Alan Turing and the Other Theory of Computation (expanded)*

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Abstract. We recognize Alan Turing's work in the foundations of numerical computation (in particular, his 1948 paper "Rounding-Off Errors in Matrix Processes"), its influence in modern complexity theory, and how it helps provide a unifying concept for the two major traditions of the theory of computation.

1 Introduction

The two major traditions of the theory of computation, each staking claim to similar motivations and aspirations, have for the most part run a parallel non-intersecting course. On one hand, we have the tradition arising from logic and computer science addressing problems with more recent origins, using tools of combinatorics and discrete mathematics. On the other hand, we have numerical analysis and scientific computation emanating from the classical tradition of equation solving and the continuous mathematics of calculus. Both traditions are motivated by a desire to understand the essence of computation, of algorithm; both aspire to discover useful, even profound, consequences.

While the logic and computer science communities are keenly aware of Alan Turing's seminal role in the former (discrete) tradition of the theory of computation, most remain unaware of Alan Turing's role in the latter (continuous) tradition, this notwithstanding the many references to Turing in the modern numerical analysis/computational

*This paper amplifies a shorter version to appear in *The Selected Works of A.M. Turing: His Work and Impact,* Elsevier [Blu12] and follows the perspective presented in "Computing over the Reals: Where Turing Meets Newton," [Blu04].

mathematics literature, e.g., [Bür10, Hig02, Kah66, TB97, Wil71]. These references are not to recursive/computable analysis (suggested in Turing's seminal 1936 paper), usually cited by logicians and computer scientists, but rather to the fundamental role that the notion of "condition" (introduced in Turing's seminal 1948 paper) plays in real computation and complexity.

In 1948, in the first issue of the *Quarterly Journal of Mechanics and Applied Mathematics*, sandwiched between a paper on "Use of Relaxation Methods and Fourier Transforms" and "The Position of the Shock-Wave in Certain Aerodynamic Problems," appears the article "Rounding-Off Errors in Matrix Processes." This paper introduces the notion of the *condition number* of a matrix, the chief factor limiting the accuracy in solving linear systems, a notion fundamental to numerical computation and analysis, and a notion with implications for complexity theory today. This paper was written by Alan Turing [Tur48].

ROUNDING-OFF ERRORS IN MATRIX PROCESSES

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SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known 'Gauss elimination process', it is found that the errors are normally quite moderate: no exponential build-up need occur.

Included amongst the methods considered is a generalization of Choleski's method which appears to have advantages over other known methods both as regards accuracy and convenience. This method may also be regarded as a rearrangement of the elimination process.

THIS paper contains descriptions of a number of methods for solving sets of linear simultaneous equations and for inverting matrices, but its main concern is with the theoretical limits of accuracy that may be obtained in the application of these methods, due to rounding-off errors.

After the war, with the anticipation of a programmable digital computing device on the horizon, it was of great interest to understand the comparative merits of competing computational "processes" and how accurate such processes would be in the face of inevitable round-off errors. Solving linear systems is basic. Thus for Turing (as it was for John von Neumann [vNG47]), examining methods of solution with regard to the ensuing round-off errors presented a compelling intellectual challenge.[1]

[1]It is clear that Turing and von Neumann were working on similar problems, for similar reasons,

In 1945, Turing had submitted an 86 page proposal to the British National Physical Laboratory (NPL) for the Automatic Computing Engine (ACE), *an automatic electronic digital computer with internal program storage* [Tur45]. This was to be an incarnation of the universal machine envisioned by his theoretical construct in "Computable Numbers" [Tur36], blurring the boundary between data and program. Thus, and in contrast to other proposed "computing machines" at the time, Turing's computer would have simplified hardware, with universality emanating from the power of programming.

Turing envisioned an intelligent machine that would learn from its teachers and from its experience and mistakes, and hence have the ability to create and modify its own programs. Turing also felt that the most conducive environment for realizing such a machine was to have mathematicians and engineers working together in close proximity, not in separate domains [Hod92].

Unfortunately, the ACE computer was never to see the light of day;[2] a less ambitious non-universal machine, the PILOT ACE, was constructed after Turing left the NPL for the University of Manchester in 1948. Although Turing's central passion during his time at the NPL was the promised realization of his universal computer, his only published paper to come out of this period (1945-1948) was "Round-Off Errors in Matrix Processes." [3]

"Rounding-Off Errors in Matrix Processes" was by no means an anomaly in Turing's creative pursuits. In his 1970 Turing Award Lecture, James Wilkinson writes [Wil71]:

> Turing's international reputation rests mainly on his work on computable numbers but I like to recall that he was a considerable numerical analyst, and a good part of his time from 1946 onwards was spent working in this field...

Wilkinson attributes this interest, and Turing's decision to go to the NPL after the war, to the years he spent at Bletchley Park gaining knowledge of electronics and "one of the happiest times of his life."

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in similar ways at the same time, probably independently. However while Turing acknowledges von Neumann, as far as I know, von Neumann never cites Turing's work in this area.

[2] An interrelated combination of personalities, rivalries, politics and bureaucracy seems to have been in play. For an in-depth chronicle of the saga, see Andrew Hodges' chapter, *Mercury Delayed* in [Hod92].

[3] The paper ends with the cryptic acknowledgement: "published with the permission of the Director of the National Physical Laboratory."

Wilkinson also credits Turing for converting him from a classical to numerical analyst. From 1946 to 1948, Wilkinson worked for Turing at the NPL on the logical design of Turing's proposed Automatic Computing Engine and the problem of programming basic numerical algorithms:

> The period with Turing fired me with so much enthusiasm for the computer project and so heightened my interest in numerical analysis that gradually I abandoned [the idea of returning to Cambridge to take up research in classical analysis].

Here I would like to recognize Alan Turing's work in the foundations of numerical computation. Even more, I would like to indicate how this work has seeded a major direction in complexity theory of real computation and provides a unifying concept for the two major traditions in the theory of computation.

2 Rounding-Off Errors in Matrix Processes

> This paper contains descriptions of a number of methods for solving sets of linear simultaneous equations and for inverting matrices, ***but its main concern is with the theoretical limits of accuracy*** that may be obtained in the application of these methods, due to round-off errors.

So begins Turing's paper [Tur48]. (Italics are mine, I'll return to this shortly.)

The basic problem at hand: Given the linear system, $Ax = b$ where A is a real nonsingular $n \times n$ matrix and $b \in \mathbb{R}^n$. Solve for $x \in \mathbb{R}^n$.

Prompted by calculations [FHW48] challenging the arguments by Harold Hotelling [Hot43] that Gaussian elimination and other direct methods would lead to exponential round-off errors, Turing introduces quantities not considered earlier to bound the magnitude of errors, showing that for all "normal" cases, the exponential estimates are "far too pessimistic."[4]

[4] In their 1946 paper, Valentine Bargmann, Deane Montgomery and von Neumann [BMvN63] also dismissed Gaussian elimination as likely being unstable due to magnification of errors at successive stages (pp. 430-431) and so turn to iterative methods for analysis. However, in 1947 von Neumann and Herman Goldstine reassess [vNG47] noting, as does Turing, that it is the computed solution, not the intermediate computed numbers, which should be the salient object of study. They re-investigated Gaussian elimination for computing matrix inversion and now give optimistic error bounds similar to those of Turing, but for the special case of positive definite symmetric matrices. Turing in his paper notes that von Neumann communicated these results to him at Princeton [during a short visit] in Jan-

In this paper, Turing introduced the notion of *condition number* of a matrix making explicit for the first time a measure that helps formalize the informal notion of ill and well-conditioned problems.[5]

3 The Matrix Condition Number: Where Turing Meets Newton

> When we come to make estimates of errors in matrix processes, we shall find that the chief factor limiting the accuracy that can be obtained is 'ill-conditioning' of the matrices involved [Tur48].

Turing provides an illustrative example:

$$\left.\begin{aligned} 1.4x + 0.9y &= 2.7 \\ -0.8x + 1.7y &= -1.2 \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} -0.786x + 1.709y &= -1.173 \\ -0.800x + 1.700y &= -1.200 \end{aligned}\right\} \tag{8.2}$$

> The set of equations (8.1) is fully equivalent to (8.2)[6] but clearly if we attempt to solve (8.2) by numerical methods involving rounding-off errors we are almost certain to get much less accuracy than if we worked with equations (8.1). ...
>
> We should describe the equations (8.2) as an *ill-conditioned* set, or, at any rate, as ill-conditioned compared with (8.1). It is characteristic of ill-conditioned sets of equations that small percentage errors in the coefficients given may lead to large percentage errors in the solution.

Turing defines two condition numbers (he calls them N and M), which in essence measure the intrinsic potential for magnification of errors. He then analyzes various standard methods for solving linear systems, including Gaussian elimination, and gets error bounds proportional to his measures of condition. Turing is "also as much interested

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uary 1947 before his own proofs were complete.

[5] In sections 3 and 4 of his paper, Turing also formulates the LU decomposition of a matrix (actually the LDU decomposition) and shows that Gaussian elimination computes such a decomposition.

[6] The third equation (in the set of four) is the second plus .01 times the first.

in statistical behaviour of the errors as in the maximum possible values" and presents probabilistic estimates; he also improves Hotelling's worst case bound from 4^{n-1} to 2^{n-1} [Tur48].

The following widely used (*spectral*) matrix condition number $\kappa(A)$, wedged somewhat between Turing's condition numbers, is often attributed to Turing, though it is unclear who first defined it. (See the discussion in Section 10, *Who invented the condition number?*) John Todd in his survey [Tod68] is vague on its genesis although he specifically credits Turing with recognizing "that a condition number should depend symmetrically on A and A^{-1}, specifically as a product of their norms."[7]

Definition. Suppose A is a real non-singular $n \times n$ matrix. The (*spectral*) *matrix condition number* of A is given by

$$\kappa(A) = \|A\| \left\|A^{-1}\right\|$$

where

$$\|A\| = \sup_{y \neq 0} \frac{|Ay|}{|y|} = \sup_{|y|=1} |Ay|$$

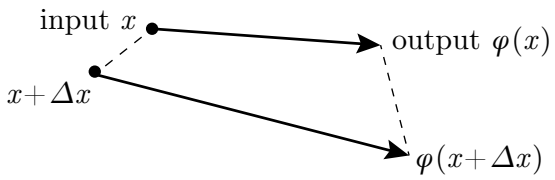
is the *operator* (spectral) *norm* with respect to the Euclidean norm. For singular matrices, define $\kappa(A) = \infty$.

This definition can be generalized using other norms. In the case of the Euclidean norm, $\kappa(A) = \sigma_1/\sigma_n$, where σ_1 and σ_2 are the largest and smallest singular values of A, respectively. It follows that $\kappa(A) \geq 1$.[8]

To see how natural a measure this is, consider a slightly more general situation. Let X and Y be normed vector spaces with associated map $\varphi : \mathrm{X} \to \mathrm{Y}$. A measure of the "condition" of *problem instance* (φ, x) should indicate how small perturbations of the *input* x (the *problem data*) can alter the *output* $\varphi(x)$ (the *problem solution*).

[7] Turing's N condition number is defined as N(*A*)N(A^{-1}) and the M condition number as nM(*A*)M(A^{-1}), where N(*A*) is the Frobenius norm of A and M(*A*) is the max norm. Turing also defines the spectral norm in this paper, so he could have easily defined the spectral condition number.

[8] For the case of computing the inverse of a positive definite symmetric matrix A by Gaussian elimination, von Neumann and Goldstine [vNG47] give an error estimate bounded by $14.2n^2(\lambda_1/\lambda_2)u$. Here λ_1 and λ_2 are the largest and smallest eigenvalues of A and u is the smallest number recognized by the machine. For the case of positive definite symmetric matrices, λ_1/λ_2 is equal to $\kappa(A)$. Thus, the spectral condition number appears implicitly in the von Neumann-Goldstine paper for this case.



So let Δx be a small perturbation of input x and $\Delta\varphi = \varphi(x + \Delta x) - \varphi(x)$. The limit as $\|\Delta x\|$ goes to zero of the ratio

$$\frac{\|\Delta\varphi\|}{\|\Delta x\|},$$

or of the *relative* ratio

$$\frac{\|\Delta\varphi\| \,/\, \|\varphi(x)\|}{\|\Delta x\| \,/\, \|x\|}$$

(favored by numerical analysts), will be a measure of the condition of the problem instance.[9] If large, computing the output with small error will require increased precision, and hence from a computational complexity point of view, increased time/space resources.

Definition.[10] The *condition number of problem instance* (φ, x) is defined by

$$\widehat{\kappa}\,(\varphi, x) = \lim_{\delta\to 0} \sup_{\|\Delta x\|\le\delta} \frac{\|\Delta\varphi\|}{\|\Delta x\|},$$

and the *relative condition number* by

$$\kappa\,(\varphi, x) = \lim_{\delta\to 0} \sup_{\|\Delta x\|\le\delta\|x\|} \frac{\|\Delta x\| \,/\, \|\varphi(x)\|}{\|\Delta x\| \,/\, \|x\|}.$$

If $\kappa\,(\varphi, x)$ is small, the problem instance is said to be *well-conditioned* and if large, *ill-conditioned*. If $\kappa\,(\varphi, x) = \infty$, the problem instance is *ill-posed*.

[9] All norms are assumed to be with respect to the relevant spaces.

[10] Here I follow the notation in [TB97], a book I highly recommend for background in numerical linear algebra.

As Turing envisaged it, the condition number measures the theoretical limits of accuracy in solving a problem. In particular, the logarithm of the condition number provides an *intrinsic* lower bound for the *loss of precision* in solving the problem instance, independent of algorithm.[11] Thus it also provides a key intrinsic parameter for specifying "input word size" for measuring computational complexity over the reals —and in connecting the two traditions of computation— as we shall see in Section 6.

If φ is differentiable, then

$$\widehat{\kappa}(\varphi, x) = \|D\varphi(x)\|$$

and

$$\kappa(\varphi, x) = \|D\varphi(x)\| \left(\|x\| / \|\varphi(x)\|\right),$$

where $D\varphi(x)$ is the Jacobian (derivative) matrix of φ at x and $\|D\varphi(x)\|$ is the operator norm of $D\varphi(x)$ with respect to the induced norms on X and Y.

Thus we have a conceptual connection between the condition number (Turing) and the derivative (Newton). Indeed, the following theorem says the matrix condition number $\kappa(A)$ is essentially the relative condition number for solving the linear system $Ax = b$. In other words, the condition number is essentially the (normed) derivative.[12]

Theorem.

1. Fix A, a real non-singular n x n matrix, and consider the map $\varphi_A : \mathbb{R}^n \to \mathbb{R}^n$ where $\varphi_A(b) = A^{-1}(b)$. Then $\kappa(\varphi_A, b) \leq \kappa(A)$ and there exist $\bar{b}$ such that $\kappa(\varphi_A, \bar{b}) = \kappa(A)$. *Thus, with respect to perturbations in b, the matrix condition number is the worst case relative condition for solving the linear system $Ax = b$.*

2. Fix $b \in \mathbb{R}^n$ and consider the partial map $\varphi_b : \mathbb{R}^{n\text{x}n} \to \mathbb{R}^n$ where, for A non-singular, $\varphi_b(A) = A^{-1}(b)$. Then for A non-singular, $\kappa(\varphi_b, A) = \kappa(A)$.

So the condition number $\kappa(A)$ indicates the number of digits that can be lost in solving the linear system. Trouble is, computing the condition number seems as hard as solving the problem itself. Probabilistic analysis can often be employed to glean information.

[11] Velvel Kahan points out that "pre-conditioning" can sometimes alter the given problem instance to a better conditioned one with the same solution. (Convert equations (8.2) to (8.1) in Turing's illustrative example.)

[12] This inspired in part the title of my paper, "Computing over the Reals: Where Turing meets Newton" [Blu04].

In the mid 1980's, in response to a challenge by Steve Smale, there was a flurry of work estimating the expected loss of precision in solving linear systems, e.g. by Adrian Ocneanu (unpublished), Eric Kostlan [Kos88], and [BS86]. The sharpest results here are due to Alan Edelman [Ede88] who showed that, with respect to the standard normal distribution, the average $\log \kappa(A) \sim \log(n)$, a result that Turing was clearly pursuing.

4 Turing's Evolving Perspective on Computing over the Reals

The Turing Machine is the canonical abstract model of a general purpose computer, studied in almost every first course in theoretical computer science. What most students of theory are not aware of, however, is that Turing defined his "machine" in order to define a theory of real computation. The first paragraph of his seminal 1936 paper [Tur36] begins and ends as follows:

> The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. ... According to my definition, a number is computable if its decimal can be written down by a machine.

Of course, the machine thus developed becomes the basis for the classical theory of computation of logic and theoretical computer science.

In the same first paragraph Turing writes, "I hope shortly to give an account of the relation of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers." As far as I know, Turing never returned to computing over the reals using this approach; *recursive analysis* (also known as *computable analysis*) was developed by others, [PER89, Wei00].

When Turing does return to computing over the reals, as in "Rounding-off Errors" written while he was preoccupied with the concrete problem of computing solutions to systems of equations, his implicit real number model is vastly different. Now real numbers are considered as individual entities and each basic algebraic operation is counted as one step. In the first section of this paper, Turing considers the "measures of work in a process:"

> It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. ... We might, for instance, count the number of additions, subtractions, multiplications, divisions, recordings of numbers, ...

This is the basic approach taken by numerical analysts, qualified as Turing also implies, by condition and round-off errors. It is also the approach taken by Mike Shub, Steve Smale and myself in [BSS89], and later with Felipe Cucker in our book, *Complexity and Real Computation* [BCSS98]. See also [Blu90], [Sma97], and [Cuc02].

5 Complexity and Real Computation in the Spirit of Turing, 1948

From the late 1930's to the 1960's, a major focus for logicians was the classification of what was computable (by a Turing Machine, or one of its many equivalents) and what was not. In the 1960's, the emerging community of theoretical computer scientists embarked on a more down to earth line of inquiry —of the computable, what was feasible and what was not— leading to a formal theory of complexity with powerful applications and deep problems, viz., the famous/infamous $P = NP?$ challenge.

Motivated to develop analogous foundations for numerical computation, [BSS89] present a model of computation over an arbitrary field R. For example, R could be the field of real or complex numbers, or $\mathbb{Z}_2$, the field of integers mod 2. In the spirit of Turing 1948, inputs to the so-called *BSS machines* are vectors over R and the basic algebraic operations, comparisons and admissible retrievals are *unit cost*. Algorithms are represented by directed graphs (or in more modern presentations, circuits) where interior nodes are labelled with basic operations, and computations flow from the input to output nodes. The *cost of a computation* is the number of nodes traversed from input to output.

As in the discrete case, *complexity* (or cost of a computation) is measured as a function of *input word size*. At the top level, input word size is defined as the vector length. When R is $\mathbb{Z}_2$, the input word size is just the bit length, as in the discrete case. Complexity classes over R, such as P, NP and EXP, are defined in natural ways. When R is $\mathbb{Z}_2$, the BSS theory of computation and complexity reduces to the classical discrete theory.

The problem of deciding whether or not a finite system of polynomial equations over R has a solution over R, so fundamental to mathematics, turns out to be a universal NP-complete problem [BSS89]. More precisely, for any field $(R, =)$, or real closed field $(R, <)$, instances of NP-complete problems over R can be coded up as polynomial systems such that an instance is a "yes" instance if and only if the corresponding polynomial system has a solution over R.[13] We call this problem the *Hilbert Nullstellensatz* over R, or HN_R.

There are many subtleties here. For example, the fact that $NP \subset EXP$ over $\mathbb{Z}_2$ is a simple counting argument on the number of possible witnesses. Over the reals or complexes, there are just too many witnesses. Indeed, it's not a priori even clear that in those cases, NP problems are decidable. Decidability in those cases follows from the decidability of HN_R by Alfred Tarski [Tar51], and membership in EXP from Jim Renegar's exponential-time decision algorithms [Ren88a].

New challenges arise: Does $P = NP?$ over the reals or complex numbers, or equivalently, is $HN_R \in P$ over those fields? And what is the relation between these questions and the classical P vs. NP challenge?

In attempt to gain new insight or to access more tools, mathematicians often position hard problems within new domains. It is tempting thus to speculate if tools of algebraic geometry might have a role to play in studying classical complexity problems. Salient transfer results: If $P = NP$ over the complex numbers, then $BPP \supseteq NP$ over $\mathbb{Z}_2$ [CSS94]. And for algebraically closed fields of characteristic 0, either $P = NP$ for all, or for none [BCSS96].

We shall return to this discussion in Section 9, but first we introduce condition into the model.

[13] The notation $(R, =)$ denotes that branching in BSS machines over R are decided by equality comparisons, while $(R, <)$ indicates that R is an ordered field and branching is now decided by order comparisons. When we talk about computing over the complex numbers or $\mathbb{Z}_2$, we are supposing our machines branch in the former sense, while over the real numbers, we mean the latter.

6 Introducing Condition into the Model: Connecting the Two Traditions

At the top level, the model of computation and complexity over the reals or complex numbers discussed above is an exact arithmetic model. But, like Turing, we note the cost of obtaining a solution (to a given accuracy) will depend on a notion of condition. Ill-conditioned problem instances will require additional resources to solve. Thus, it would seem natural to measure cost as a function of an *intrinsic* input word size which depends on the condition as well as the input dimension and desired accuracy of solution and not "on some perhaps whimsically given input precision" [Blu90, Blu91].

This perspective helps connect the two major traditions of computation.

To illustrate, consider the linear programming problem (LPP): The problem is to optimize a linear function on a polytope in Euclidean space defined by linear inequalities. The discovery of the first practical polynomial time algorithms for linear programming in the 1980's, the so-called *interior point methods*, crystalized for me and for others, flaws in the discrete theory's analysis of algorithms for problems whose natural domains are continuous.

For suppose some coefficient of an LPP instance is 1. If the instance is well-conditioned, the answer to a slightly perturbed instance should not vary wildly, nor should the cost of computation. However, when cost is measured as a function of input word size in bits, as the discrete theory prescribes, the complexity *analysis* allows much more time to solve a very slightly perturbed instance, e.g., when 1 is replaced by $1 + 10^{-10^{10}}$.

When the underlying space is the real numbers, this makes no sense.[14] What is clearly desired: a more intrinsic notion of input word size that takes into account condition.

But how to measure condition of a linear program? [15]

[14] The interior point methods are not even finite as real number algorithms. They are iterative processes. In the discrete case, a stopping rule halts the iteration in a timely manner and then a diophantine "jump" to an exact solution is performed. Such a jump does not work over the reals. For these algorithms over the real numbers, solutions to within a prescribed accuracy will have to suffice. This contrasts with George Dantzig's Simplex Method [Dan47] which, although exponential in the worst case [KM72], is a finite exact arithmetic algorithm. This conundrum yields the open question: Is linear programming *strongly polynomial* in the sense that, is there an algorithm for LPP that is polynomial in both the discrete and the exact arithmetic models?

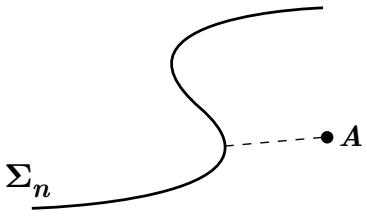
[15] In 1990, with this question in mind, I posed the challenge [Blu90] : "For more general classes of problems over continuous domains, an important research direction is to develop analogous measures

7 The Condition Number Theorem Sets the Stage

Let Σ_n be the variety of singular (*ill-posed*) $n \times n$ matrices over $\mathbb{R}$, i.e.,

$$\Sigma_n = \{A \in \mathbb{R}^{n\times n} | A \text{ is not invertible } \}.$$

We might expect that matrices close to Σ_n would be ill-conditioned while those at a distance, well-conditioned. That is what the *Condition Number Theorem* says. It provides a geometric characterization of the matrix condition number which suggests, for other computational problems, how condition could be measured.



The Condition Number Theorem.

$$\kappa\,(A) = \frac{\|A\|}{dist(A, \Sigma_n)}$$

Here $dist(A, \Sigma_n) = \inf\{\|A - B\|\,|B \in \Sigma_n\}$ where $dist$ is measured respect to the operator norm or the Frobenius norm. (The Frobenius norm is given by $\|A\|_F = \sqrt{\sum {a_{ij}}^2}$, where $A = [a_{ij}]$.)

The Condition Number Theorem is a re-interpretation of a classical theorem of Carl Eckart and Gale Young [EY36]. Although published in 1936, Turing and von Neumann seem not to have been aware of it. Velvel Kahan [Kah66], and later his student Jim Demmel [Dem87]), were the first to exploit it connecting condition with distance to ill-posedness.

Jim Renegar, inspired by the Condition Number Theorem, answers our query on how to measure condition of a linear program, and then uses this measure as a key parameter in the complexity analysis of his beautiful algorithm [Ren88b, Ren95a, Ren95b].[16]

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of condition, as well as other intrinsic parameters, and to study their relationship to computational complexity."

[16] Jim tells me that he started thinking about this after a talk I gave at at MSRI during the 1985-1986 Computational Complexity Year [Ren88b].

Recall, the linear programming problem (A, b, c) is to maximize $c^T x$ such that $Ax \geq b$. Here A is a real $m \times n$ matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let (A, b) denote the above inequalities, which also define a polytope in $\mathbb{R}^n$. In the following, we assume that $m \geq n$. We call

$$\Sigma_{m,n} = \{(A, b) | (A, b) \text{ is on the boundary between the feasible and infeasible}\}$$

the space of ***ill-posed*** linear programs.

Let $C_P(A, b) = \|(A, b)\|_F / dist_F((A, b), \Sigma_{m,n})$. Here $\|\|_F$ is the Frobenius norm, and $dist_F$ is measured with respect to that norm. Similarly, define $C_D(A, c)$ for the dual program.

Definition. The Renegar ***condition number*** [Ren95b] for the linear program (A, b, c) is given by

$$C(A, b, c) = max[C_P(A, b), C_D(A, c)].$$

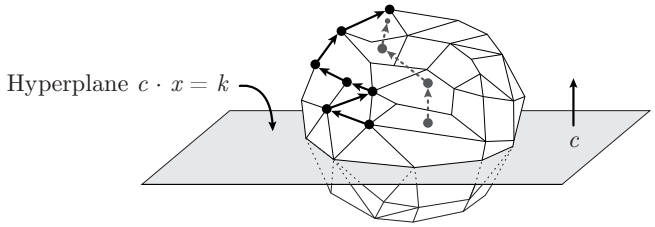
Renegar's algorithm for the LPP [Ren88b] imagines that each side of a (bounded, non-empty) polytope, given by inequalities, exerts a force inward, yielding the "center of gravity" of the polytope. Specifically, the center ξ is gotten by maximizing

$$\sum_{i=1}^{m} \ln (\alpha_i \cdot x - b_i)$$

where α_i is the ith row vector of the matrix A, and b_i is the ith entry in b.

In essence, the algorithm starts at this initial center of gravity ξ. It then follows a path of centers (approximated by Newton) generated by adding a sequence of new inequalities $c \cdot x \geq k^{(j)}$ ($k^{(j)}$, $j = 1, 2, ...$, chosen so that each successive new polytope is bounded and non-empty). Let $k^{(1)} = c \cdot \xi$. If the new center is $\xi^{(1)}$, then $k^{(2)}$ is chosen so that $k^{(1)} \leq k^{(2)} \leq c \cdot \xi^{(1)}$. And so on.

Conceptually, the hyperplane $c \cdot x = k^{(1)}$ is successively moved in the direction of the vector c, pushing the initial center ξ towards optimum.



Theorem (Renegar). $\sharp((A,b,c),\epsilon) = O(\sqrt{m}(\log m + \log C(A,b,c) + \log 1/\epsilon)$.

Here $\sharp(A,b,c),\epsilon)$ is the ***number of iterates*** for Renegar's algorithm to return a solution to within accuracy ϵ, or declare that the linear program is infeasible or unbounded.

The total number of ***arithmetic operations*** is

$$O(m^3 \log\,(mC(A,b,c)/\epsilon)).$$

For numerical analysis, it makes sense to define an algorithm to be ***polynomial time*** if there are positive constants k and c such that for all input instances x, the number of steps $T(x)$ to output with accuracy ϵ satisfies,

$$T(x,\epsilon) \leq k(dim(x) + \log \mu(x) + \log\,(1/\epsilon))^c.$$

Here $dim(x)$ is its vector length and $\mu(x)$ is a number representing the condition of x. "Bit size" has been replaced with a more intrinsic input word size.

Renegar's algorithm is polynomial time in this sense.

John Dunagan, Dan Spielman and Shang-Hua Teng [DSTT02] showed that on average, $\log C(A,b,c) = O(\log m)$, with respect to the standard normal distribution.

This implies that the expected number of iterates of Renegar's algorithm is

$$O(\sqrt{m}(\log m/\epsilon)),$$

and the expected number of arithmetic operations,

$$O(m^3 \log m/\epsilon).$$

Thus in expectation, condition has been eliminated as a parameter.

8 Condition Numbers and Complexity

The results above illustrate a two-part scheme for complexity of numerical analysis proposed by Smale [Sma97]:

1. Estimate the running time $T(x,\epsilon)$ of an algorithm as a function of $(dim(x), \log \mu(x), \log 1/\epsilon)$.

2. Estimate Prob$\{x|\mu(x) \geq t\}$, assuming a given probability distribution on the input space.

Taken together, the two parts give a probability bound on the expected running time of the algorithm, eliminating the condition μ.

So, how to estimate the (tail) probability that the condition is large?

Suppose (by normalizing) that all problem instances live within the unit sphere. Suppose Σ is the space of ill-posed instances and that condition is inversely proportional to "distance" to ill-posedness. Then the ratio of the volume of a thin tube about Σ to the volume of the unit sphere provides an estimate that the condition is large. To calculate these volumes, techniques from integral geometry [San04] and geometric measure theory [Fed69] are often used as well as volume of tube formulas of Hermann Weyl [Wey39].

This approach to estimating statistical properties of condition, was pioneered by Smale [Sma81]. Mike Shub and I used these techniques to get log linear estimates for the average loss of precision in evaluating rational functions [BS86]. It is the approach employed today to get complexity estimates in numerical analysis, see e.g., [BCL08].

Many have observed that average analysis of algorithms may not necessarily reflect their typical behavior. In 2001, Spielman and Tang introduced the concept of *smoothed analysis* which, interpolating between worst and average case, suggests a more realistic scheme [ST01].

The idea is to first smooth the complexity measure locally. That is, rather than focus on running time at a problem instance, compute the average running time over all slightly perturbed instances. Then globally compute the worst case over all the local "smoothed" expectations.

More formally, assuming a normal distribution on perturbations, *smoothed running time* (or *cost*) is defined as

$$T_s(n, \epsilon) = \sup_{\bar{x} \in \mathbb{R}^n} \mathbf{E}_{x \sim \mathcal{N}(\bar{x}, \sigma^2)} T(x, \epsilon).$$

Here $\mathcal{N}(\bar{x}, \sigma^2)$ designates the distribution of $\bar{x}$ with variance σ^2, and $x \sim \mathcal{N}(\bar{x}, \sigma^2)$ means x is chosen according to this distribution.

If $\sigma = 0$, smoothed cost reduces to worst case cost; if $\sigma = \infty$ then we get the average cost.

Part 2 of Smale's scheme is now replaced by:

2*. Estimate $\sup_{\bar{x}\in\mathbb{R}^n} \text{Prob}_{x\sim\mathcal{N}(\bar{x},\sigma^2)}\{x|\mu(x) \geq t\}$

Now, 1. and 2*. combine to give a *smoothed complexity analysis* eliminating μ. Estimating 2* employs techniques described above, now intersecting tubes about Σ with discs about $\bar{x}$ to get local probability estimates.

Dunagan, Spielman and Teng [DSTT02] also give a smoothed analysis of Renegar's condition number. Assuming $\sigma \leq 1/m)$, the smoothed value of $\log C(A, b, c)$ is $O(\log m/\sigma)$. This in turn yields smoothed complexity analyses of Renegar's linear programming algorithm. For the number of iterates:

$$\sup_{\|(\bar{A},\bar{b},\bar{c})\|\leq 1} \mathbf{E}_{(A,b,c)\sim\mathcal{N}(\bar{A},\bar{b},\bar{c}),\,\sigma^2 I)} \sharp((A, b, c), \epsilon) = O(\sqrt{m}(\log m/\sigma\epsilon)).$$

And for the smoothed arithmetic cost:

$$T_s(m, \epsilon) = O(m^3 \log m/\sigma\epsilon).$$

9 What does all this have to do with the classical P vs. NP challenge?

This is (essentially) the question asked by Dick Karp in the Preface to our book, *Complexity and Real Computation* [BCSS98].

As noted in Section 5, the problem of deciding the solvability of finite polynomial systems, HN, is a universal NP-Complete problem. Deciding quadratic (degree 2) polynomial systems is also universal [BSS89]. If solvable in polynomial time over the complex numbers $\mathbb{C}$, then any classical NP problem is decidable in probabilistic polynomial time in the bit model [CSS94]. While premise and conclusion seem unlikely, understanding the complexity of $HN_{\mathbb{C}}$ is an important problem in its own right. Much progress has been made here, with condition playing an essential role.

During the 1990's, in a series of papers dubbed "Bezout I-V," Shub and Smale showed that the problem of finding *approximate zeros* to "square" complex polynomial systems can be solved probabilistically in polynomial time on average over $\mathbb{C}$ [SS94, Sma97].

The notion of an approximate zero means that Newton's method converges quadratically, immediately, to an actual zero. Achieving output accuracy to within ϵ requires only $\log\log 1/\epsilon$ additional steps.

For a system $f = (f_1, ..., f_n)$ of n polynomials in n variables over $\mathbb{C}$, the Shub-Smale homotopy algorithm outputs an approximate solution in cN^5 arithmetic steps. Here N is the number of coefficients in the system and c is a universal constant. For quadratic systems, this implies that the number of arithmetic operations is bounded by a polynomial in n (since in this case, $N \leq n^3$).

The trouble is, the algorithm is non-uniform. The starting point depends on the degree $d = (d_1, ..., d_n)$ of the systems, and has a probability of failure.

In *Math Problems for the Next Century*, Smale [Sma00] posed the following question (his 17th problem):

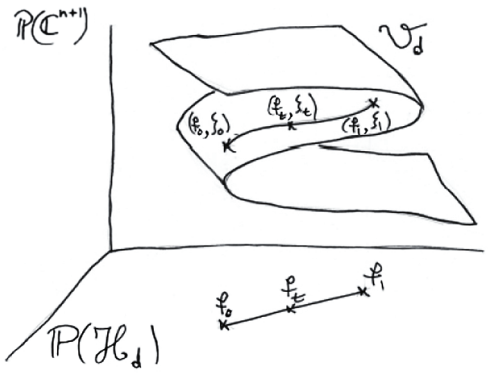
> Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?

Building on the Shub-Smale homotopy algorithm, there has been exciting progress here. As with Shub-Smale, the newer path-following algorithms approximate a path, starting from a given pair (g, ξ) where $\xi = (\xi_0, ..., \xi_n)$ and $g(\xi) = 0$, and output (f, ξ^*) where ξ^* is an approximate zero of f. [17]

The idea is to lift the "line" segment

$$f_t = (1-t)g + tf,\ t \in [0,1]$$

to the variety $\mathcal{V}_d = \{(f, \xi) | f(\xi) = 0\}$. Note that $f_0 = g$, and that $f_1 = f$, the system to be solved.



Sketch by Jean-Pierre Dedieu

[17] The polynomial systems considered are homogeneous (for good scaling properties) and the ambient spaces, projective (for compactness). Here g and f are systems of n homogeneous polynomials in $n+1$ variables of degree $d = (d_1, ..., d_n)$.

By the *implicit function theorem*, this lift exits if the line does not intersect the *discriminant variety* of polynomial systems that have zeros with multiplicities. These algorithms approximate the lifting.

To steer clear of the *singular variety* of ill-posed pairs (i.e., pairs (f, ξ) where ξ is a multiple zero of f), they take into account the condition along the way. The condition will determine appropriate step size at each stage and hence running time.

Major considerations are: how to choose good initial pairs, how to construct good partitions (for approximating the path and steering clear of the singular variety), how to define measures of condition.

In two additional papers Bezout VI [Shu09] and Bezout VII [BS09], Shub and Carlos Beltrán present an Adaptive Linear Homotopy (ALH) algorithm with incremental time step dependent on the inverse of a normalized condition number squared.

Beltrán and Luis Miguel Pardo [BP11] show how to compute a random starting pair yielding a uniform Las Vegas algorithm, polynomial time on average. Utilizing the numerical algebraic geometry package Macaulay2, the randomized algorithm was implemented by Beltrán and Anton Leykin [BL12].

Bürgisser and Cucker give a hybrid deterministic algorithm which is almost polynomial time on average [BC11]. Let D be the maximum of the degrees, d_i. Then for $D \leq n$ the algorithm is essentially the ALH of Beltrán and Shub with initial pair:

$$g = (g_1, ..., g_n) \text{ where } g_i(x_0, ..., x_n) = 1/\sqrt{2n}({x_0}^{d_i} - {x_i}^{d_i}) \text{ and } \zeta = (1, ..., 1).$$

And for $D > n$, the algorithm calls on Renegar's symbolic algorithm [Ren89].

The algorithm takes $N^{(loglogN)}$ arithmetic steps on average, coming close to answering Smale's question in the affirmative.

For a tutorial on the subject of this section, see [BS12].

10 Postscript: Who Invented the Condition Number?

It is clear that Alan Turing was first to explicitly formalize a measure that would capture the informal notion of condition (of solving a linear system) and to call this measure a *condition number*. Formalizing what's "in the air" serves to illuminate essence and chart new direction. However, ideas in the air have many proprietors.

To find out more about the origins of the *spectral condition number*, I emailed a number of numerical analysts.

I also looked at many original papers. The responses I received, and related readings, uncover a debate concerning the origins of the (concept of) condition number not unlike the debate surrounding the origins of the general purpose computer —with Turing and von Neumann figuring central to both. (For an insightful assessment of the latter debate see Mike Shub's article, "Mysteries of Mathematics and Computation" [Shu94].)

Gauss himself [Gau03] is referenced for considering perturbations and preconditioning. Pete Stewart points to Helmut Wittmeyer [Wit36] in 1936 for some of the earliest perturbation bounds where products of norms appear explicitly. In 1949, John Todd [Tod50] explicitly focused on the notion of condition number, citing Turing's N and M condition numbers and the implicit von Neumann-Goldstine measure, which he called the P-condition number (P for Princeton).

Beresford Parlett tells me that "the notion was 'in the air' from the time of Turing and von Neumann et. al.," that the concept was used by George Forsythe in a course he took from him at Stanford early in 1959 and that Wilkinson most surely "used the concept routinely in his lectures in Ann Arbor [summer, 1958]." The earliest explicit definition of the spectral condition number I could find in writing was in Alston Householder's 1958 SIAM article [Hou58] (where he cites Turing) and then in Wilkinson's book [Wil63], p.91).

By far, the most informative and researched history can be found in Joe Grcar's 76 page article, "John von Neumann's Analysis of Gaussian Elimination and the Origins of Modern Numerical Analysis" [Grc11]. Here he uncovers a letter from von Neumann to Goldstine (dated January 11, 1947) that explicitly names the ratio of the extreme singular values as ℓ. Why this was not included in their paper [vNG47] or made explicit

in their error bounds is a mystery to me.[18] Grcar chalks this up to von Neumann using his speeches (e.g. [vN89]) to expound on his ideas, particularly those given to drum up support for his computer project at the Institute for Advanced Study.[19]

Grcar's article definitely puts von Neumann at center stage. Of von Neumann's role as a key player in this area there is no doubt. However, Grcar also implies, that Turing's work on rounding error, and the condition number, was prompted by Turing's meeting with von Neumann in Princeton in January 1947. Indeed, prominently on page 630 he says, "No less than Alan Turing produced the first derivative work from the inversion paper."

This flies in the face of all we know about Alan Turing's singular individuality, both in personality and in research. In their personal remembrances of Turing, both Wilkinson [Wil71], who worked closely with him at the NPL in Teddington, and Max Newman [New55], earlier at Cambridge and Bletchley and later in Manchester, point to Turing's "strong predilection for working everything out from first principles, usually in the first instance without consulting any previous work on the subject, and no doubt it was this habit which gave his work that characteristically original flavor."

It also flies in the face of fact. As recounted by Wilkinson, Turing's experience with his team at the NPL, prior to meeting von Neumann in Princeton in 1947, was the stimulus for his paper:

> ... some time after my arrival [at the NPL in 1946], a system of 18 equations arrived in Mathematics Division and after talking around it for some time we finally decided to abandon theorizing and to solve it. ... The operation was manned by [Leslie] Fox, [Charles] Goodwin, Turing, and me, and we decided on Gaussian elimination with complete pivoting. Turing was not particularly enthusiastic, partly because he was not an experienced performer on a desk machine and partly because he was convinced that it

[18] Joe was also kind enough to illuminate for me in detail how one could unravel von Neumann's and Goldstine's error analysis for the general case in their paper.

Many authors have cited the obtuseness and non-explicitness. For example, Edelman, in his PhD thesis [Ede89], recasts von Neumann's and Goldstine's ideas in modern notation given "the difficulty of extracting the various ideas from their work" and cites Wilkinson's referring to the paper's "indigestibility."

[19] An unpublished and undated paper by Goldstine and von Neumann [GvN63] containing material presented by von Neumann in various lectures going back to 1946, but clearly containing later perspectives as well, explicitly singles out (on page 14) ℓ as the "figure of merit." Also interesting to me, in the same paragraph, are the words "loss of precision" connected to the condition number, possibly for the first time.

> would be a failure. ... the system was mildly ill-conditioned, the last equation had a coefficient of order 10^{-4} ... and the residuals were ... of order 10^{-10}, that is of the size corresponding to the exact solution rounded to ten decimals. ...
>
> ... I'm sure that this experience made quite an impression on him and set him thinking afresh on the problem of rounding errors in elimination processes. About a year later he produced his famous paper "Rounding-off errors in matrix process" ...[20]

Velvel Kahan (also a Turing Award recipient), in his 1966 paper [Kah66] and in a lengthy phone conversation (August 2011), asserts that von Neumann and Goldstine were misguided in their approach to matrix inversion (by computing A^{-1} from the formula $A^{-1} = (A^T A)^{-1} A^T$).

Kahan's assessment of Turing is a fitting conclusion to this paper:

> A more nearly modern error-analysis was provided by Turing (1948) in a paper whose last few paragraphs foreshadowed much of what was to come, but his paper lay unnoticed for several years until Wilkinson began to publish the papers which have since become a model of modern error-analysis.[21]

11 Acknowledgements

Much of what I know about Alan Turing's life comes from Andrew Hodges' definitive biography, *Alan Turing: the Enigma* [Hod92]. I had the pleasure to discuss the material in this paper with Hodges at the Turing Centenary Conference, CiE 2012 - *How the World Computes*, at the University of Cambridge, England. I thank the organizers of the conference, and in particular, Barry Cooper, for inviting me to speak.

Mike Shub and Steve Smale are the architects, sans pareil, of the most fundamental results in the area of complexity and real computation. My perspectives are clearly

[20] On June 23, 2012, at the Turing Centenary Conference in Cambridge, England, Andrew Hodges told me that in Turing's 1945 paper on the design of the ACE programmable computer (classified for many years) there is a discussion of the need for accurate numerical computations, and Gaussian Elimination is one of the examples given.

[21] Kahan is referring to *backward error analysis* which, rather than estimating the errors in a computed solution of a given problem instance (i.e., *forward error analysis*), estimates the closeness of a nearby problem instance whose exact solution is the same as the approximate computed solution of the original. Grcar [Grc11] also points to the von Neumann and Goldstine paper as a precursor to this notion as well.

fashioned by my long time collaboration with them. Jim Renegar, who constantly amazes with his deep insights and beautiful algorithms, has been a constant sounding board for me.

I would like to thank the Mathematics Department of the University of Toronto for inviting me to be Dean's Distinguished Visitor during the Fields Institute Program on the Foundations of Computational Mathematics, Fall 2009. There I co-taught a course with my collaborator Felipe Cucker, and learned about new advances concerning condition from lectures given by Felipe and Peter Bürgisser whose influence is apparent here. Their forthcoming book, *Condition*. [BC12], is certain to become the bible in this area. For an excellent survey paper see [Bür10].

Jean-Pierre Dedieu, who celebrated his 60th birthday during the Fields program, inspired us all with his beautiful lectures on *Complexity of Bezout's Theorem and the Condition Number*. We lost Jean-Pierre during the Centenary Year of Alan Turing. I dedicate this paper to him.

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