16-782
Planning & Decision-making in Robotics

Planning under Uncertainty: Minimax Formulation

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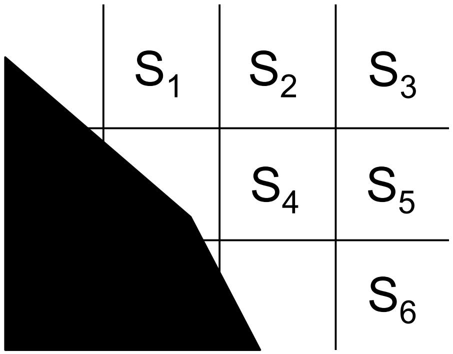
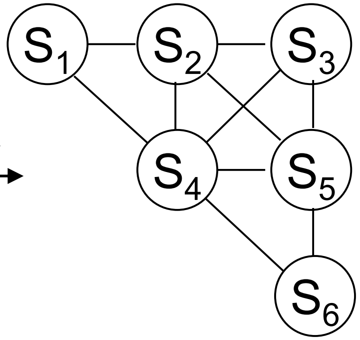
Uncertainty in Robotics

- So far our planners assumed no uncertainty
  - execution is perfect



convert into a graph →



search the graph for a least-cost path from s_{start} to s_{goal}

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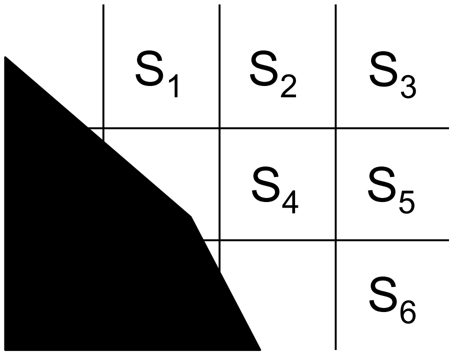

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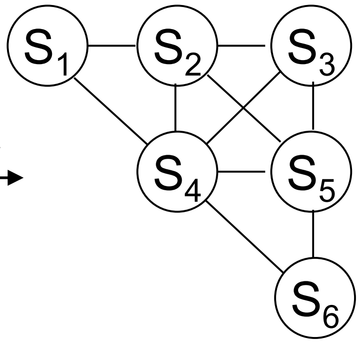


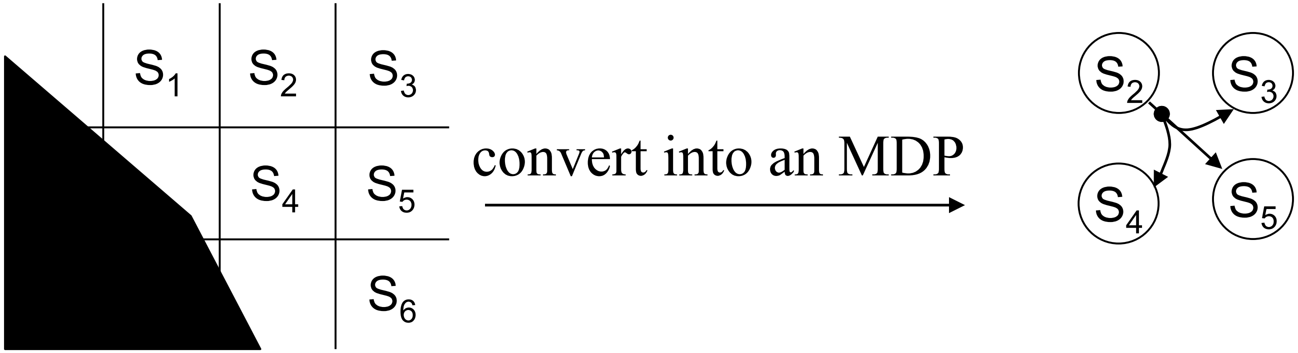
search the graph for a least-cost path from s_{start} to s_{goal}

- Any deviations from the plan are dealt by re-planning
- Could be quite suboptimal and sometimes dangerous
  - planning a path along cliff does not take into account slippage
  - others examples???

Uncertainty in Robotics

- Modeling uncertainty in execution <u>during planning</u>

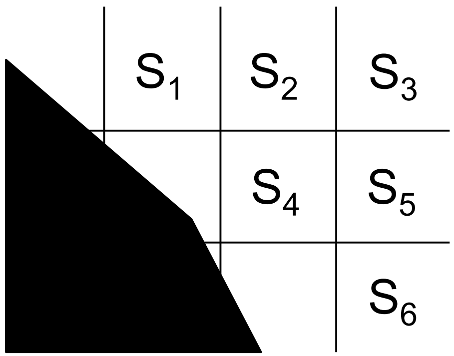
Markov Decision Processes (MDP)



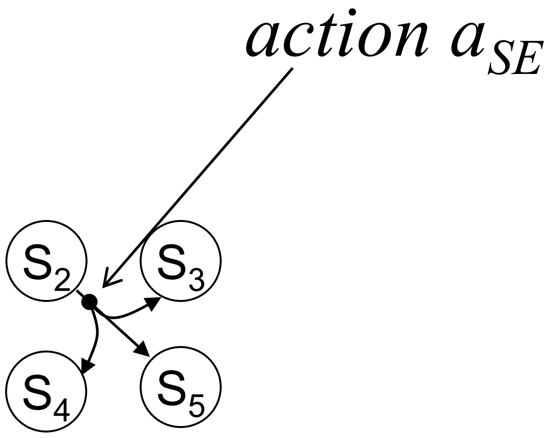
- at least one action in the graph has more than one outcome
- each outcome is associated with probability and cost

- Modeling uncertainty in execution <u>during planning</u>

Markov Decision Processes (MDP)



convert into an MDP →



- at least one action in the graph has more than one outcome

- each outcome is associated with probability and cost

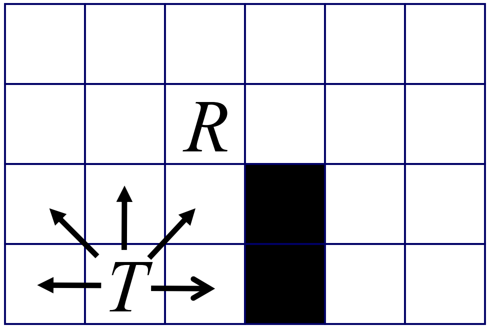
example: $s_3, s_4, s_5 \in succ(s_2, a_{SE})$,

$$P(s_5|a_{se},s_2) = 0.9, \quad c(s_2,a_{se},s_5) = 1.4$$
$$P(s_3|a_{se},s_2) = 0.05, \quad c(s_2,a_{se},s_3) = 1.0$$
$$P(s_4|a_{se},s_2) = 0.05, \quad c(s_2,a_{se},s_4) = 1.0$$

Moving-Target Search Example

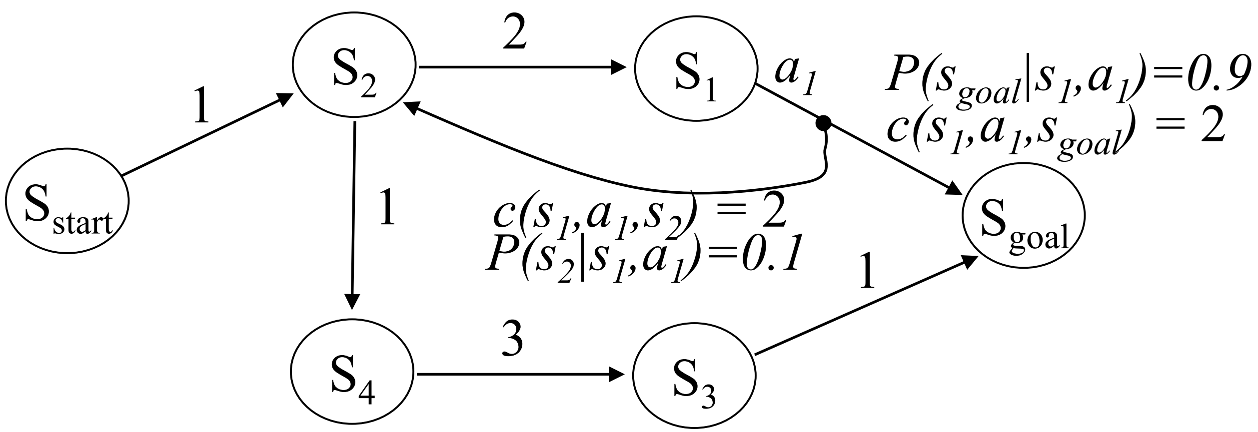
- Uncertainty in the target moves

- What is a state-space and action space?



Planning in MDPs

- What plan to compute?
  - Plan that minimizes the worst-case scenario (minimax plan)
  - Plan that minimizes the expected cost



- Without uncertainty, plan is a single path:
  a sequence of states (a sequence of actions)
- In MDPs, plan is a policy π:
  mapping from a state onto an action

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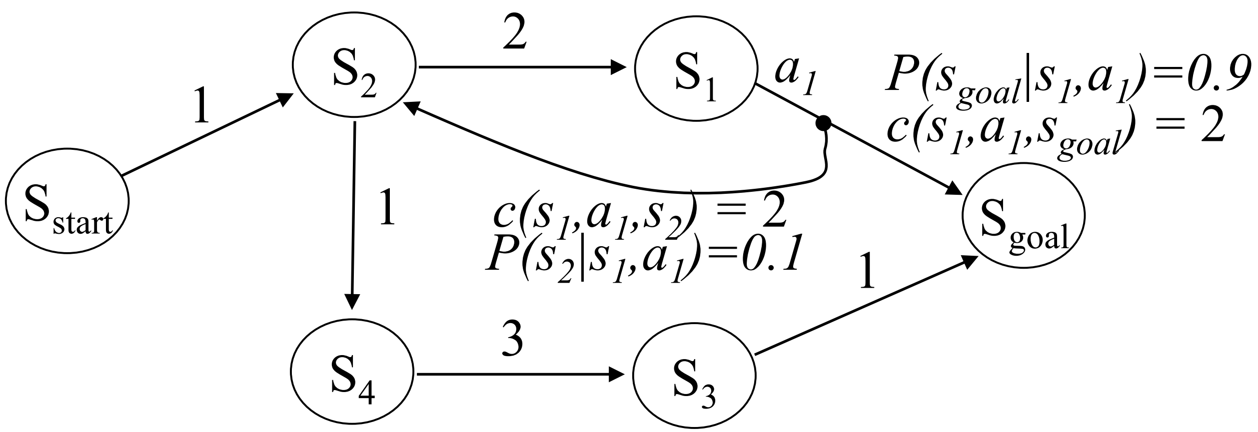

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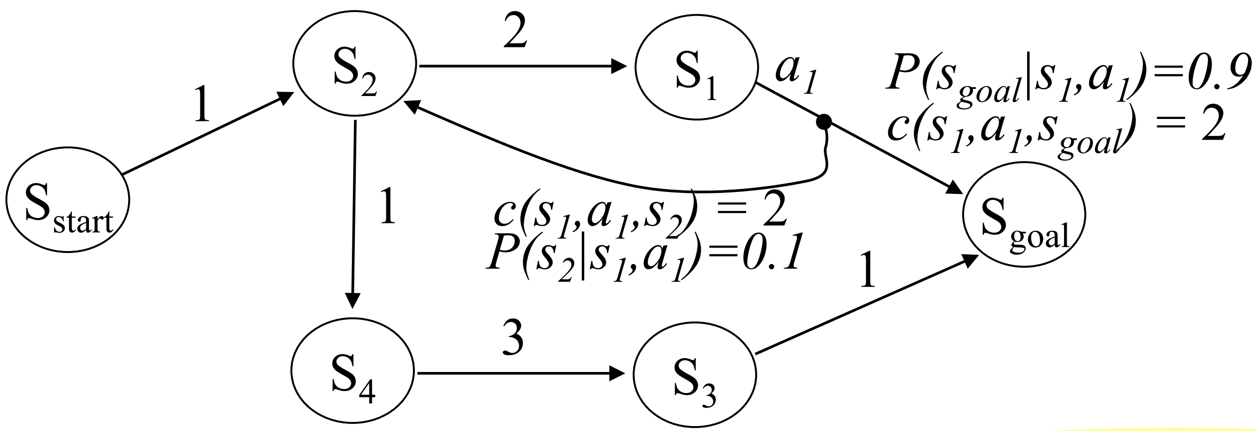
Why?

Minimax Formulation



What is the best plan?

- Optimal policy π^*:
  minimizes the *worst* cost-to-goal
  $\pi^* = argmin_{\pi}\ max_{outcomes\ of\ \pi}\{cost\text{-}to\text{-}goal\}$

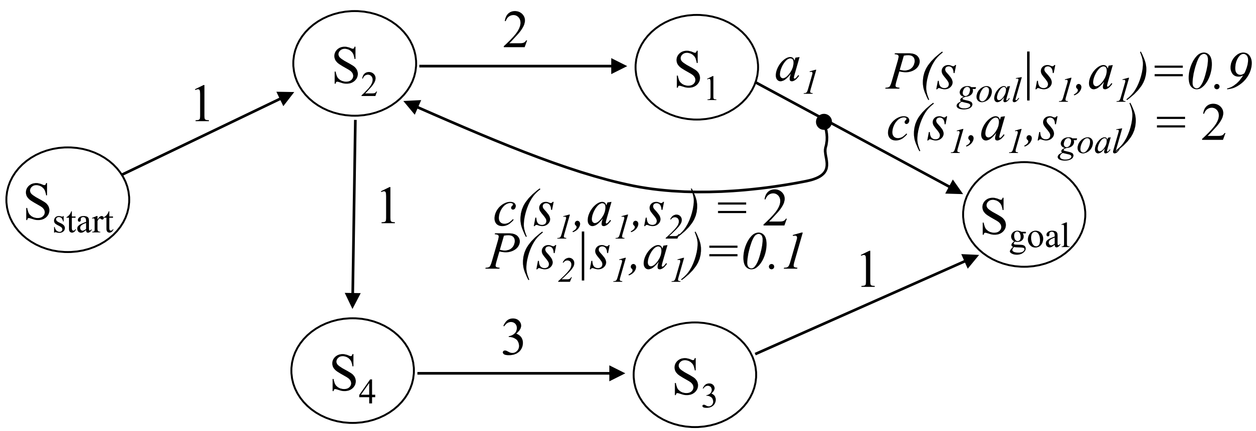
- worst cost-to-goal for π_1=(go through s_4) is:

  1+1+3+1 = 6

- worst cost-to-goal for π_2=(try to go through s_1) is:

  $1+2+2+2+2+2+2 + \ldots = \infty$

Minimax Formulation



- Optimal policy π^*:
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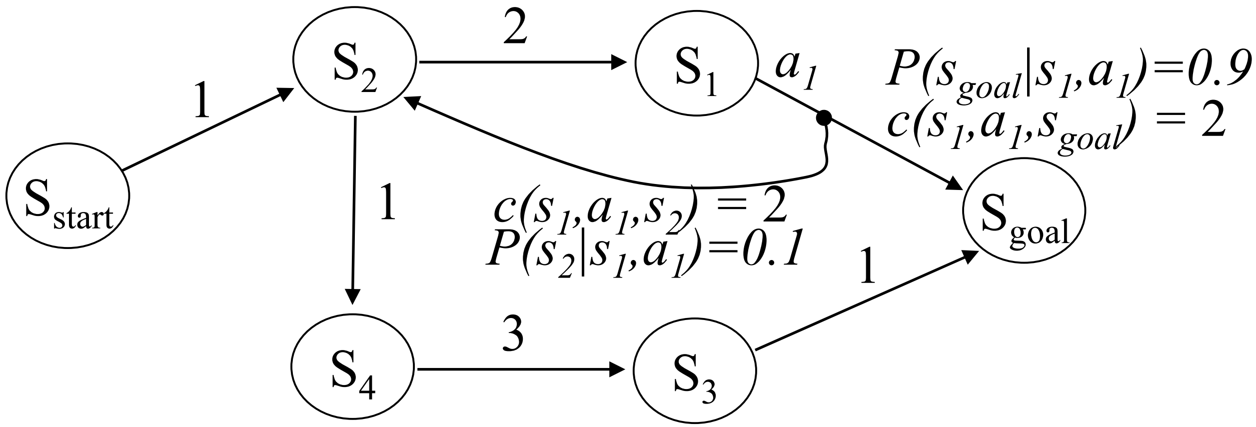
  $\pi^* = argmin_{\pi}\, max_{outcomes\ of\ \pi}\{cost\text{-}to\text{-}goal\}$

- Optimal minimax policy π^* = (go through s_4) =
$[\{s_{start}, a_{ne}\}, \{s_2, a_{south}\}, \{s_4, a_{east}\}, \{s_3, a_{ne}\}, \{s_{goal}, null\}]$

Computing Minimax Plans



- Minimax backward A*:

$g(s_{goal}) = 0;$ all other g-values are infinite; $OPEN = \{s_{goal}\};$
while(s_{start} *not expanded*)
remove s with the smallest $[f(s) = g(s)+h(s)]$ from *OPEN*;
insert s into *CLOSED*;
for every s' s.t $s \in succ(s', a)$ for some a and s' not in *CLOSED*
if $g(s') > \max_{u \in succ(s', a)} c(s',u) + g(u)$
$g(s') = \max_{u \in succ(s', a)} c(s',u) + g(u);$
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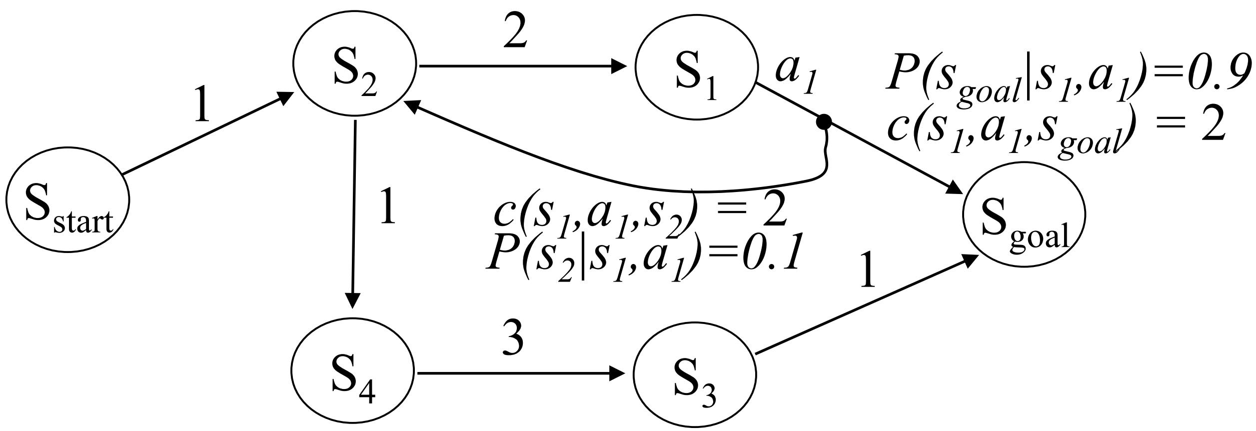

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      insert s' into *OPEN*;

reduces to usual backward A if no uncertainty in outcomes*

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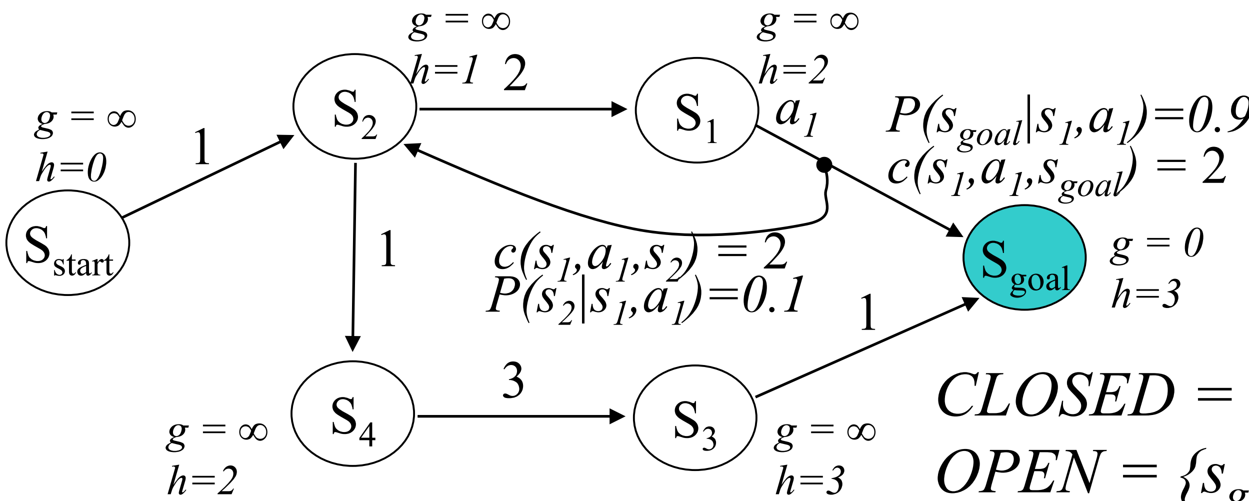

$CLOSED = \{\}$

$OPEN = \{s_{goal}\}$

next state to expand: s_{goal}

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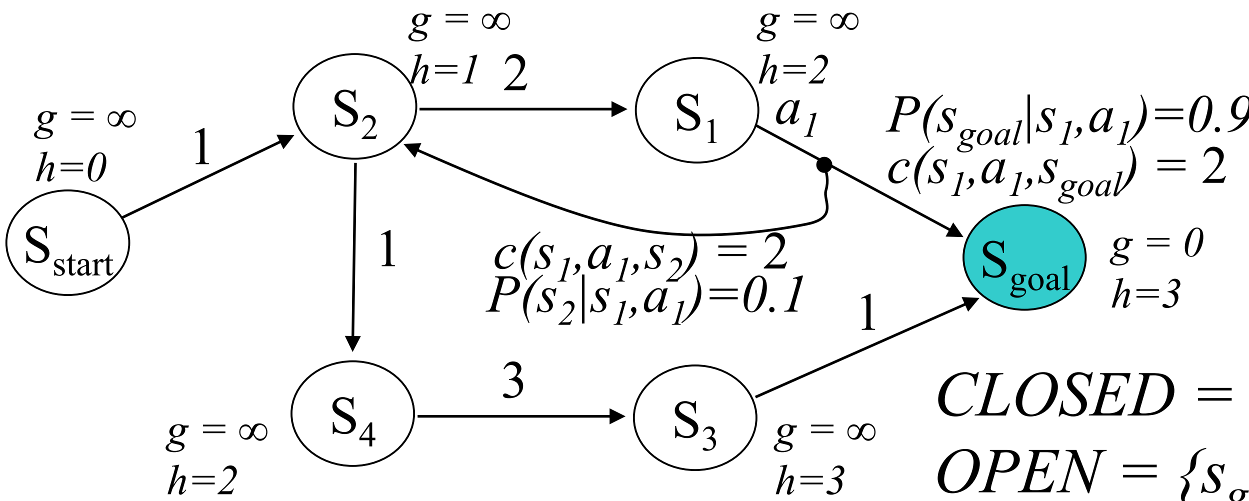

CLOSED = {}

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next state to expand: s_{goal}

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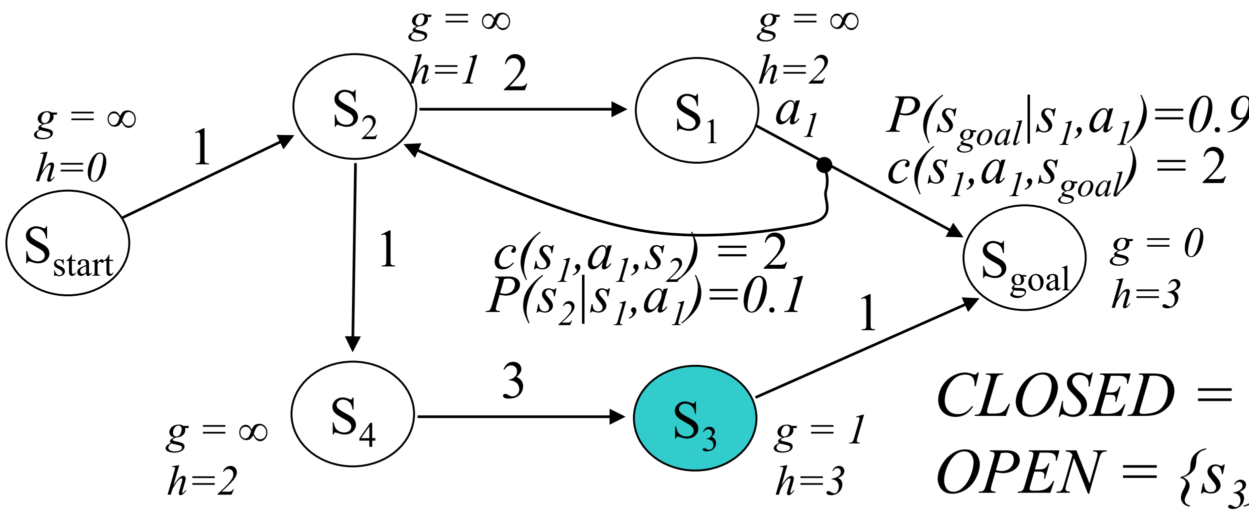
  insert s' into *OPEN*;

After s_{goal} *expanded, what are* $g(s_3)$ *and* $g(s_1)$*?*

Computing Minimax Plans



$CLOSED = \{s_{goal}\}$

$OPEN = \{s_3\}$

next state to expand: s_3

- Minimax backward A*:

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while(s_{start} *not expanded*)

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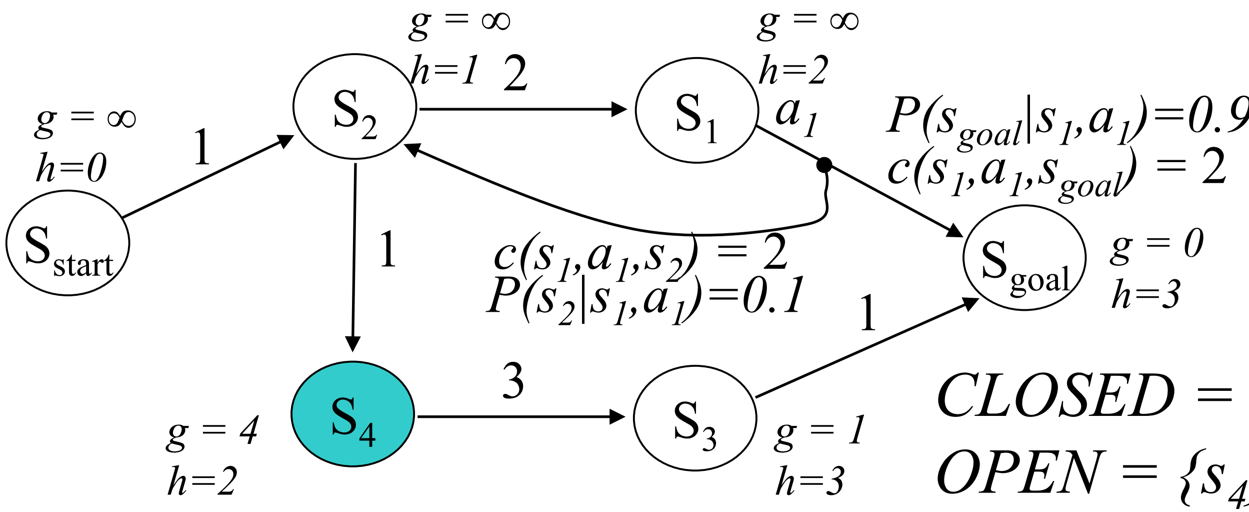

$CLOSED = \{s_{goal}, s_3\}$

$OPEN = \{s_4\}$

next state to expand: s_4

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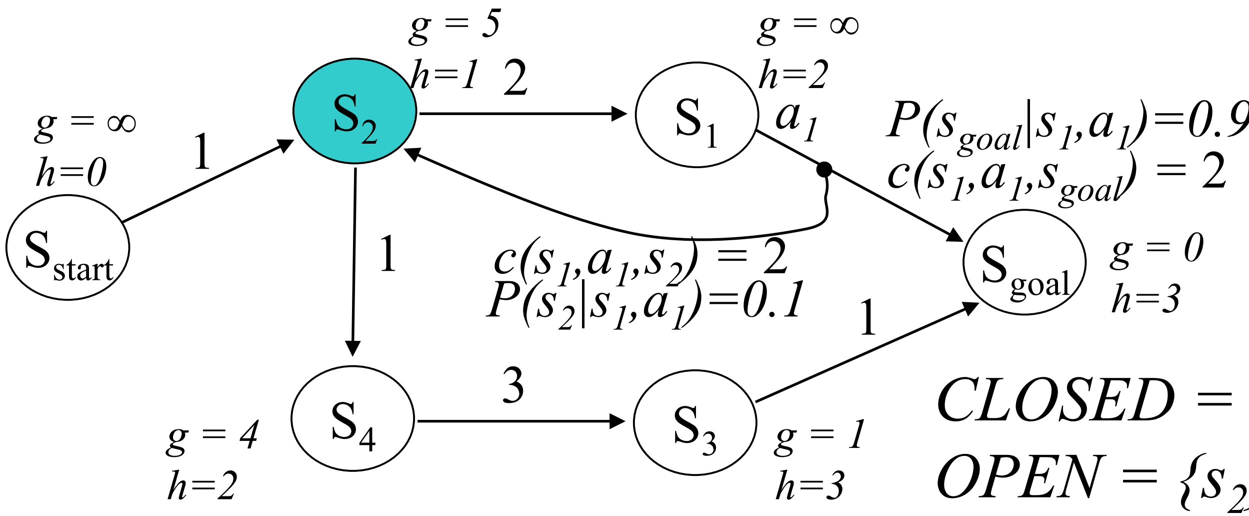

$CLOSED = \{s_{goal}, s_3, s_4\}$

$OPEN = \{s_2\}$

next state to expand: s_2

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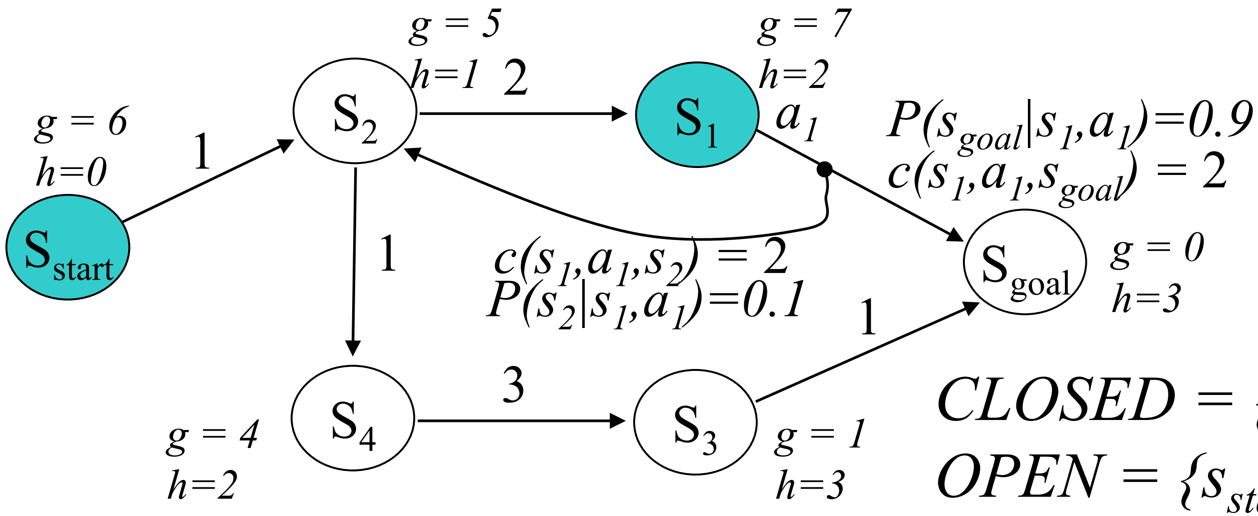

$CLOSED = \{s_{goal}, s_3, s_4, s_2\}$
$OPEN = \{s_{start}, s_1\}$
next state to expand: s_{start}

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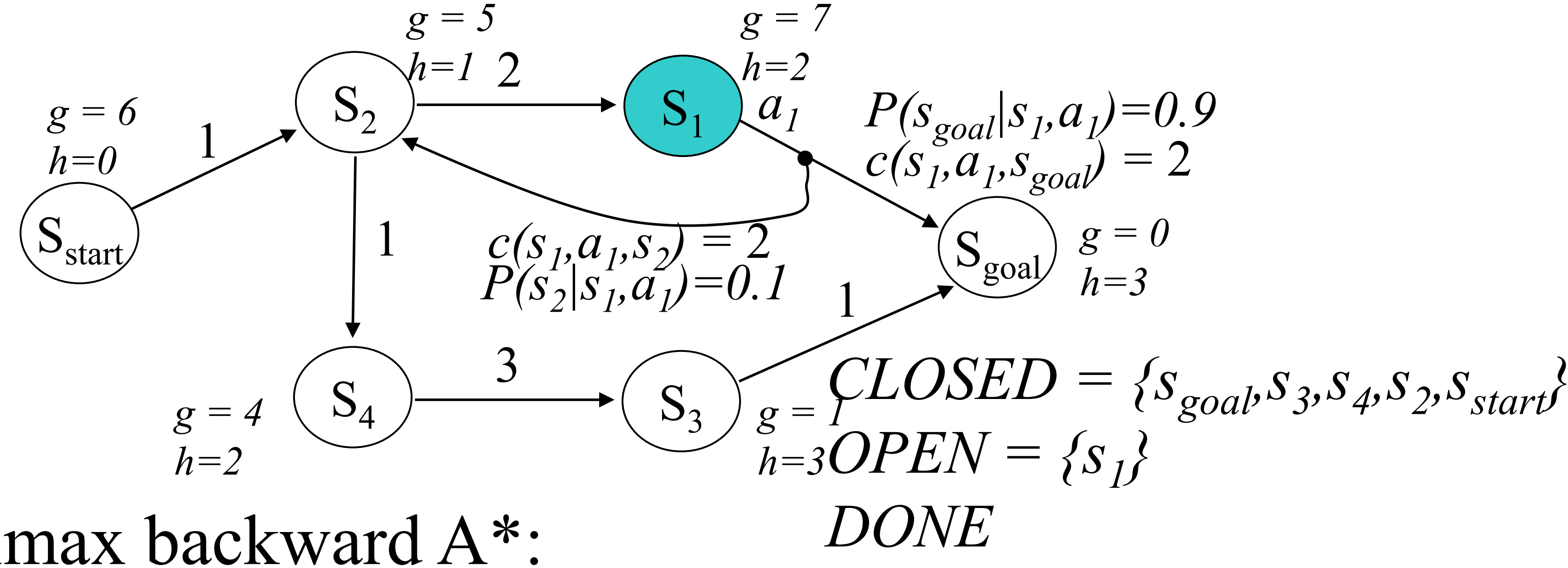

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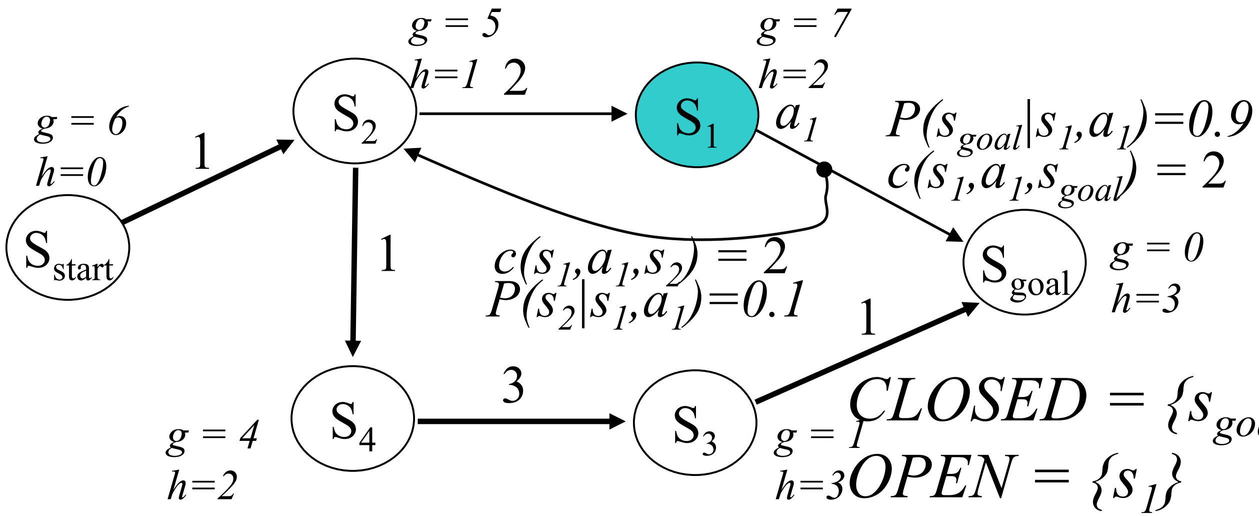

$CLOSED = \{s_{goal},s_3,s_4,s_2,s_{start}\}$

$OPEN = \{s_1\}$

DONE

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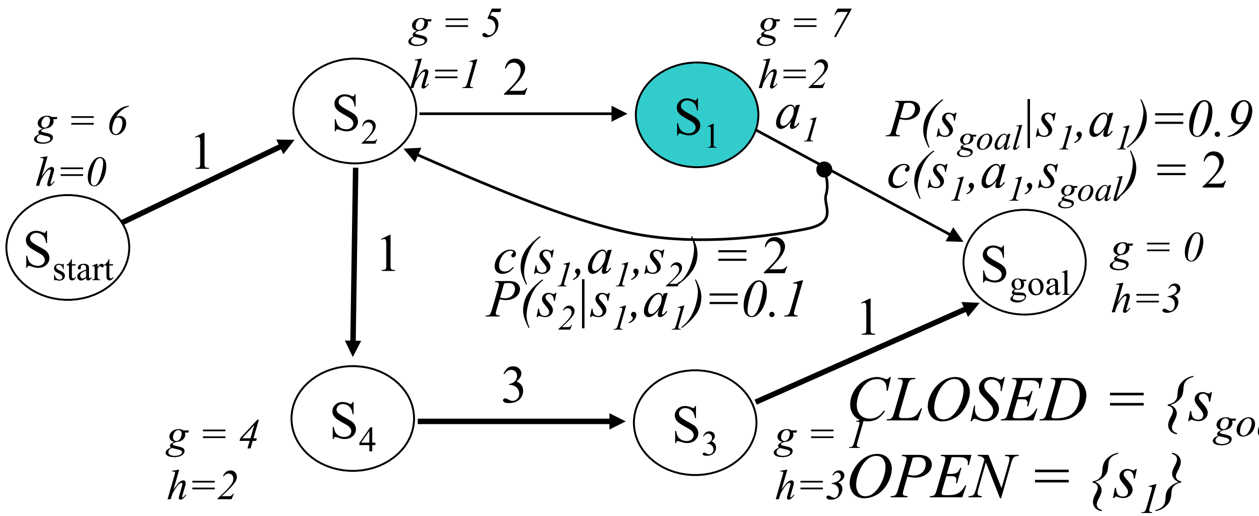
in this example, the computed policy is a path, but in general it is a Directed Acyclic Graph

Computing Minimax Plans



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What are its branches?
Why no cycles?

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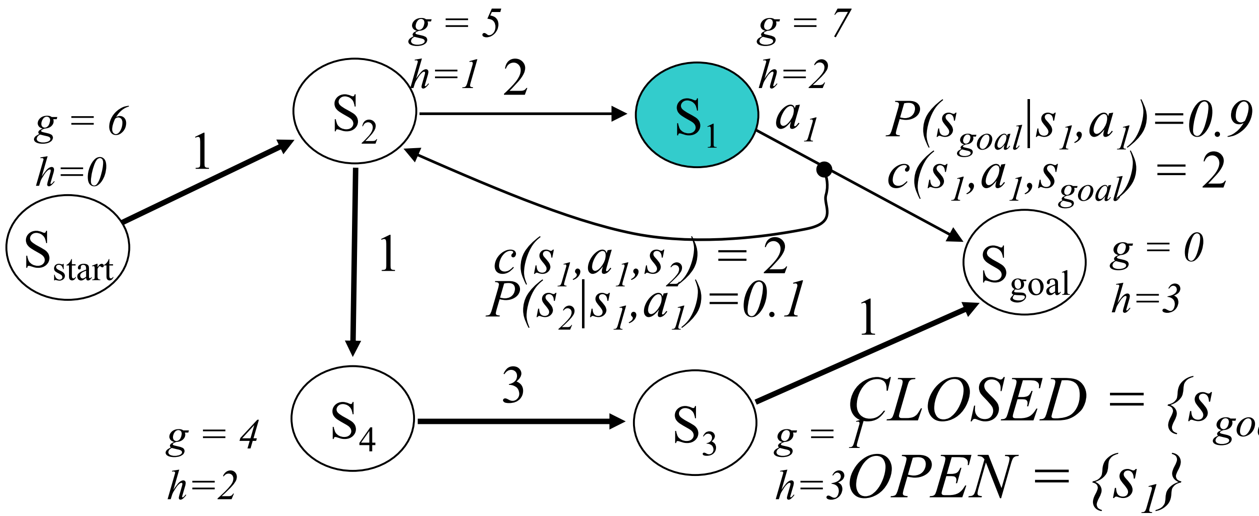
Computing Minimax Plans



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  if $g(s') > max_{u \in succ(s', a)}\ c(s',u) + g(u)$

  $g(s') = max_{u \in succ(s', a)}\ c($

  insert *s’* into *OPEN*;

Minimax A guarantees to find an optimal path, and never expands a state more than once, provided heuristics are consistent (just like A*)*

Computing Minimax Plans

- Pros/cons of minimax plans
  - robust to uncertainty
  - overly pessimistic
  - harder to compute than normal paths
    - especially if backwards minimax A* does not apply
    - even if backwards minimax A* does apply, still more expensive than computing a single path with A* (heuristics are not guiding well)

Why?

What You Should Know…

- What is Markov Decision Processes (MDP)
- Minimax formulation of planning under uncertainty
- The operation of Minimax backward A*
- Pros and cons of planning with Minimax formulation