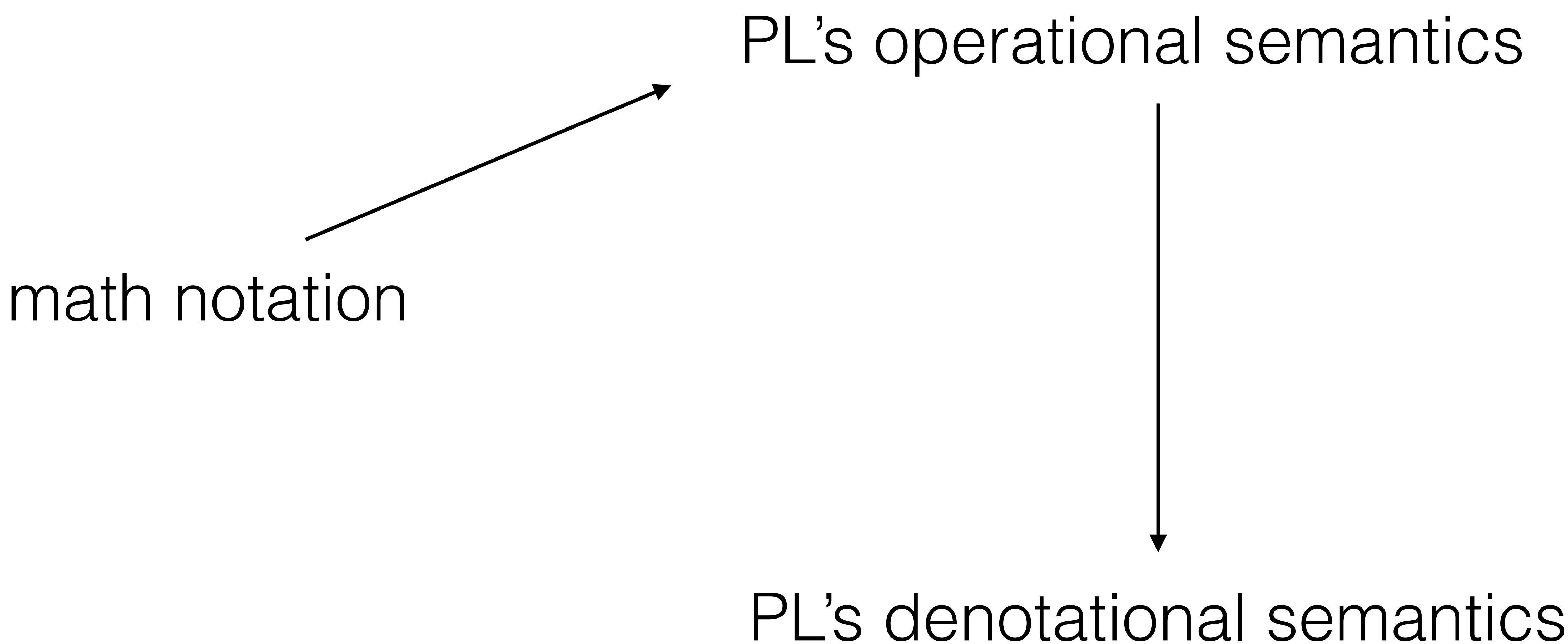
Another way to give semantics to mathematical notation…

Once formalized as a procedure, a mathematical idea becomes a tool that can be used directly to compute results.

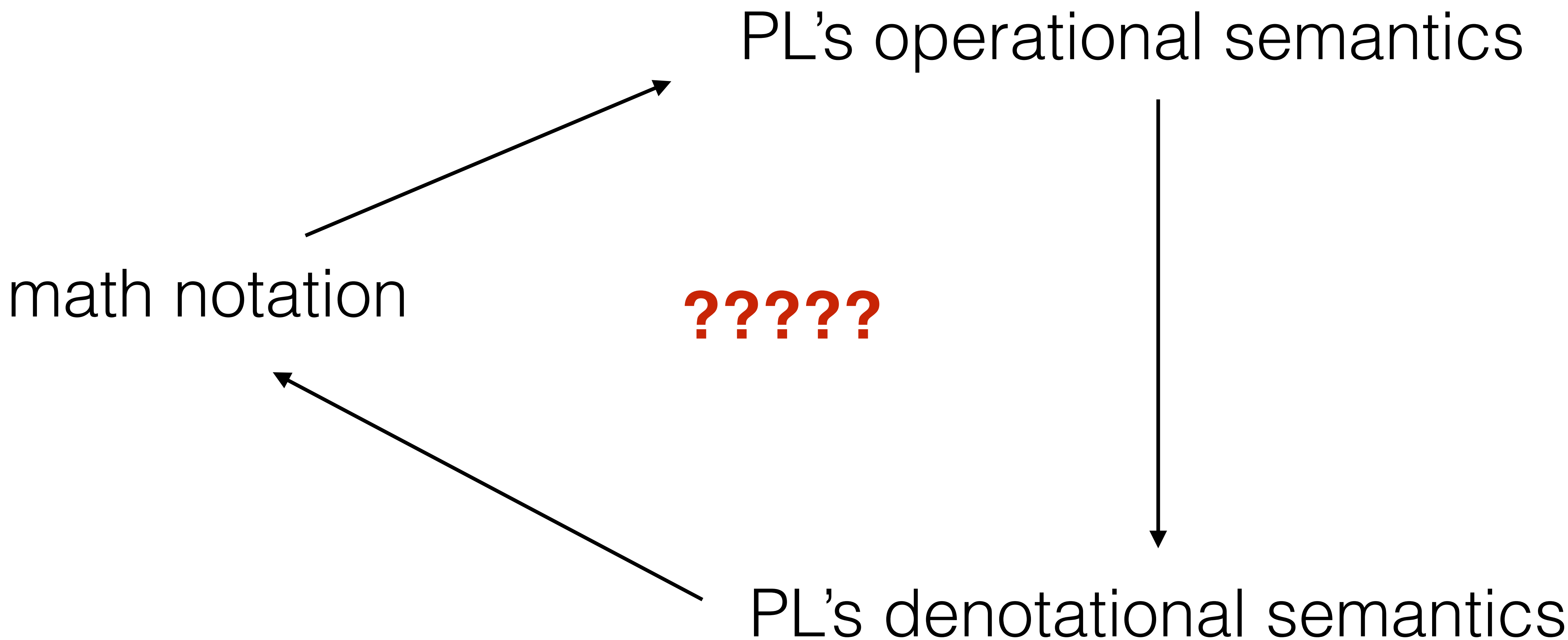
Another way to give semantics to mathematical notation…



Another way to give semantics to mathematical notation…



Another way to give semantics to mathematical notation…



Modularity, compositionality

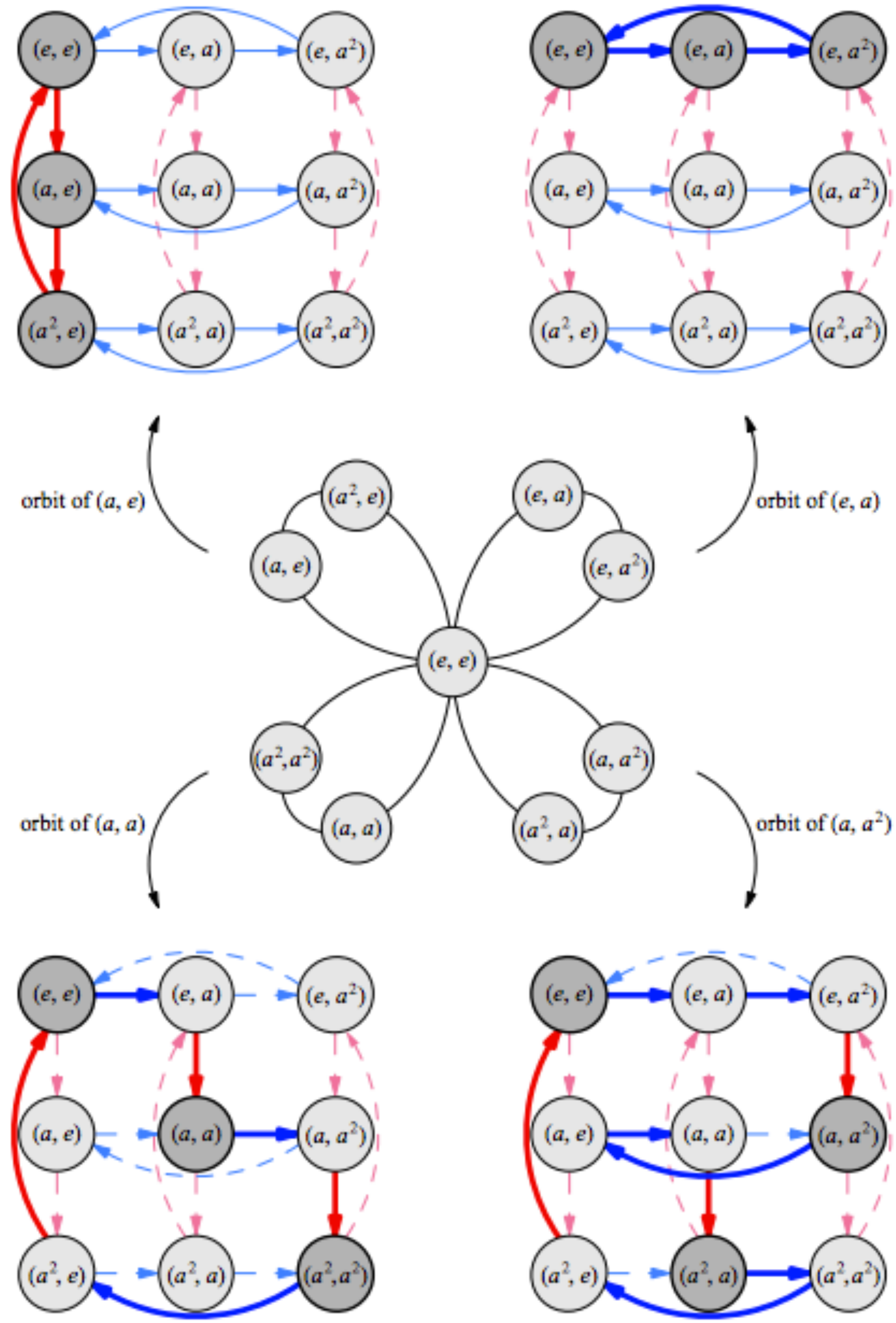
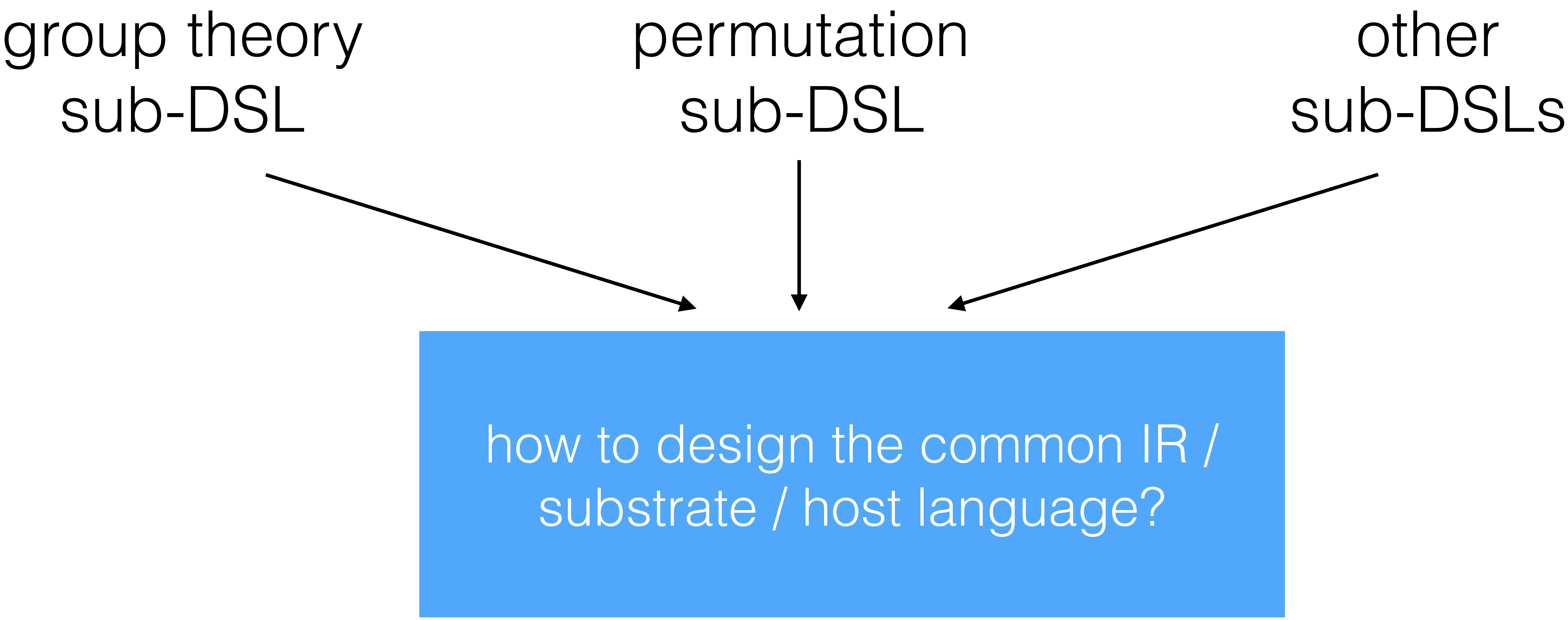


Figure 5.14. In the center is the cycle graph of $C_3 \times C_3$, and next to each of the four orbits is a copy of the Cayley diagram for $C_3 \times C_3$ with the corresponding orbit highlighted. These Cayley diagrams are structured like the original in Figure 2.10 on page 22.

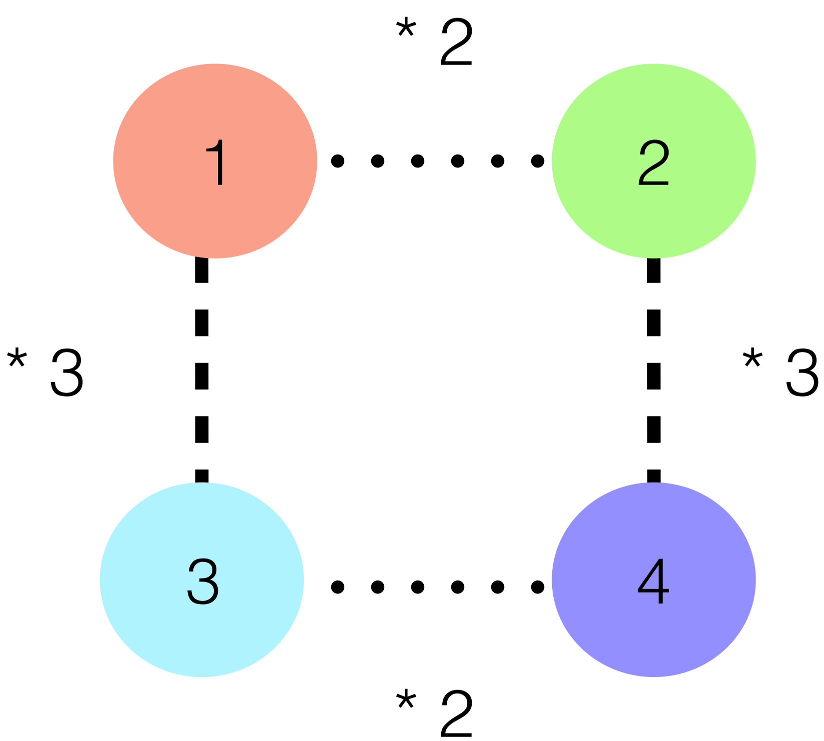
Source:
Visual Group Theory

Dynamic Symbols in Illustrator

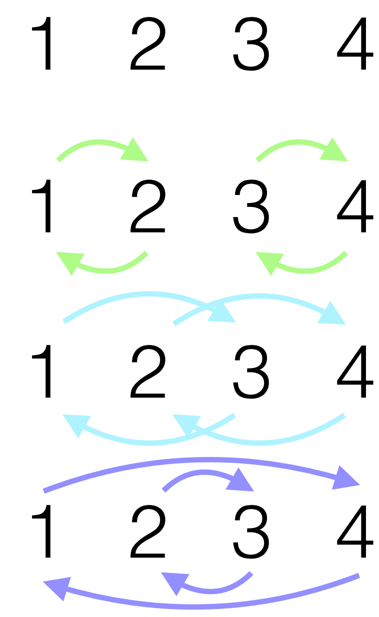
Interoperability


group theory sub-DSL
permutation sub-DSL
other sub-DSLs
how to design the common IR / substrate / host language?

Cayley's theorem



Cayley diagram for K_4



Permutation group

Extensibility

…for each component of Penrose

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

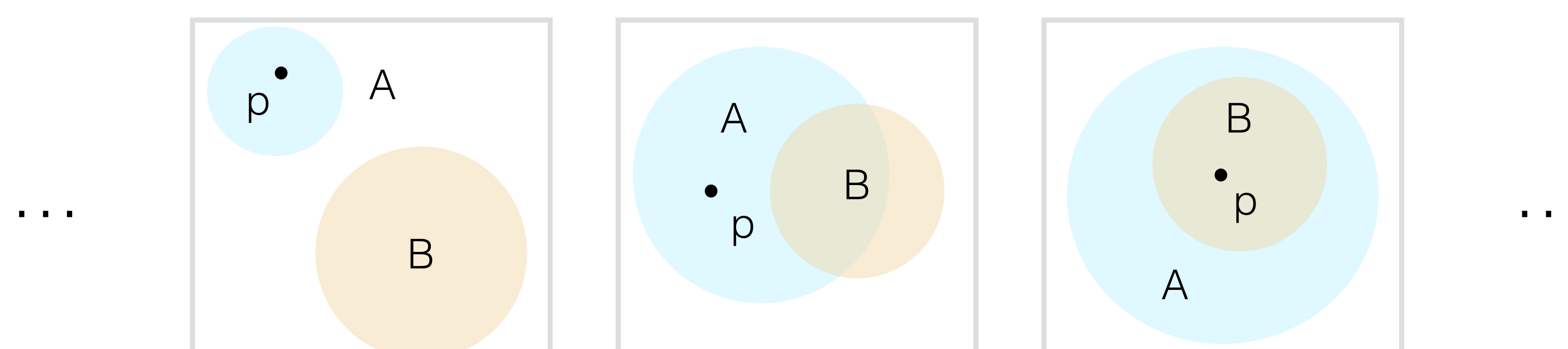
↘ ↓ ↙

Intermediate representations

↓

Sampling and optimization

↓



Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

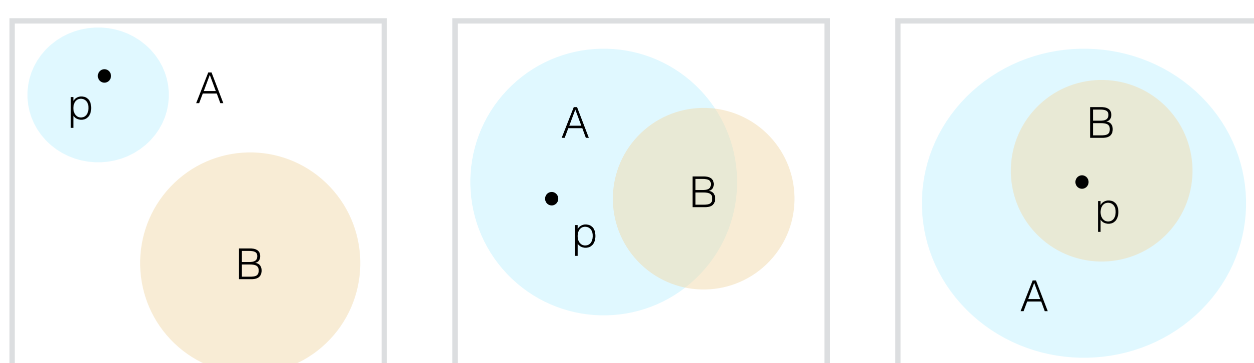
Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

Intermediate representations

Sampling and optimization

. . .



. . .

Penrose architecture

"Let A and B be sets, and p be a point in A."

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```
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Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

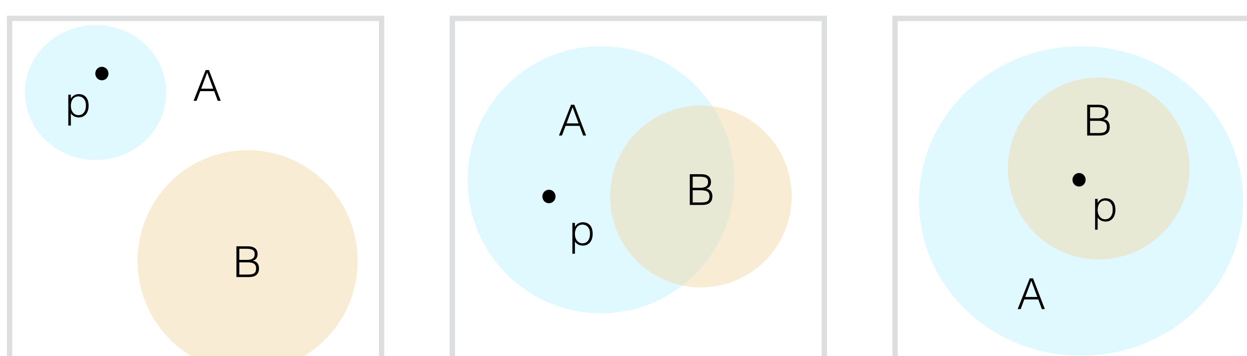
Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

Intermediate representations

Sampling and optimization

. . .



. . .

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

```
Dimension 2
Coordinates None
```

Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

↘ ↓ ↙

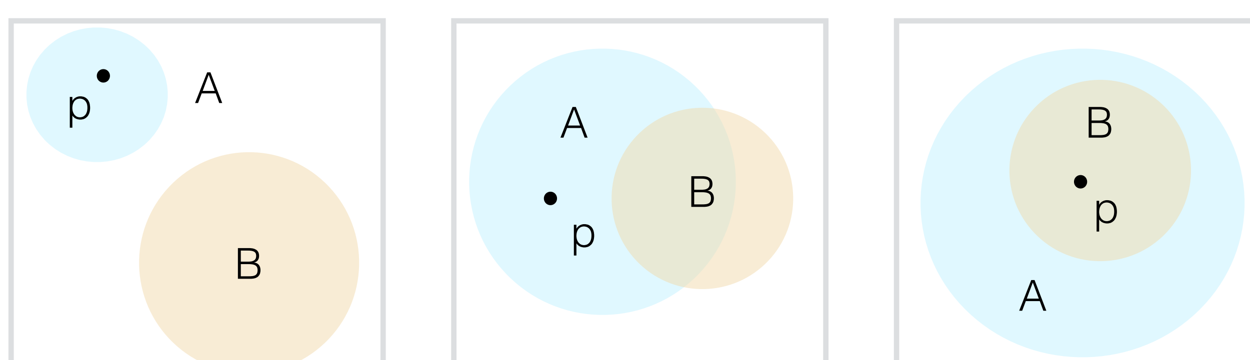
Intermediate representations

↓

Sampling and optimization

↓

. . .



. . .

Layout and runtime challenges

Penrose architecture

"Let A and B be sets, and p be a point in A."

Substance

```
Set A
Set B
Point p
In p A
```

View

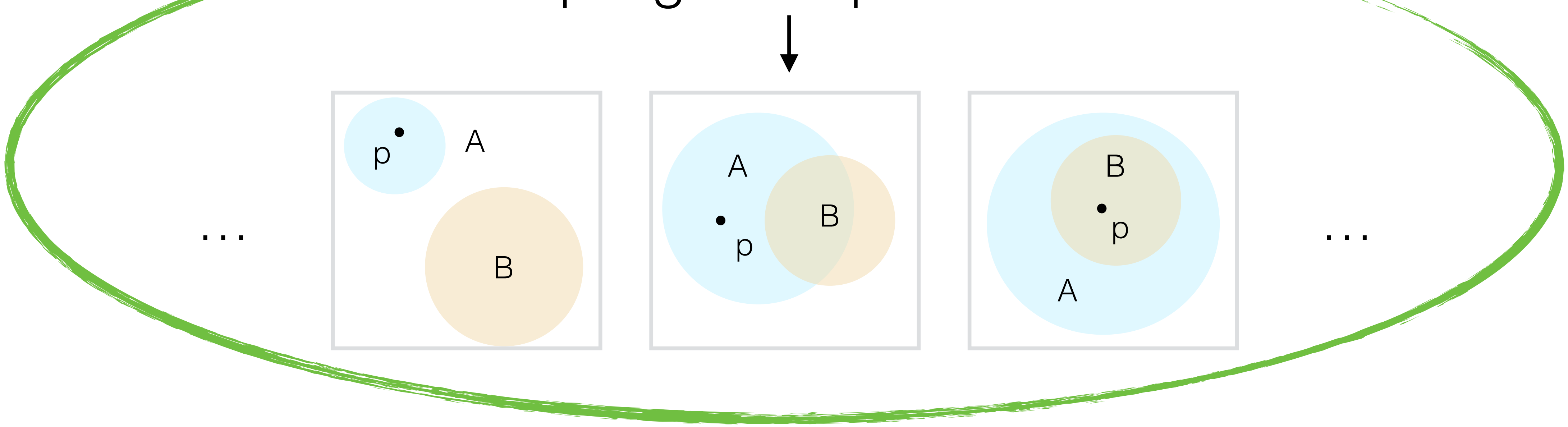
```
Dimension 2
Coordinates None
```

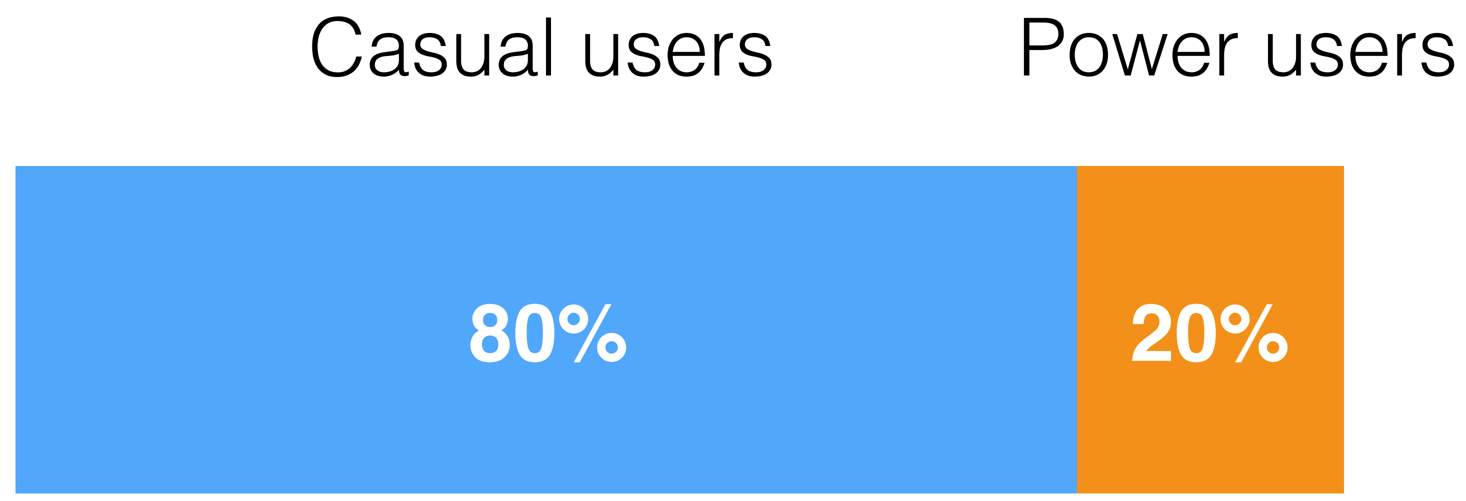
Style

```
Shape Set Circle
Color A Blue
Color B Orange
Shape Point Circle
Label All Auto
```

Intermediate representations

Sampling and optimization



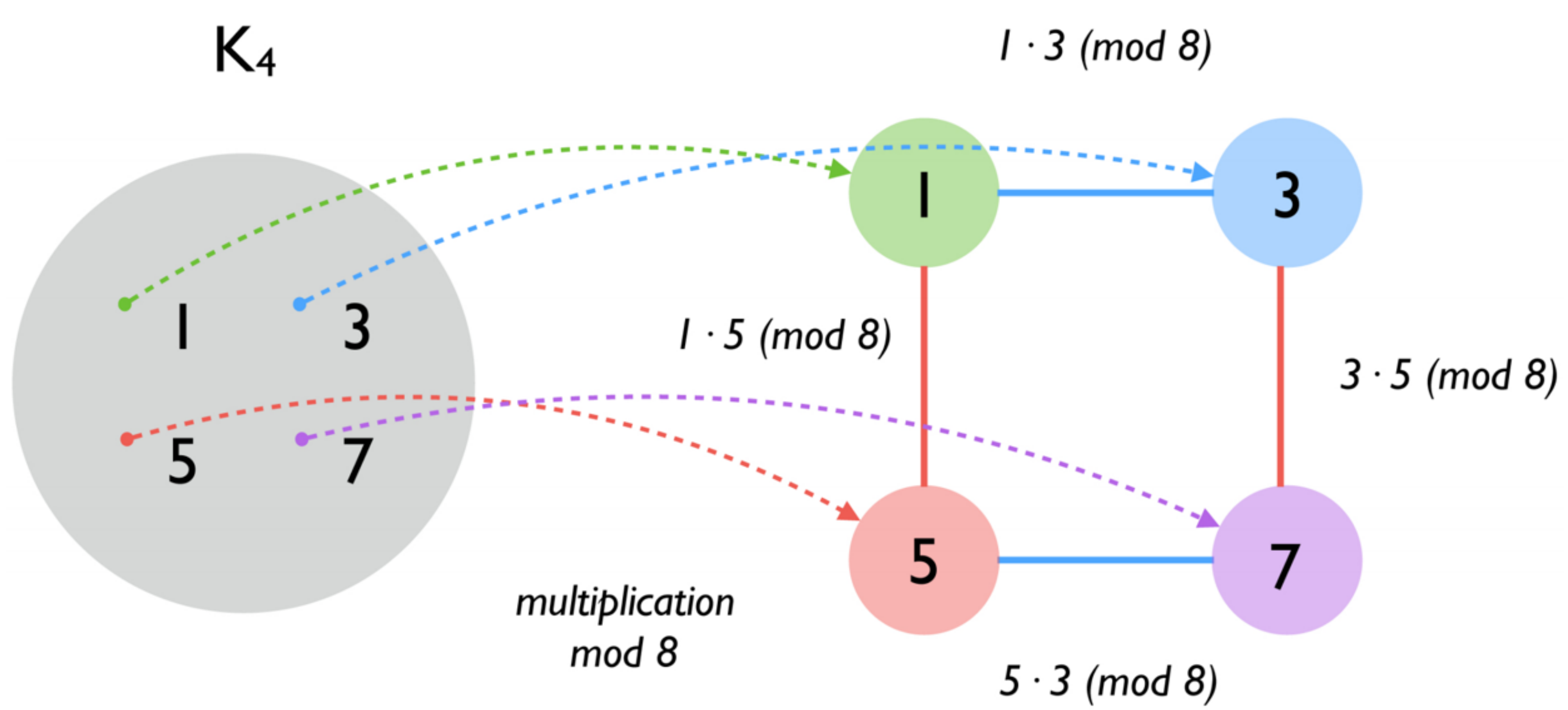

Casual users
Power users
80%
20%

Constraints

```
p in A
A and B do not intersect
curve 2 is not homotopic to 0
B is contained in the preimage of f
```

Style

```
linked.layout = left_to_right
```



Style

```
Label All Auto
Position All Auto
```

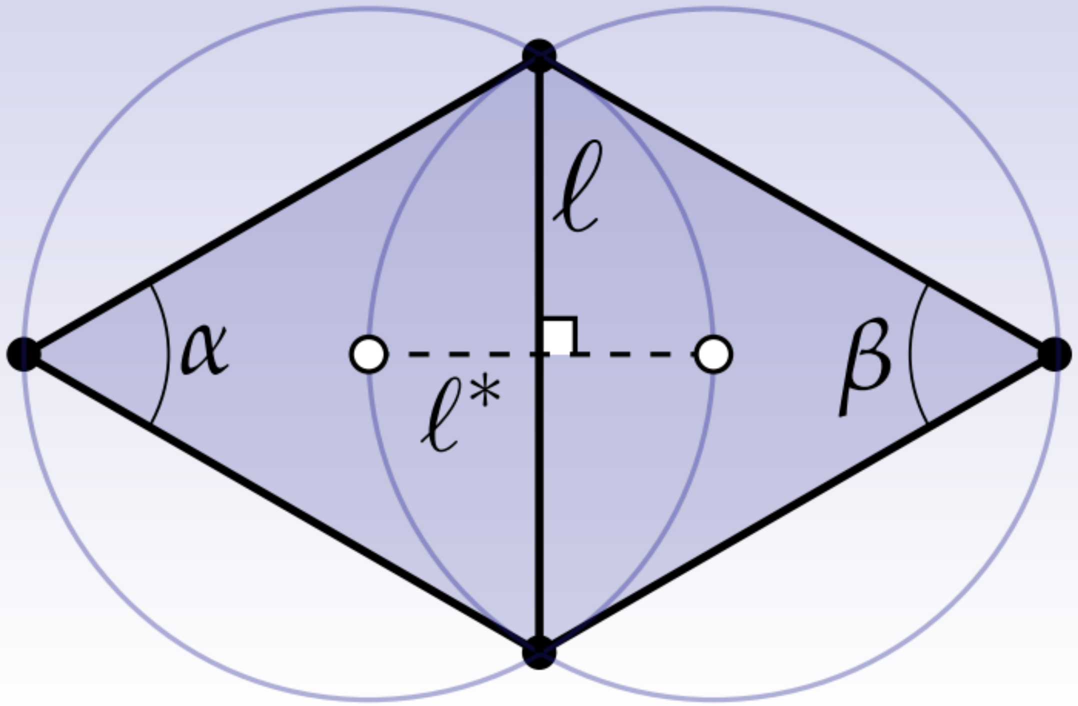
```
import Expertise
```

Many common decisions for geometry

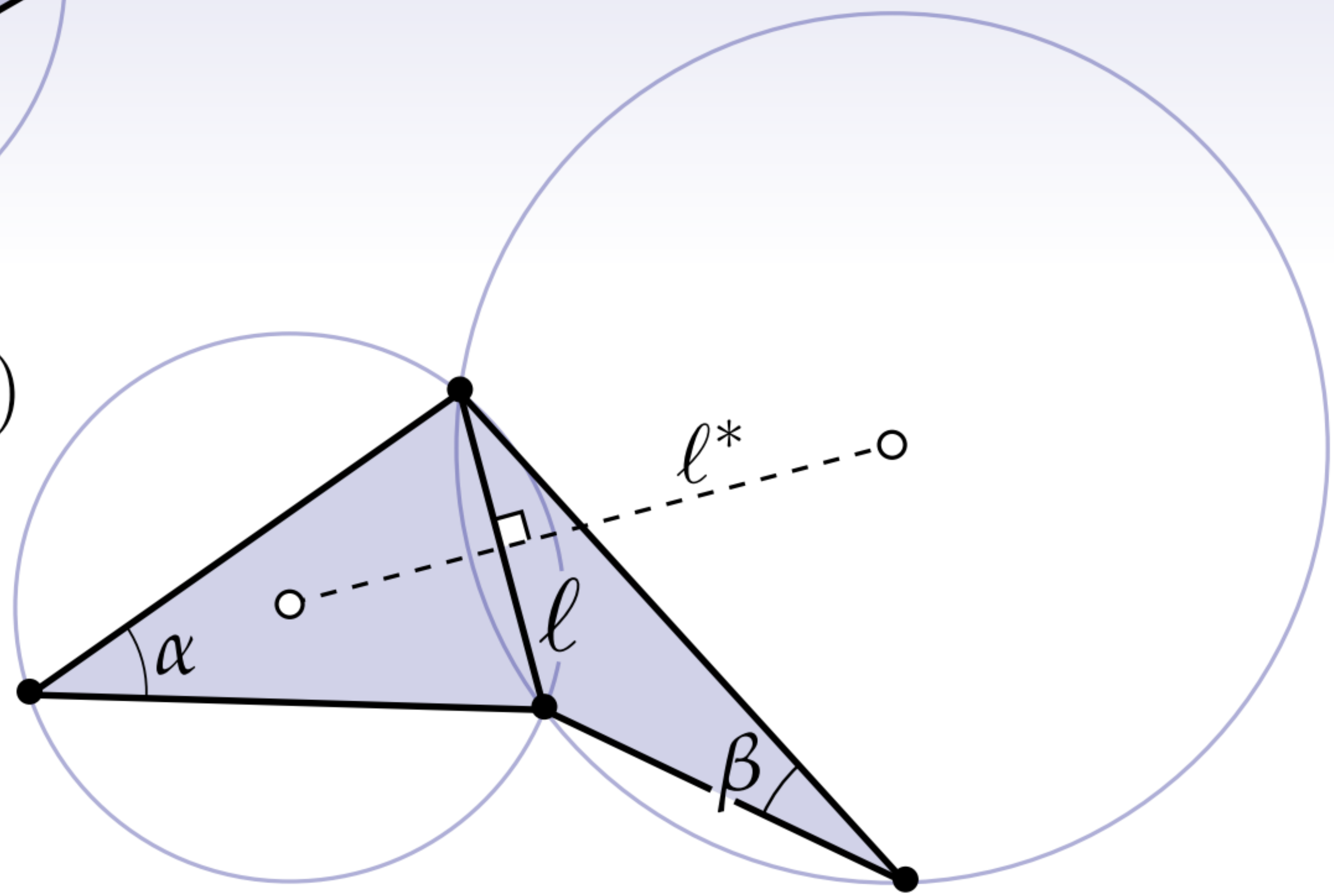
What's the geometry?
Where does the camera go, and what's the focal length?
Where's the light/shadow?
 Do we want to include shading?
 Do we want a separate light for the shadow vs. the shading?
Which "geometric" lines should be included?
 Always the silhouette.
 What about other contours?
Which "combinatorial" lines should be included?
 E.g., grid lines.
Should we automatically trace contours, or do it by hand?
Are there any components that can be algorithmically generated?
What's the line thickness, for various classes of lines?
What's the line style for hidden lines?
What's the line style for partial lines?
What's the fill color for various regions?
 Should we use different colors/brightnesses to indicate lighting?
 Should we use gradients for shading?
 How should we color occluded regions?
 How dark is the shadow?
 What direction does the shadow gradient go?
How do we label the figure?
 What gets labeled?
 Where do labels go? (How) should they occlude/be occluded by geometry?

Slide borrowed from Keenan Crane

Is This Figure Misleading?



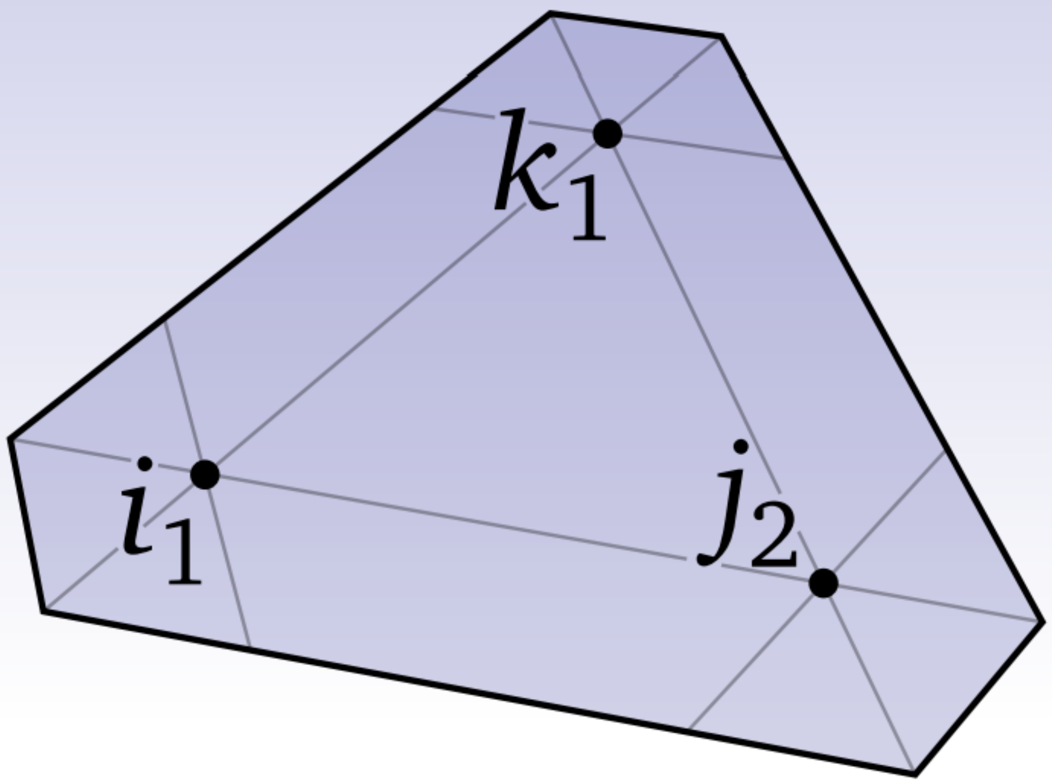
$$\frac{\ell^*}{\ell} = \frac{1}{2}(\cot\alpha + \cot\beta)$$

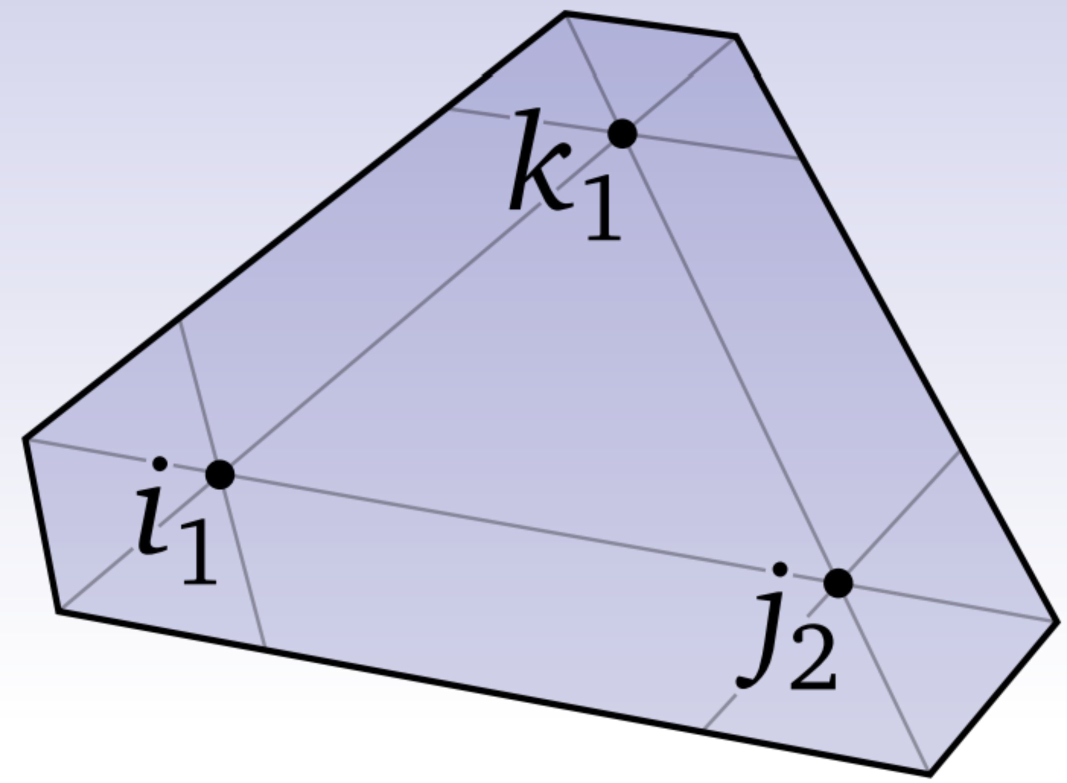


$$\frac{\ell^*}{\ell} = \frac{1}{2}(\cot\alpha + \cot\beta)$$

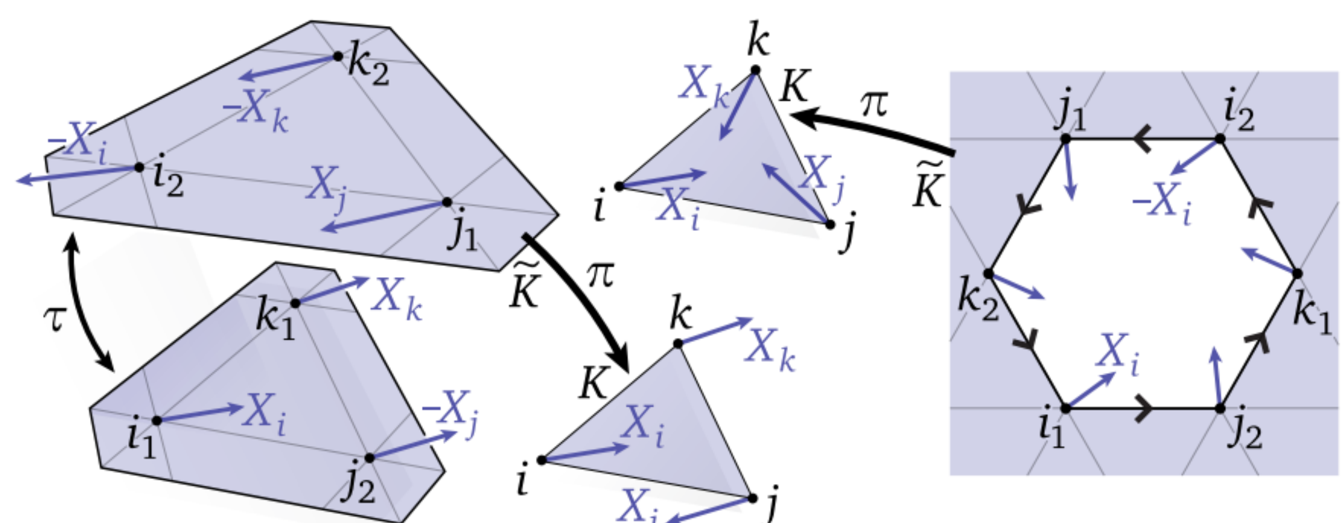
Slide borrowed from Keenan Crane

Labels - Global Placement





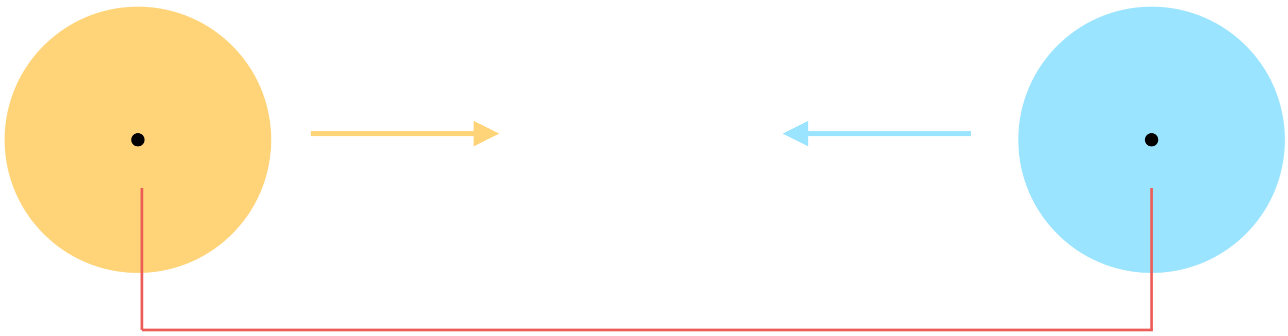
...In context, lots of constraints!



Slide borrowed from Keenan Crane

Objectives

Objectives



Objectives

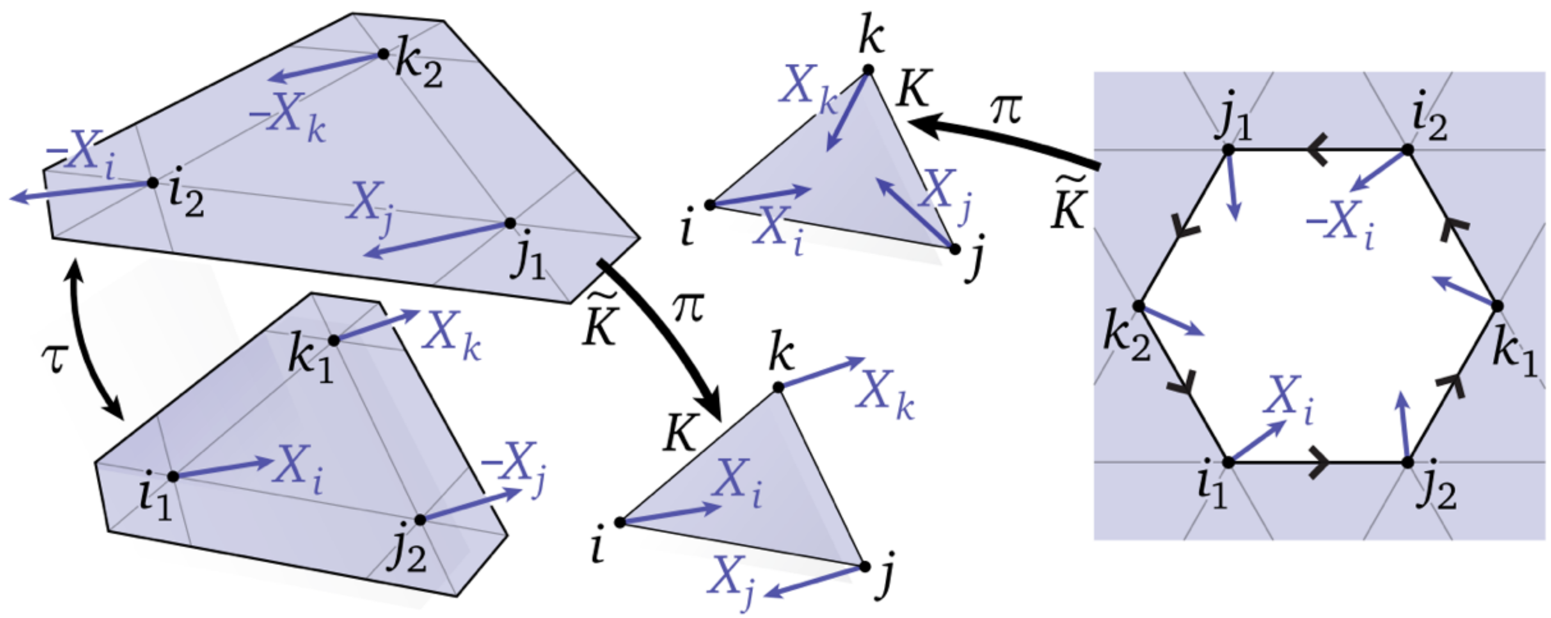
```
lay it out left to right
center sets, but keep them away from each other
position labels close to their object but far from others'
smart line breaks
```

Objectives

```
lay it out left to right
center sets, but keep them away from each other
position labels close to their object but far from others’
smart line breaks
```

$$
\begin{aligned}
centerOrRadius(x_1, x_2, y_1, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}^3 \\
-(c_1 + c_2)((x_1 - x_2)^2 + (y_1 - y_2)^2) \\
+c_1 c_2 \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2)}
\end{aligned}
$$

Objectives

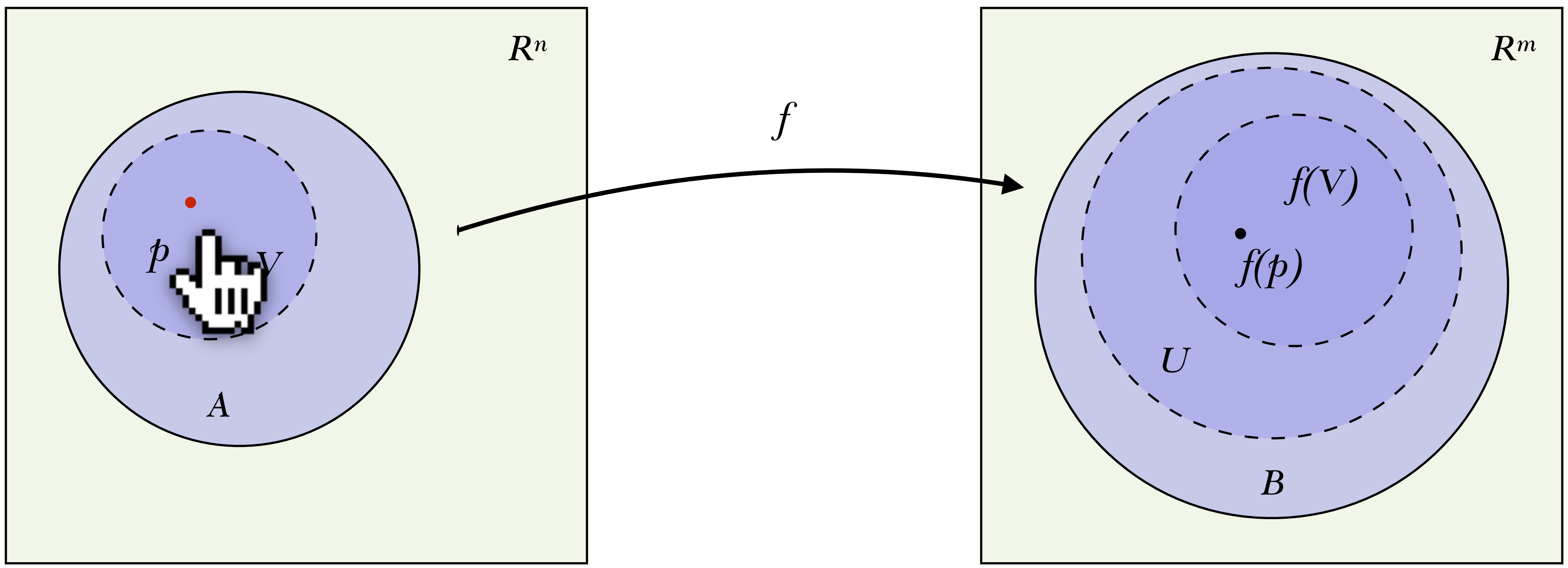


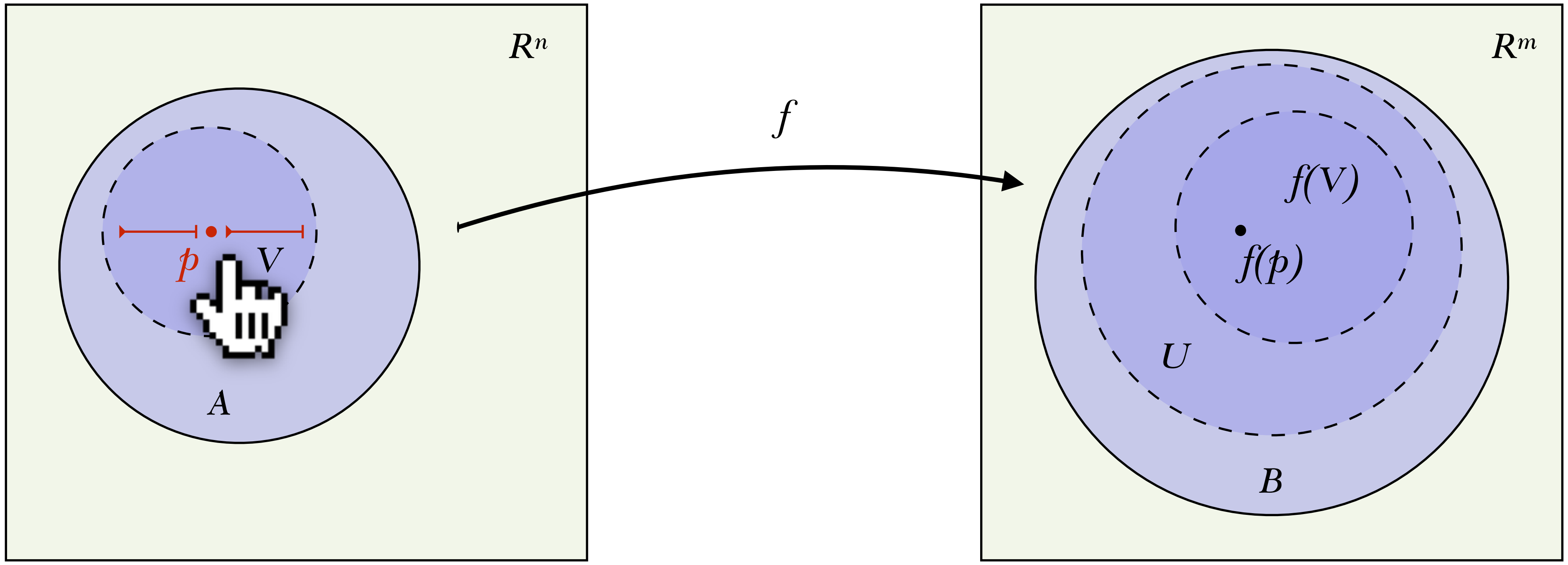
?

Source: Keenan Crane

Prodirect manipulation: center the points

want to center this point





Rn
p
V
A
f
Rm
f(V)
f(p)
U
B

Substance

```
Set A
Set B
Set R^n
Set R^m
Subset A R^n
Subset B R^m
Map f A B
OpenSet U
Subset U B
```

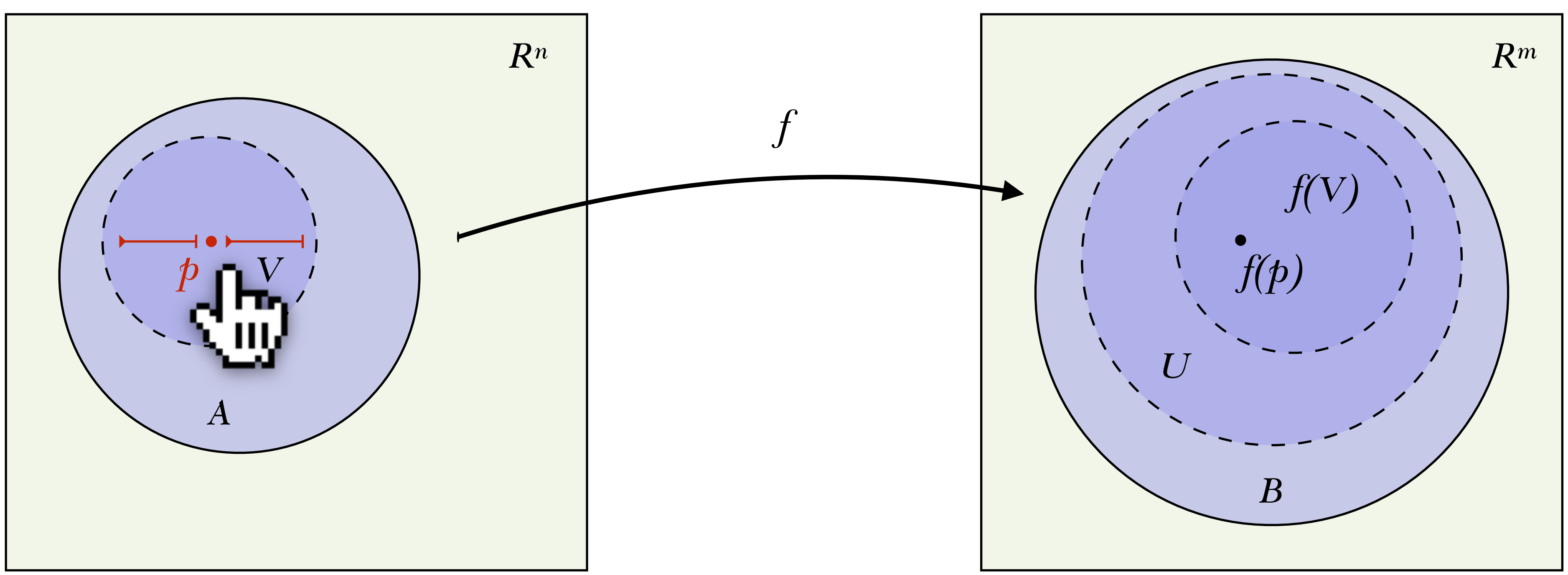
```
Point p
In p A
Point f(p)
In f(p) U
OpenSet V p
Subset V A
In p V
Set f(V)
Subset f(V) U
In f(p) f(V)
```

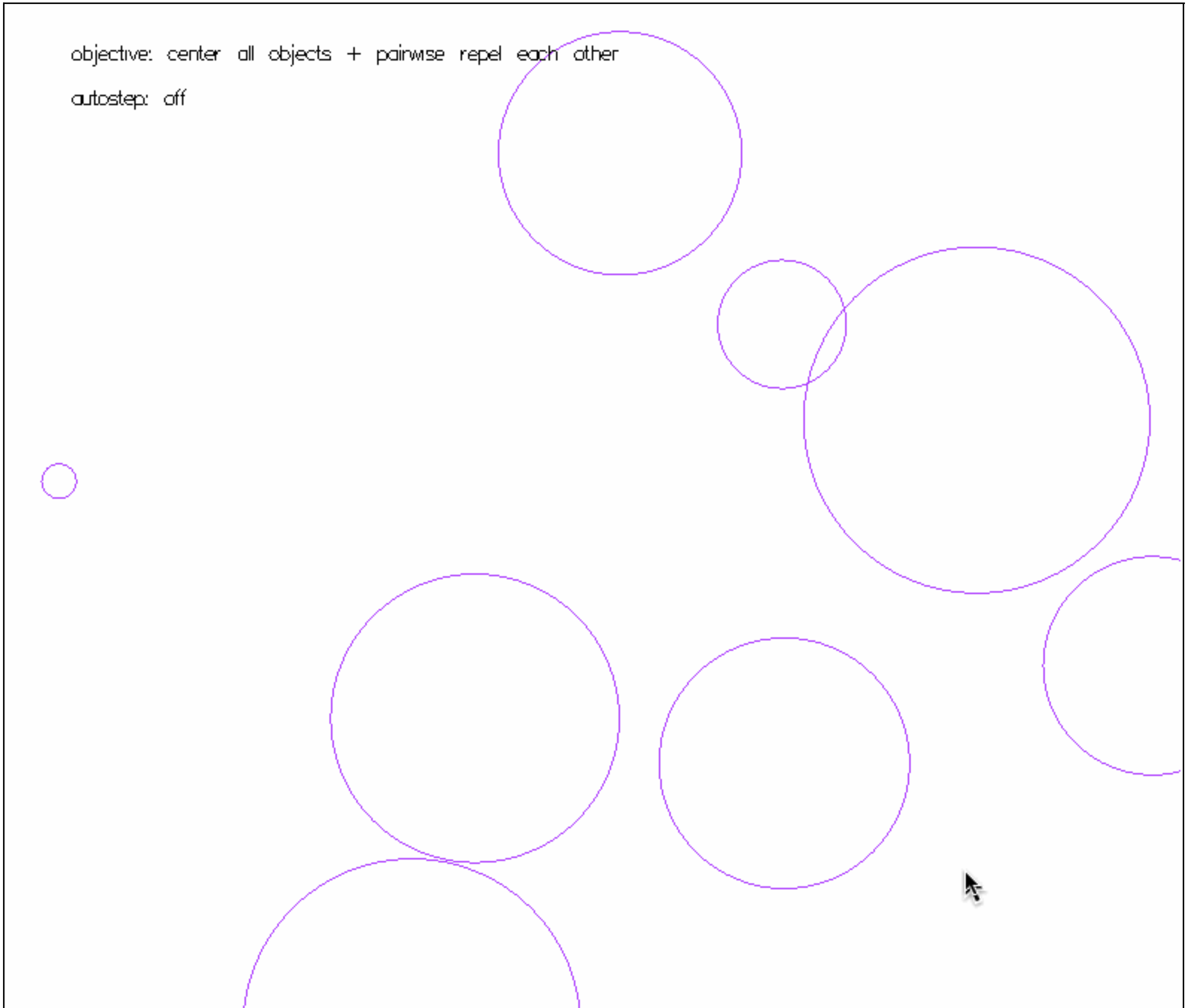
now centered at p

Style

```
Style All Auto
Shape R^n Square
Shape R^m Square
Color R^n Yellow
Color R^m Yellow
```

runtime infers code changes!




objective: center all objects + pairwise repel each other
autostep: off

```
import Expertise
```

Our broader vision

Automatically parse and visualize mathematics

MEAN VALUE THEOREMS

5.7 Definition Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

5.8 Theorem *Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.*

The analogous statement for local minima is of course also true.

Proof Choose δ in accordance with Definition 5.7, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

5.9 Theorem *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \qquad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b), and

(12) $$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$.

If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum (Theorem 4.16). By (12), $x \in (a, b)$, and Theorem 5.8 shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a *generalized mean value theorem*; the following special case is usually referred to as "the" mean value theorem:

5.10 Theorem *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof Take $g(x) = x$ in Theorem 5.9.

5.11 Theorem *Suppose f is differentiable in (a, b).*

(*a*) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(*b*) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(*c*) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Proof All conclusions can be read off from the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

which is valid, for each pair of numbers x_1, x_2 in (a, b), for *some* x between x_1 and x_2.

THE CONTINUITY OF DERIVATIVES

We have already seen [Example 5.6(*b*)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Theorem 4.23). The precise statement follows.

5.12 Theorem *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

A similar result holds of course if $f'(a) > f'(b)$.

Proof Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attains its minimum on $[a, b]$ (Theorem 4.16) at some point x such that $a < x < b$. By Theorem 5.8, $g'(x) = 0$. Hence $f'(x) = \lambda$.

mathoverflow

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Computation of π_4 of simple Lie groups

7

4

Below we assume any simple Lie group G to be simply connected.

$\pi_3(G) = \mathbb{Z}$ for any simple Lie group G and there is a uniform proof for that.

Now the textbooks say $\pi_4(G)$ is trivial except for $G = Sp(n)$, for which it is $\mathbb{Z}/2\mathbb{Z}$.

My question is the following: is there a uniform way to derive this fact, in such a way to show which property of the symplectic group is so special?

at.algebraic-topology | lie-groups | homotopy-theory

share cite improve this question | edited 3 hours ago | asked 8 hours ago Yuji Tachikawa 2,290 ● 1 ● 12 ● 34

2 you probably mean "for G locally isomorphic to $Sp(n)$" – YCor 8 hours ago

Thanks, corrected. – Yuji Tachikawa 3 hours ago

add a comment

2 Answers

active | oldest | votes

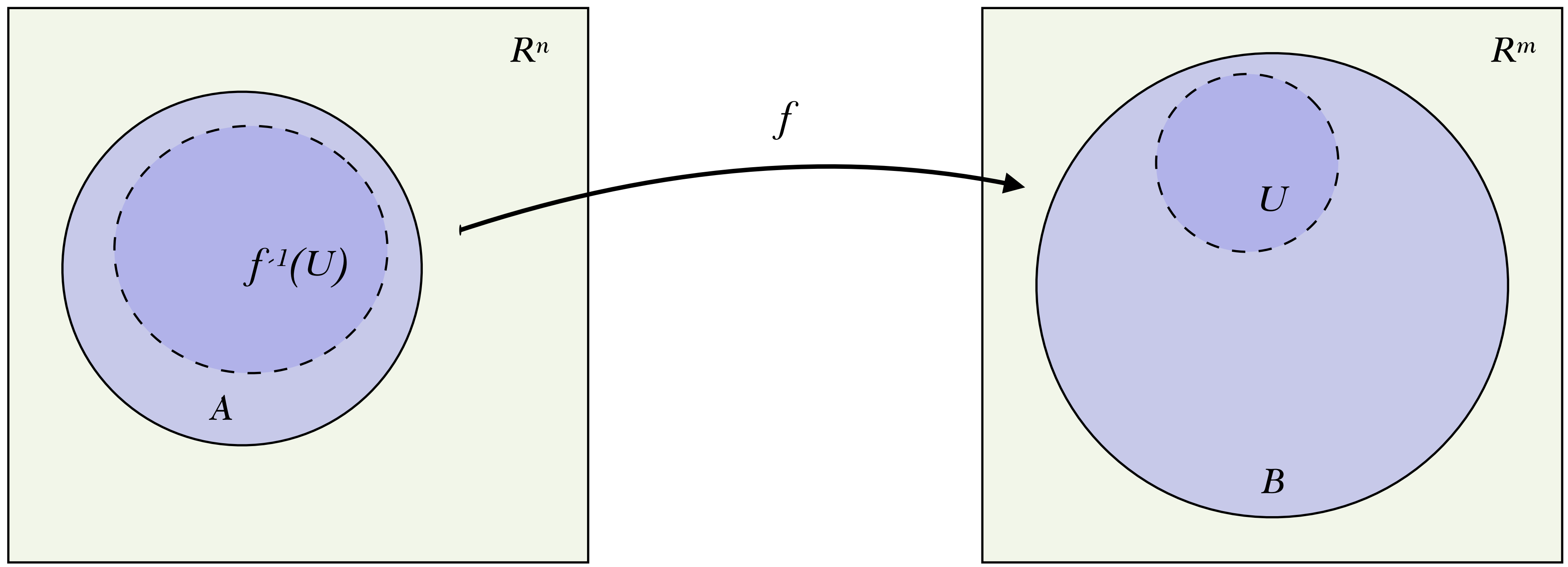
2

As a partial answer, here's at least a uniform statement, which can be found as Theorem 3.10 in Mimura's survey "Homotopy theory of Lie groups" in the Handbook of Algebraic Topology:

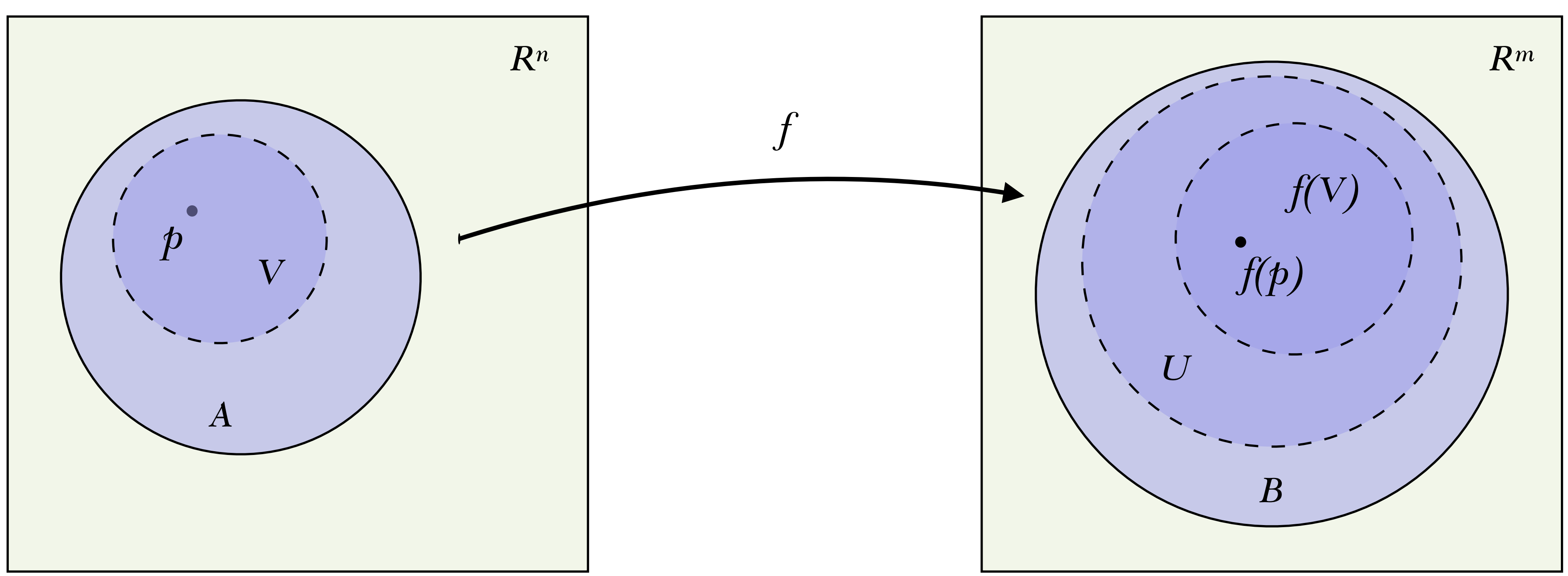
Let G be a compact, connected, simply connected, simple Lie group, T a maximal torus of G,

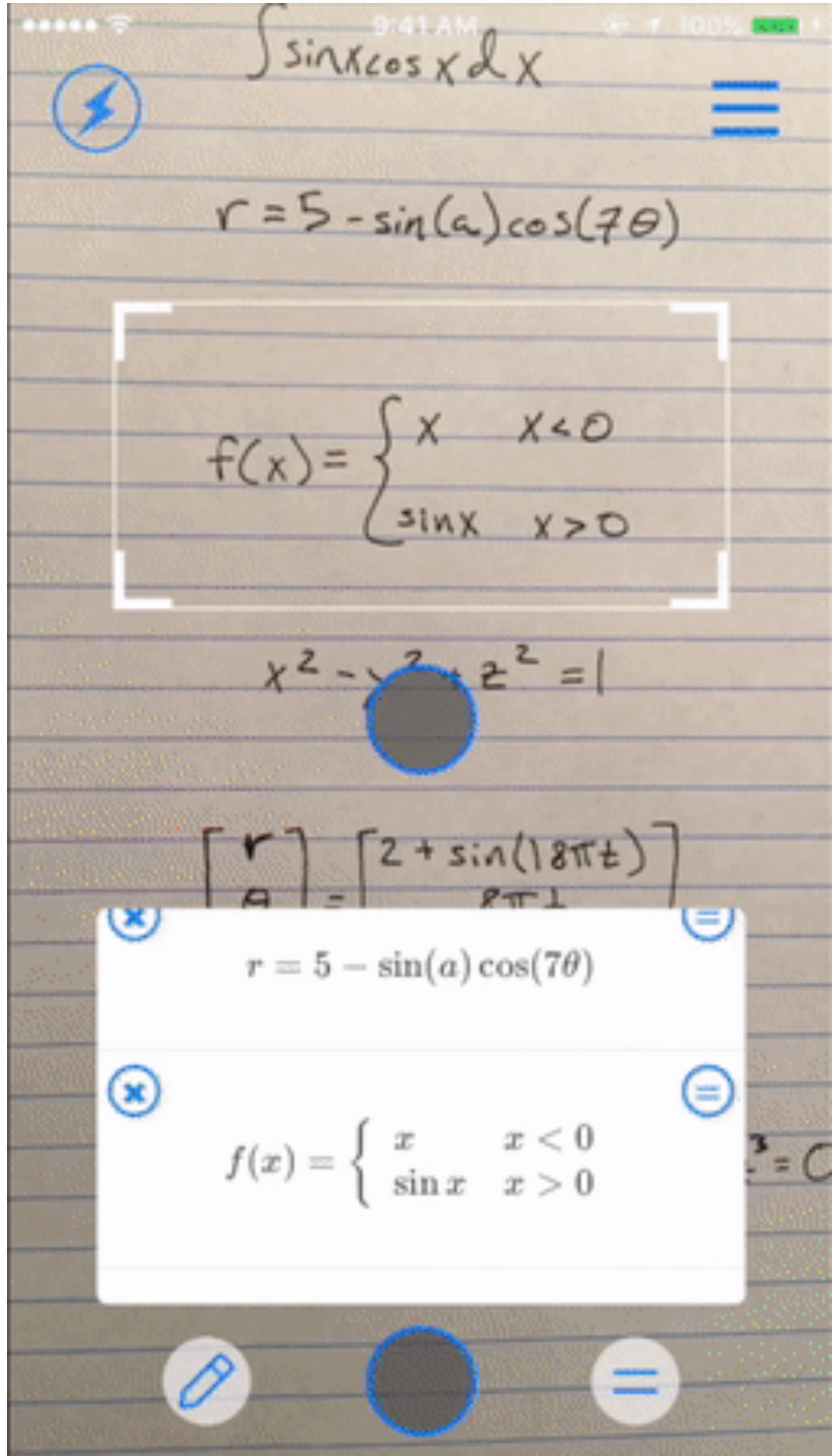
Proposition 1.3.3. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be sets, and let $f: A \to B$ be a map. The following statements are equivalent.*

(1) for every open subset $U \subset B$, the set $f^{-1}(U)$ is open in A.



(2) *For every point $p \in A$, and every open subset $U \subset B$ containing $f(p)$, there is an open subset $V \subset A$ containing p such that $f(V) \subset U$.*

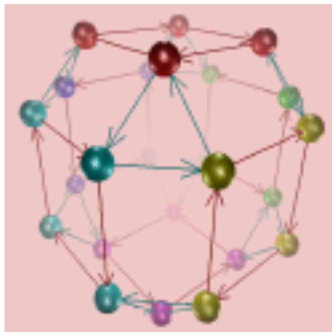


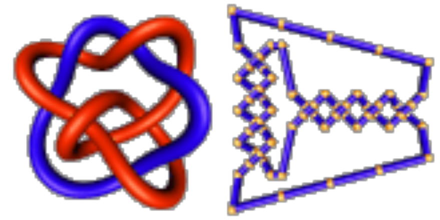


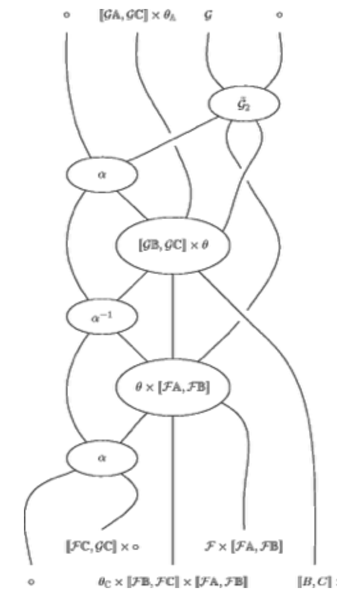
Source: https://www.reddit.com/r/math/comments/5i845f/mathpix_a_phonecamera_powered_3d_grapher

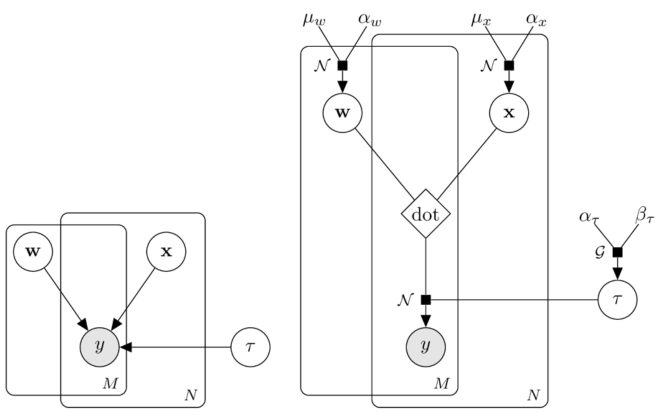
Next time you want to illustrate your paper or talk…

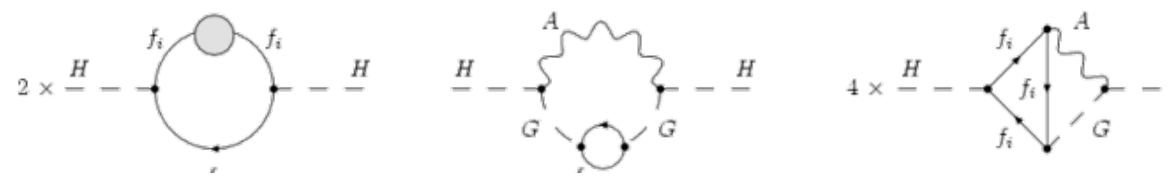
…reach for Penrose!











Gallery

illustrating a theorem by example

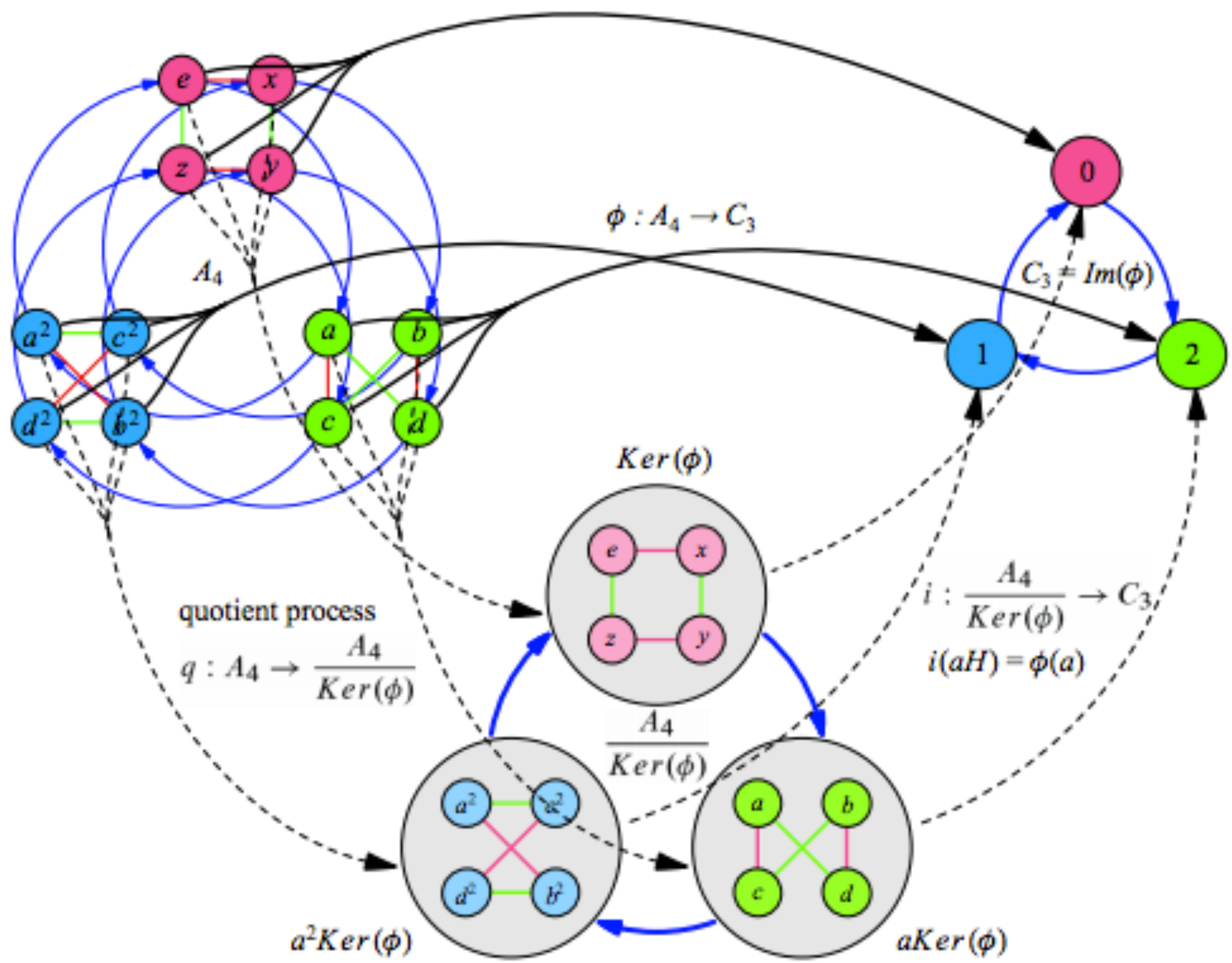


Figure 8.14. The Fundamental Homomorphism Theorem (Theorem 8.5) exemplified using the group A_4 and the quotient map ϕ whose kernel is the subgroup $\{e, x, y, z\}$ of A_4

Source:
Visual Group Theory

illustrating an algorithm

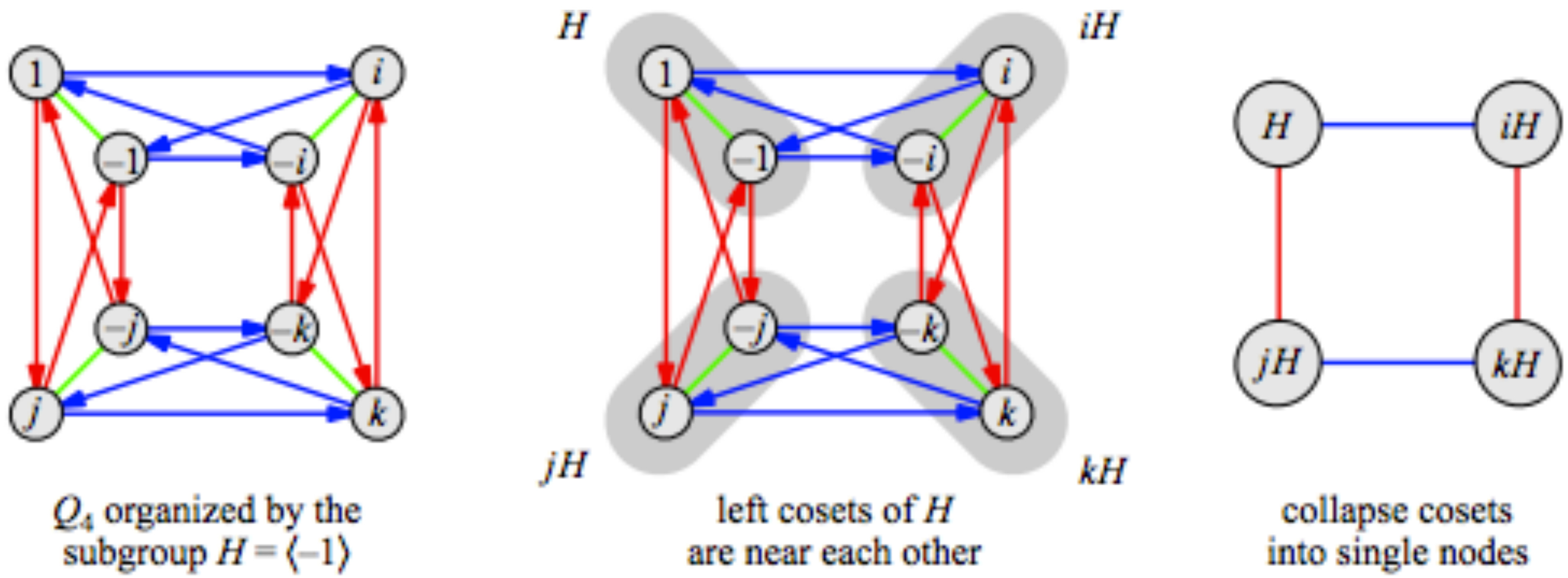


Figure 8.12. The quotient of Q_4 by C_2 corresponding to the homomorphism τ_1 in Figure 8.9. It follows the quotient procedure from Chapter 7, as depicted in Figure 7.20 (and others).

Source:
Visual Group Theory

illustrating an algorithm

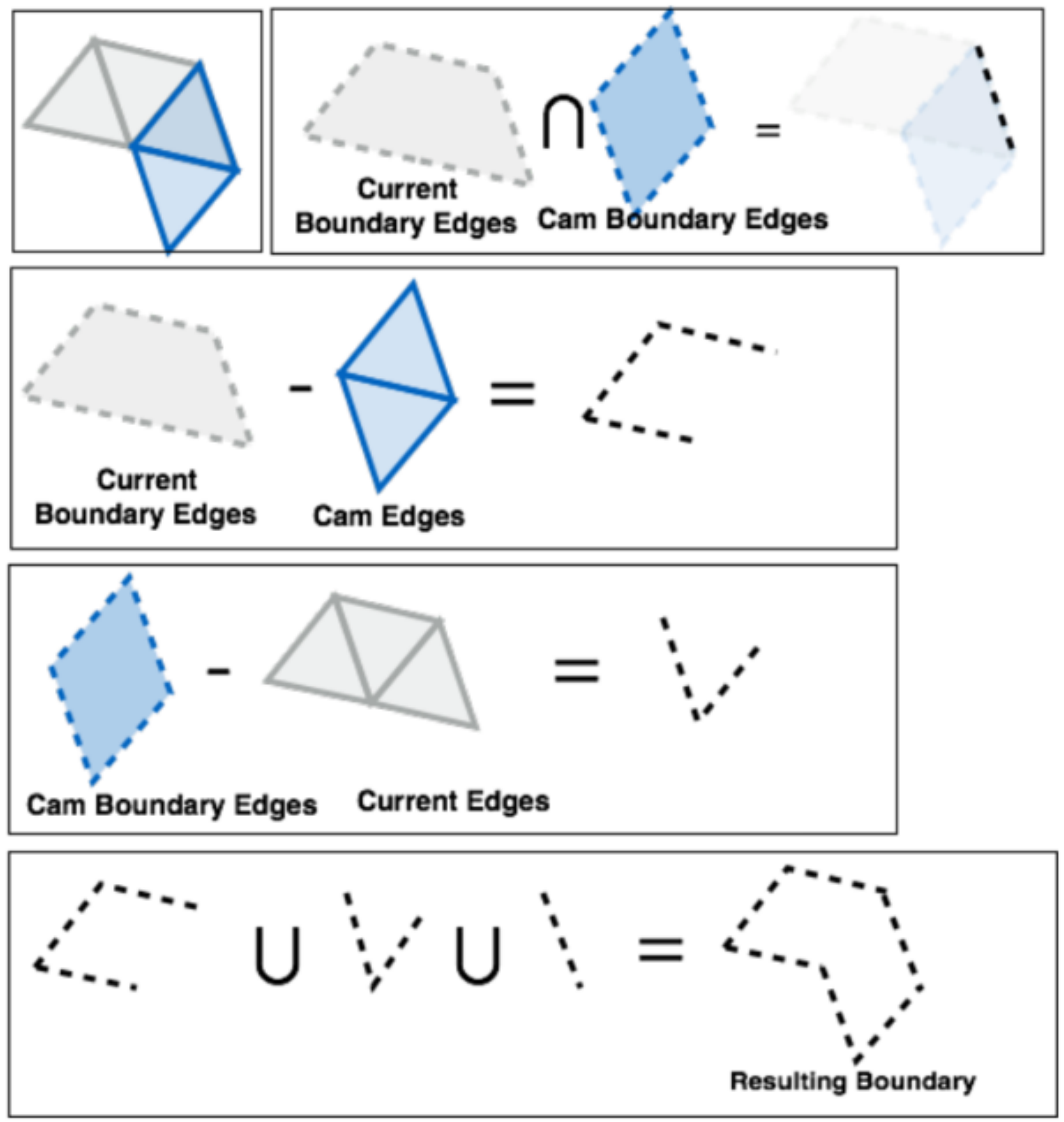


Figure 3. Illustration of calculating the new boundary of the surface area seen by cameras after adding a new camera.

Source:
https://arxiv.org/pdf/1610.06204.pdf

more geometric diagrams

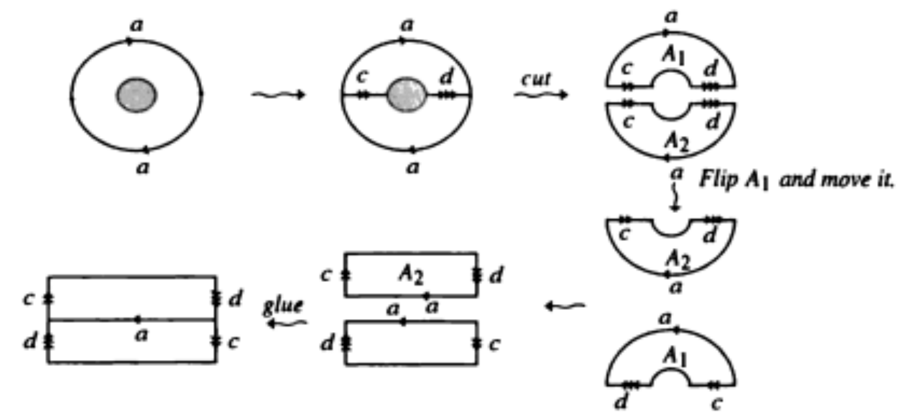


Figure 2.4.13

Figure 2.4.14

Exercises

2.4.1. Give a polygonal disk and a gluing scheme that will yield the surface pictured in Figure 2.4.15.

Figure 2.4.15

2.4.2*. Prove Lemma 2.4.5.

We need to find simplicial complexes whose underlying spaces are familiar objects such as T^2 and P^2. For T^2 this is easy, since it sits in $\mathbb{R}^3$; see Figure 3.3.5. Surfaces that do no sit in $\mathbb{R}^3$, such as P^2, are harder to work with. We will eventually solve this problem using the following construction, which is a simplicial analog of quotient maps and identification spaces (discussed in Section 1.4).

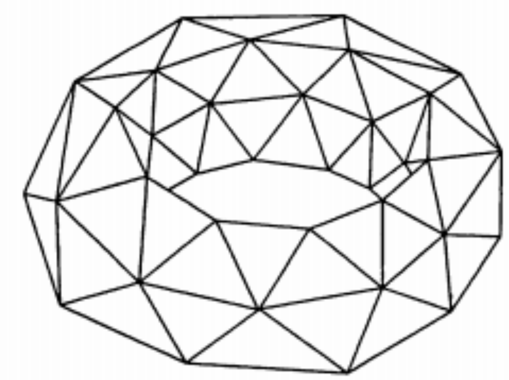


Figure 3.3.5

Consider the way in which T^2 is formed out of gluing the edges of a square, as described in Section 2.4. If we want to obtain a simplicial complex with an underlying space that is a torus, it would be tempting to break up the square shown in Figures 2.4.2 and 2.4.3 into two 2-simplices, as shown in Figure 3.3.6 (i). Unfortunately, when the edges of this square are identified as prescribed by the gluing scheme, all three vertices of both triangles are identified to a single point; since a 2-simplex must have three distinct vertices, we have not produced a simplicial complex by this process of breaking up the original square and then gluing. However, if we break up the original square into 2-simplices a bit more

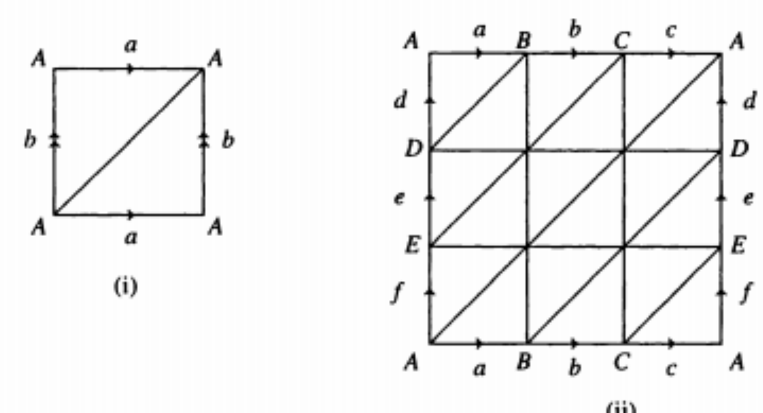
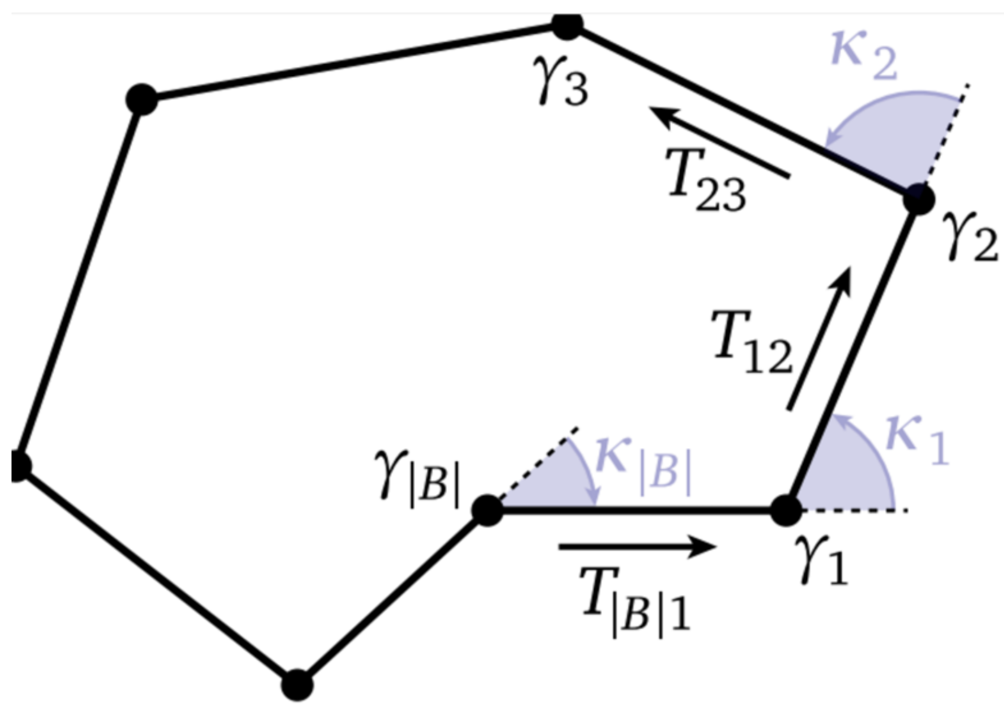


Figure 3.3.6

Source:
A first course in geometric topology

geometric diagram described by notation



$G = (V, E) \qquad G = C_{|B|}$

$\gamma : V \to \mathbb{R}^2$

$\kappa : V \to \mathbb{R}$

$T : E \to S^1 \subset \mathbb{R}^2$

$$T_{ij} := \frac{\gamma_j - \gamma_i}{|\gamma_j - \gamma_i|}$$

$$\kappa_j = \arg(T_{jk} T_{ij}^{-1})$$

Source: Keenan Crane

illustrating maps (functions) between objects

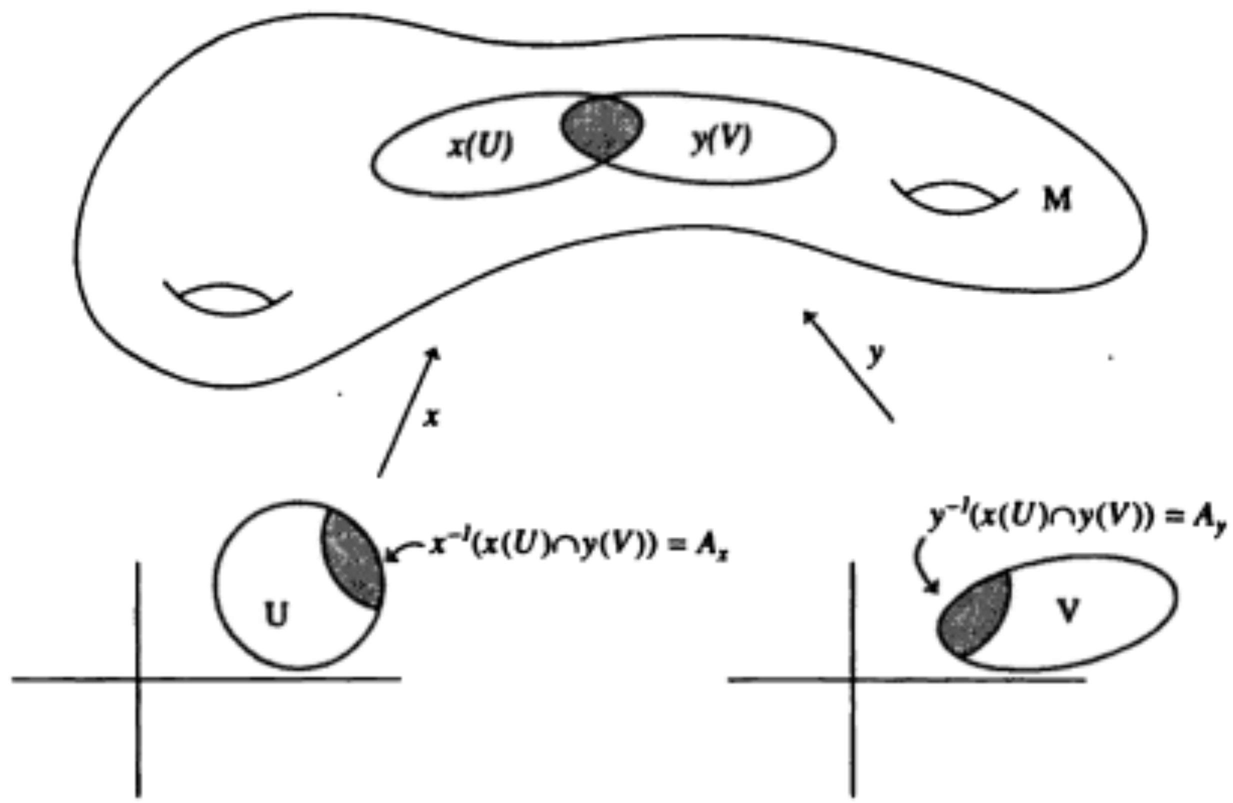
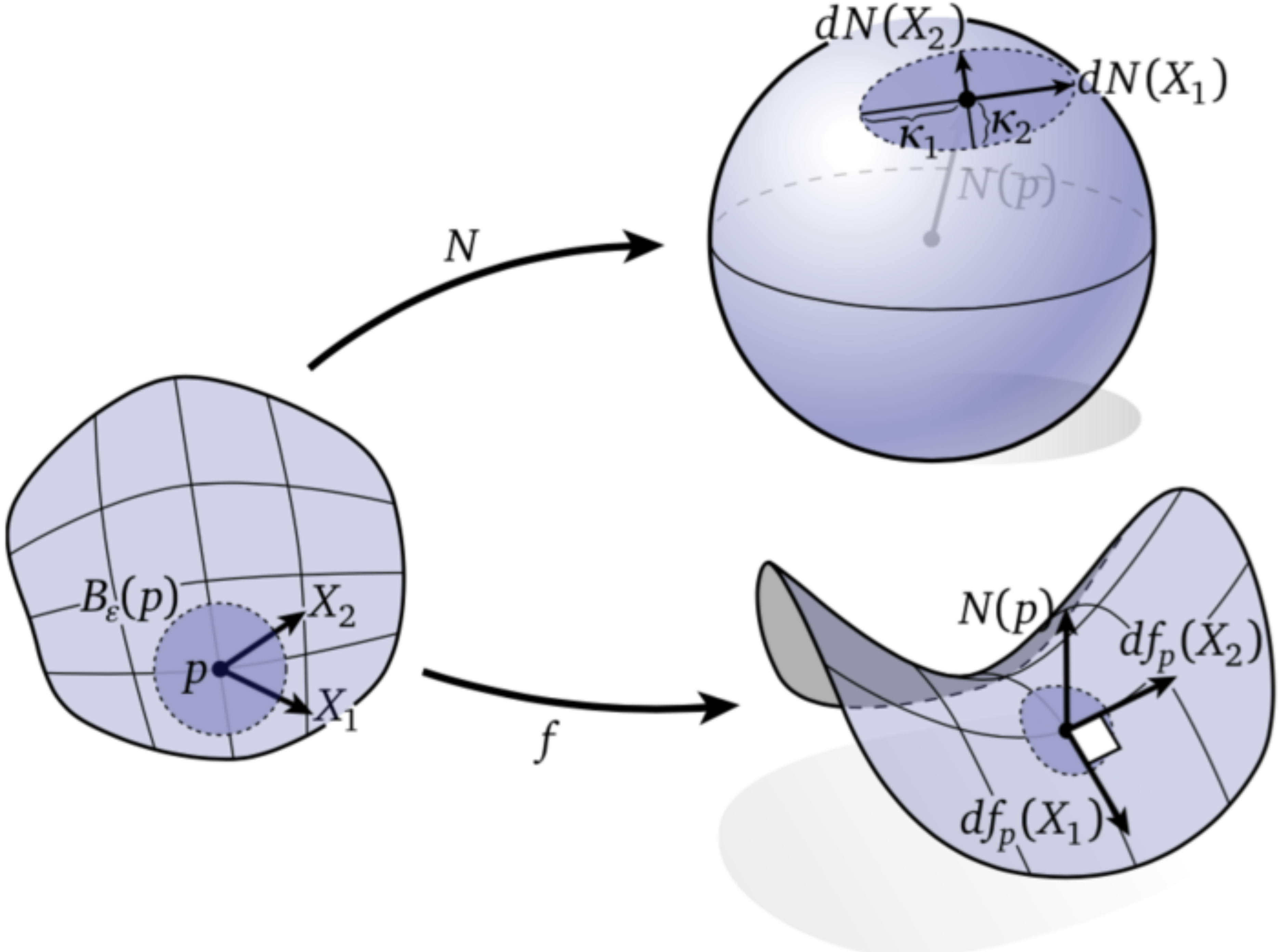


Figure 5.2.2

Since x and y in the above definition are injective, they are bijections onto their images, so we can validly refer to maps x^{-1} and y^{-1} in the above definition (though only as maps of sets, with no mention of differentiability). It is easy to see that $\phi_{x,y}$ is bijective. Also, note that

Source:
A first course in geometric topology

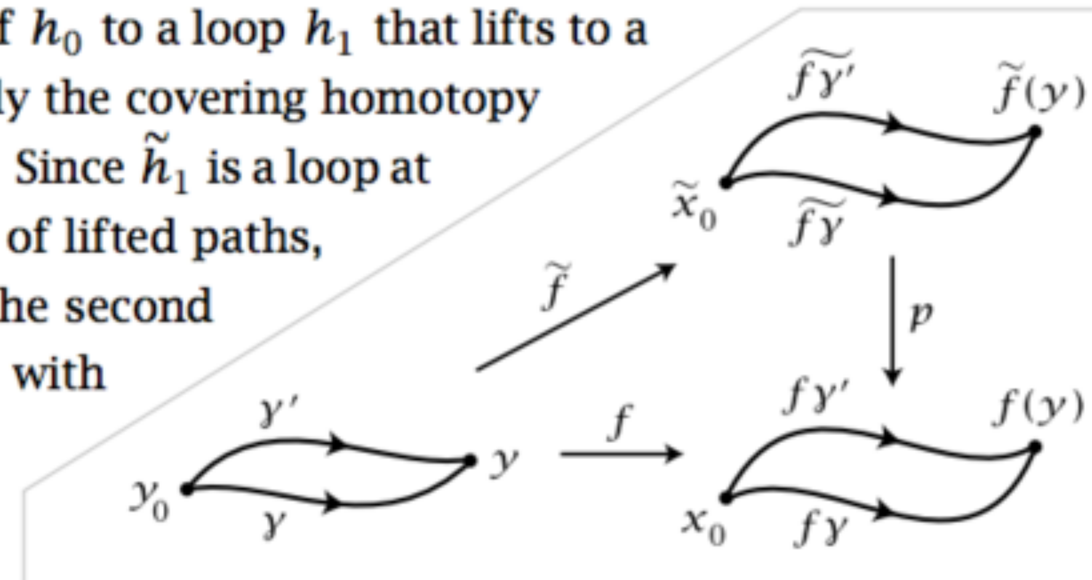
illustrating maps (functions) between objects



Source: Keenan Crane

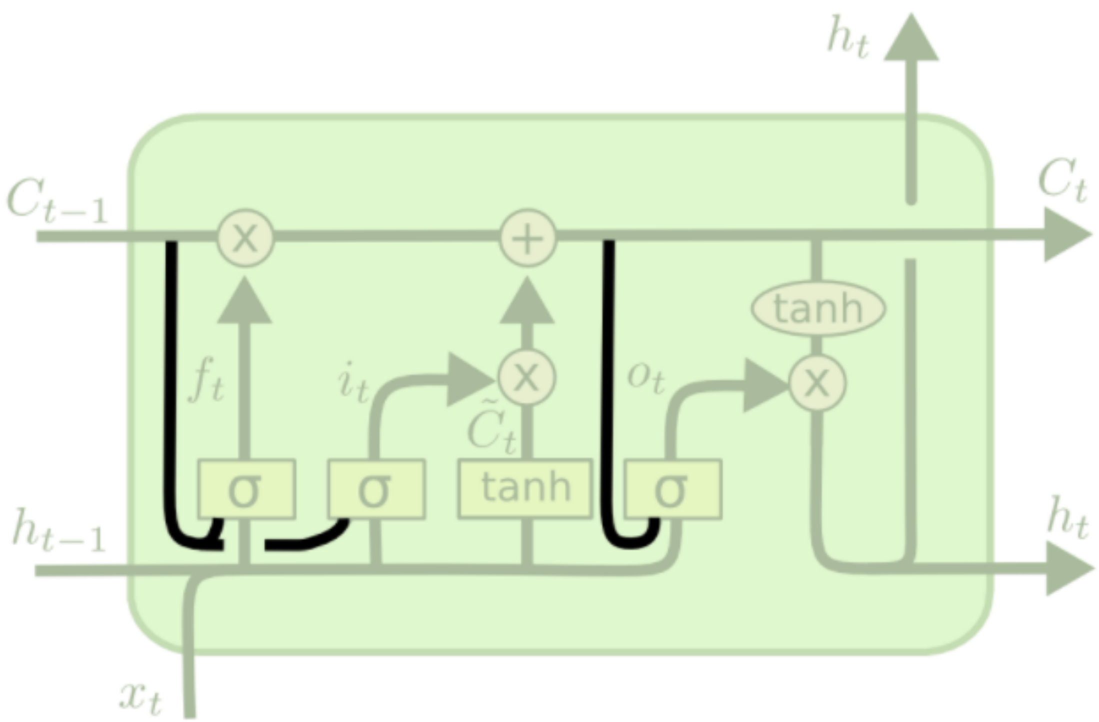
function composition

Proof: The 'only if' statement is obvious since $f_* = p_* \widetilde{f}_*$. For the converse, let $y \in Y$ and let γ be a path in Y from y_0 to y. The path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\widetilde{x}_0$. Define $\widetilde{f}(y) = \widetilde{f\gamma}(1)$. To show this is well-defined, independent of the choice of γ, let γ' be another path from y_0 to y. Then $(f\gamma') \cdot \overline{(f\gamma)}$ is a loop h_0 at x_0 with $[h_0] \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. This means there is a homotopy h_t of h_0 to a loop h_1 that lifts to a loop $\widetilde{h}_1$ in $\widetilde{X}$ based at $\widetilde{x}_0$. Apply the covering homotopy property to h_t to get a lifting $\widetilde{h}_t$. Since $\widetilde{h}_1$ is a loop at $\widetilde{x}_0$, so is $\widetilde{h}_0$. By the uniqueness of lifted paths, the first half of $\widetilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows that $\widetilde{f}$ is well-defined.



Source:
Algebraic Topology
Hatcher

circuit-type diagrams for equations



$$f_t = \sigma\left(W_f \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \ + \ b_f\right)$$

$$i_t = \sigma\left(W_i \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \ + \ b_i\right)$$

$$o_t = \sigma\left(W_o \cdot [\boldsymbol{C_t}, h_{t-1}, x_t] \ + \ b_o\right)$$

Source:
"Understanding LSTM Networks," Chris Olah

circuit-type diagrams for equations

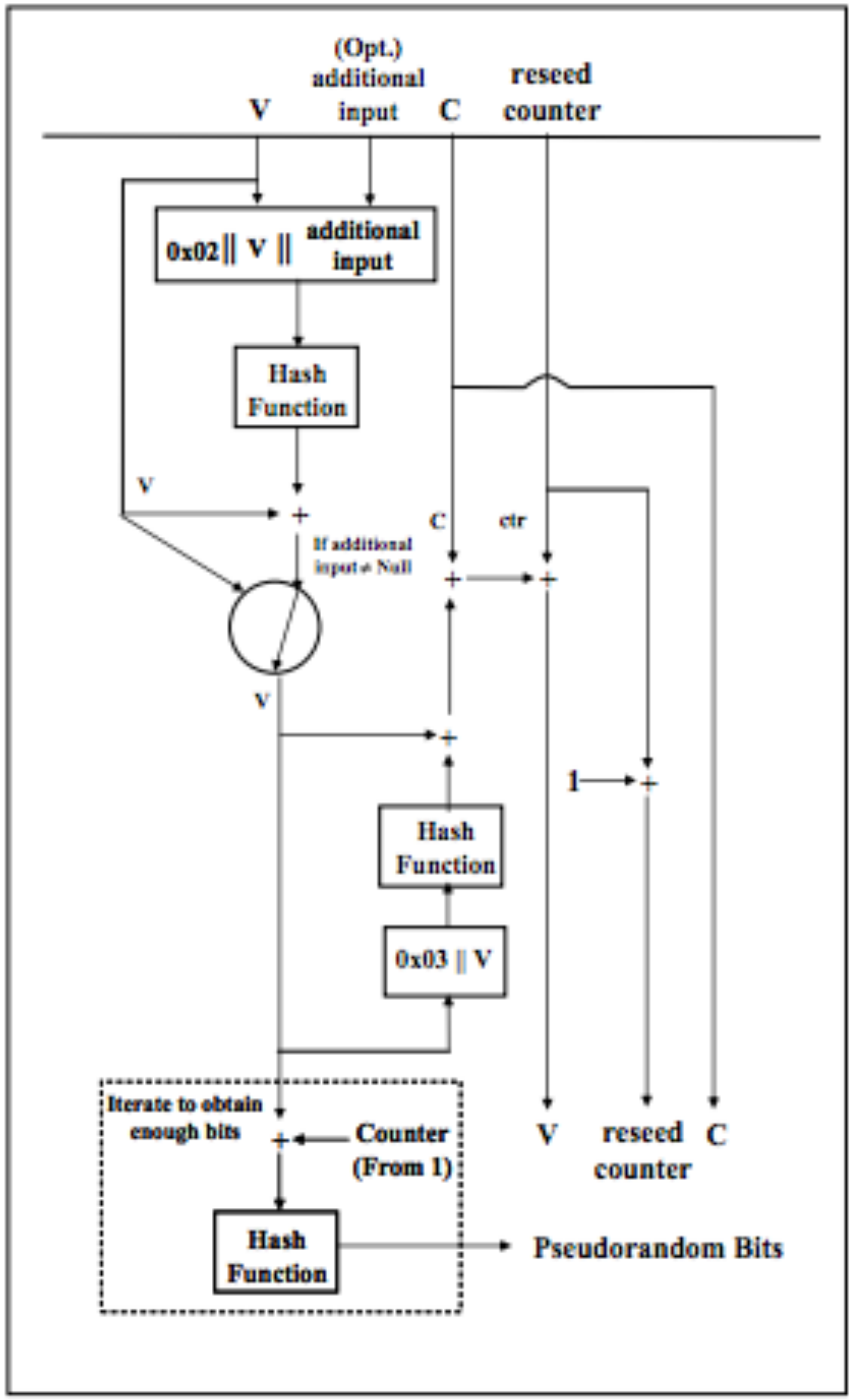


Figure 8: Hash_DRBG

Hash_DRBG Generate Process:

1. If *reseed_counter* > *reseed_interval*, then return an indication that a reseed is required.
2. If (*additional_input* ≠ *Null*), then do

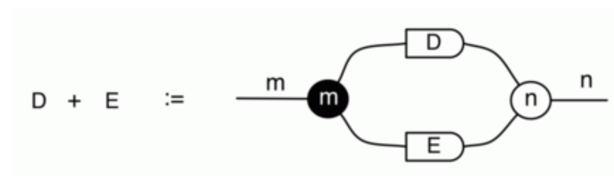
   2.1 w = **Hash** (0x02 || V || *additional_input*).

   2.2 $V = (V + w) \bmod 2^{seedlen}$.
3. (*returned_bits*) = **Hashgen** (*requested_number_of_bits*, V).
4. H = **Hash** (0x03 || V).
5. $V = (V + H + C + reseed\_counter) \bmod 2^{seedlen}$.
6. *reseed_counter* = *reseed_counter* + 1.
7. Return **SUCCESS**, *returned_bits*, and the new values of V, C, and *reseed_counter* for the *new_working_state*.

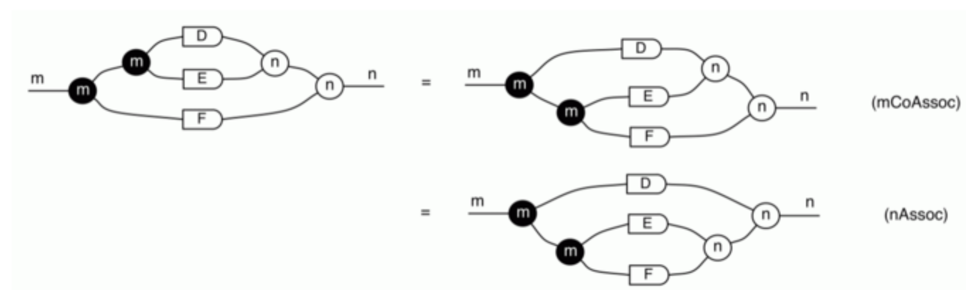
Source: NIST 800-90A

math done via diagrams

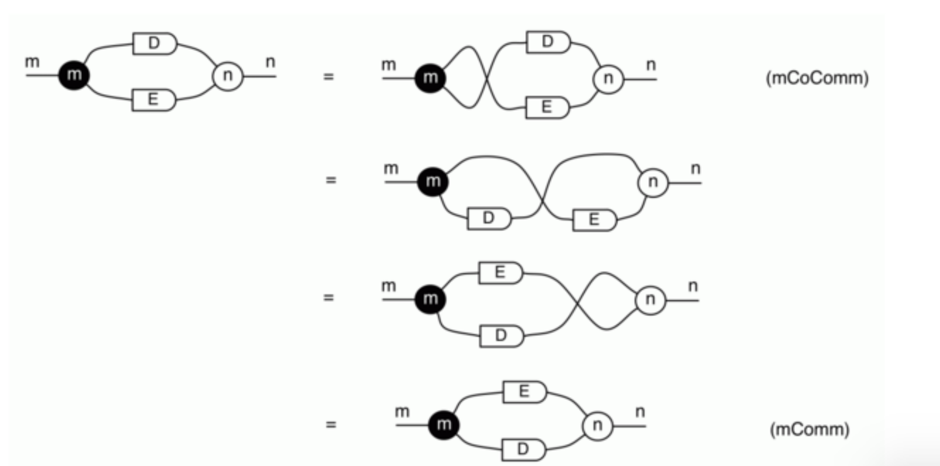
These properties intrigued Fibonchi, and he went ahead and defined an operation on *compatible* diagrams—those that agreed in their respective numbers of dangling wires on the left and right. He called the operation **sum**. This operation would be rediscovered in different notation a few centuries later and given the name **matrix addition**.



He proved that sum was associative



and commutative:



Source:
Graphical Linear Algebra

sometimes we don't know what we want to draw…

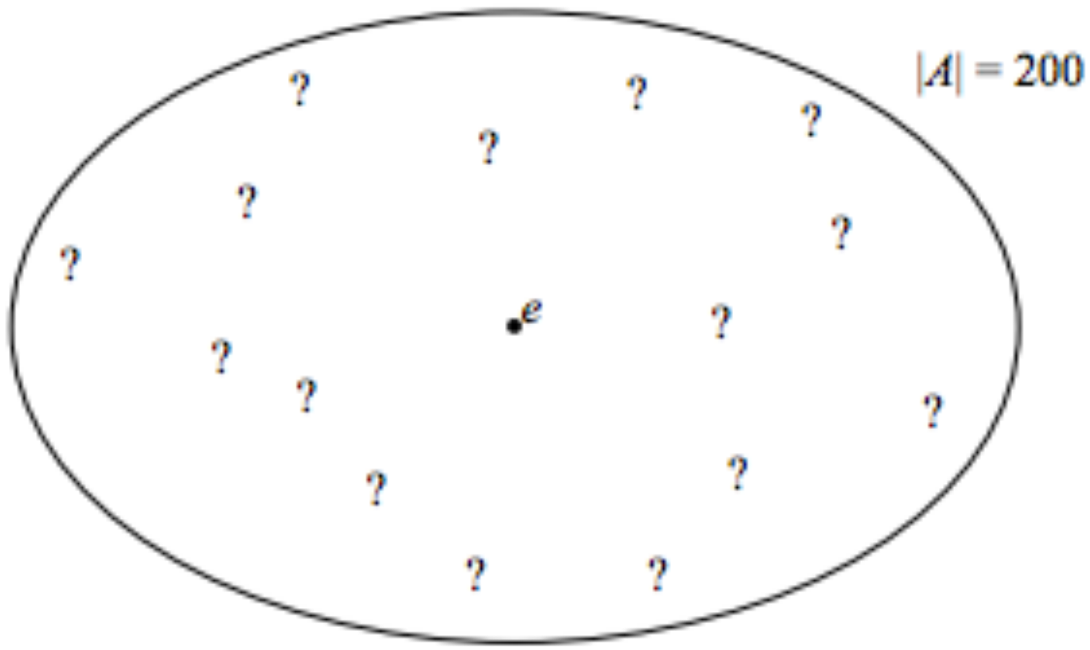


Figure 9.1. If we assume only that a group A has order 200, we do not yet know anything about its internal structure. This chapter teaches several theorems that will reveal much of that structure.

Source:
Visual Group Theory

“Testimonials”

(from real people!)

I would certainly be an eager user of such a system.

The current state of the art for my workflow is a lot of copy-pasting and tweaking of TikZ code.

Edward Morehouse
category theorist

This looks fabulous! (…)

I'd love to use it to rewrite Group Explorer for the web!

Nathan Carter
author of Visual Group Theory

Thanks!
Questions?

FAQ

How will you evaluate the effectiveness of Penrose?

How does Penrose compare to Asymptote, TikZ, Ipe, etc.?

How does Penrose compare to Mathematica, Wolfram Alpha, GeoGebra, graphing calculators, etc.?

What other domains of math might you target?

How will Penrose deal with continuous domains?

How much of the system have you actually implemented so far?

Do you plan to put Penrose on the web, or integrate it with TeX?

What rendering backend will you use?

Have you considered <related work>?

Can I use Penrose right now? No? Then when??

How much about mathematics does Penrose need to “know”?

Have you considered hooking it up to a system for formal reasoning, such as a theorem prover (e.g. Coq)?

What about programs that represent invalid, incorrect, or inconsistent diagrams?

How can Penrose enable counterfactual or nonconstructive reasoning?

(Comment: “I found the anti-example of continuity more helpful than the illustrations of the definition.”)

How will you deal with quantifiers (e.g. for all, there exists) and their nesting and ordering?

How will you separate Substance from Style? I don’t find HTML/CSS to be very convincing.

Should all branches of mathematics be illustrated? Some seem more amenable to illustration than others.

How easy would it be for you to extend Penrose to do interactive diagrams, animated diagrams, sequential diagrams, algorithmic diagrams, or parameterized diagrams? What about illustrating theorems and proofs?

I don’t know!

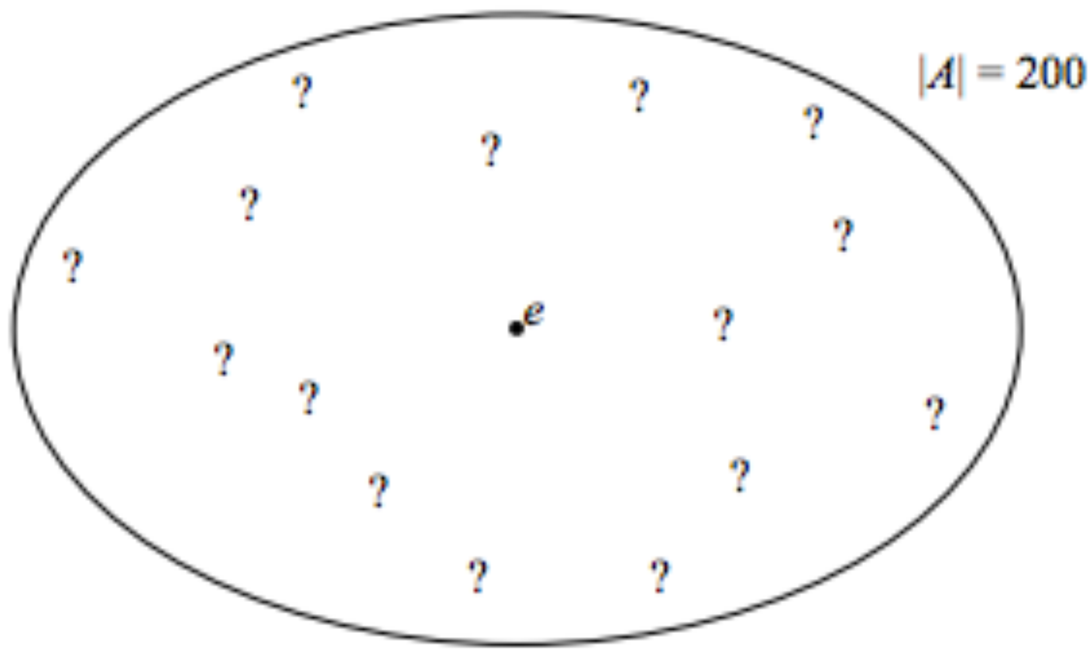


Figure 9.1. If we assume only that a group A has order 200, we do not yet know anything about its internal structure. This chapter teaches several theorems that will reveal much of that structure.