10-423/10-623 Generative AI

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Variational Autoencoders (VAEs)

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Lecture 9
Feb. 14, 2024

Reminders

- Homework 2: Generative Models of Images
  - Out: Thu, Feb 8
  - Due: Tue, Feb 20 at 11:59pm

Recall…

KL DIVERGENCE

KL Divergence

Recall…

- *Definition*: for two distributions q(x) and p(x) over x ∈ $\mathcal{X}$, the **KL Divergence** is:

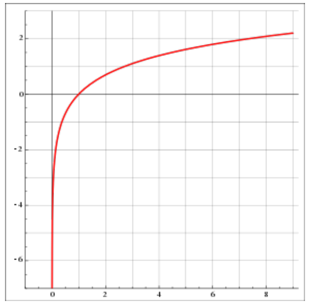
$$\text{KL}(q||p) = E_{q(x)}\left[\log \frac{q(x)}{p(x)}\right] = \begin{cases} \sum_x q(x) \log \frac{q(x)}{p(x)} \\ \int_x q(x) \log \frac{q(x)}{p(x)} dx \end{cases}$$

- *Properties*:
  - KL(q || p) measures the **proximity** of two distributions q and p
  - KL is **not** symmetric: KL(q || p) ≠ KL(p || q)
  - KL is minimized when q(x) = p(x) for all x ∈ $\mathcal{X}$

Recall...

$$\mathrm{KL}(q||p) = E_{q(x)}\left[\log \frac{q(x)}{p(x)}\right]$$

KL Divergence



Understanding the Behavior of KL as an objective function

Example 1: Keeping all else constant, consider the effect of a particular x’ on KL(q || p)

x’	q(x’)	p(x’)	q(x’) log(q(x’)/p(x’))	effect on KL(q || p)
1	0.9	0.9	0	no increase
2	0.9	0.1	1.97	big increase
3	0.1	0.9	-0.21	little decrease
4	0.1	0.1	0	little decrease

KL **does** insist on good approximations for values that have **high** probability in q

KL **does not** insist on good approximations for values that have **low** probability in q

Example 2: Which q distribution minimizes KL(q || p)?

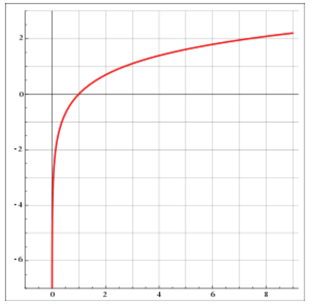
$$\mathbf{p} = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} \quad \mathbf{q}^{(1)} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad \mathbf{q}^{(2)} = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} \quad \mathbf{q}^{(3)} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

Q: If we’re minimizing KL, why not return $q^{(3)}$?

A: Because it’s not a distribution!

$\text{KL}(q||p) = E_{q(x)}\left[\log \frac{q(x)}{p(x)}\right]$

KL Divergence



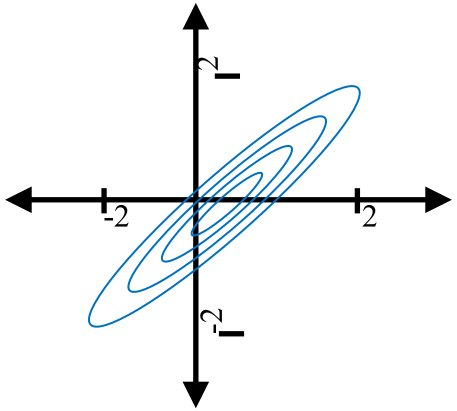
Recall...

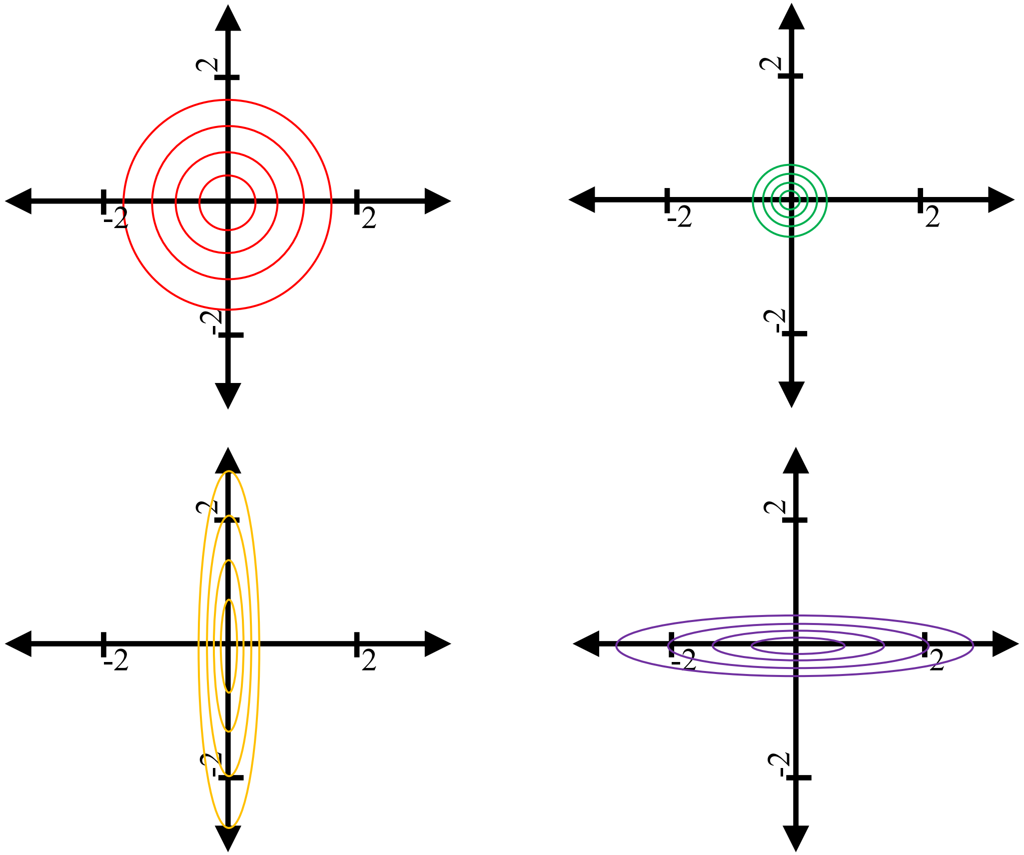
Understanding the Behavior of KL as an objective function

Example 3: Which q distribution minimizes KL(q || p)?

$$p(\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu} = [0,0]^T, \boldsymbol{\Sigma})$$

$$q(x_1, x_2) = \mathcal{N}_1(x_1 \mid \mu_1, \sigma_1^2)\mathcal{N}_2(x_2 \mid \mu_2, \sigma_2^2)$$





VARIATIONAL DIFFUSION MODELS AND VARITIONAL AUTOENCODERS (VAES)

Diffusion Models

- Next we will consider (1) **diffusion models** and (2) **variational autoencoders (VAEs)**
  – Although VAEs came first, we're going to dive into diffusion models since they will receive more of our attention
- The steps in defining these models is roughly:
  – Define a probability distribution involving Gaussian noise
  – Use a variational lower bound as an objective function
  – Learn the parameters of the probability distribution by optimizing the objective function
- So what is a variational lower bound?

HIGH-LEVEL INTRO TO VARIATIONAL INFERENCE

Variational Inference

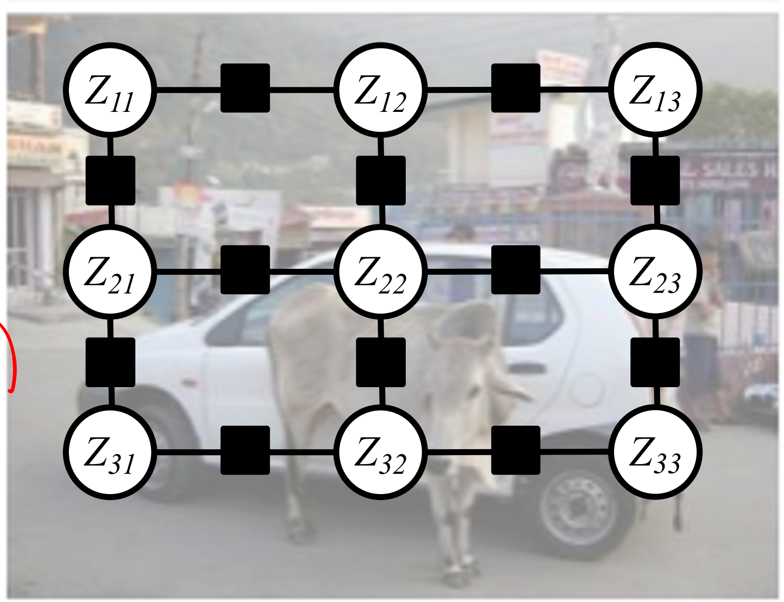
Problem:

– For observed variables **x** and latent variables **z**, estimating the posterior p(**z** | **x**) is intractable

Z = pixel labels for sem. seg.

X = image

$p_\alpha(x,z)$ follow $p(x|z) = f_\alpha(z)$
where $z \sim \text{Categorical}(\phi)$
f_α is an MLP



$p(z|x)$

Narrative adapted from Jason Eisner's High-Level Explanation of VI: https://www.cs.jhu.edu/~jason/tutorials/variational.html

Variational Inference

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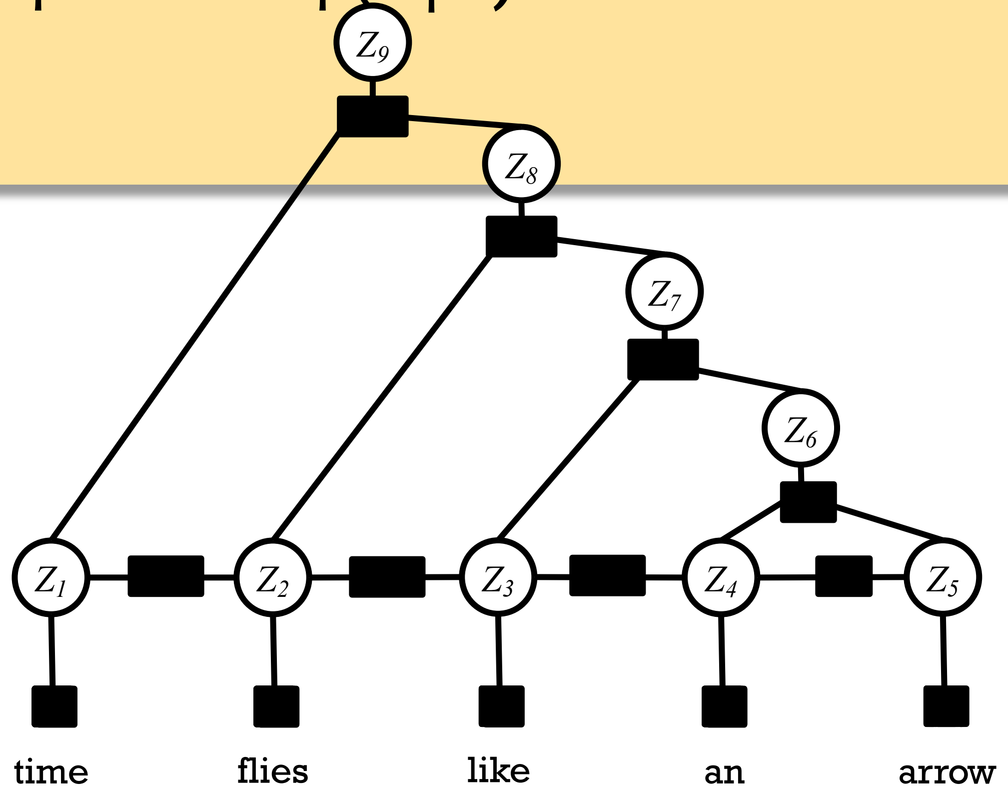

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Variational Inference

Problem:

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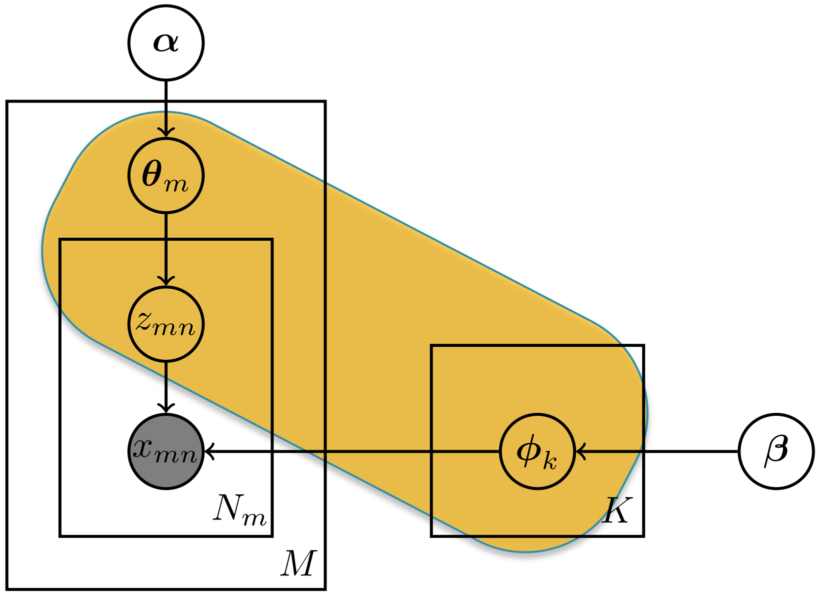
– For training data **x** and parameters **z**, estimating the posterior p(**z** | **x**) is intractable



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Variational Inference

Problem:

- For observed variables **x** and latent variables **z**, estimating the posterior p(**z** | **x**) is intractable
- For training data **x** and parameters **z**, estimating the posterior p(**z** | **x**) is intractable

Solution:

- Approximate p(**z** | **x**) with a simpler q(**z**)
- Typically q(**z**) has more independence assumptions than p(**z** | **x**) – fine b/c q(**z**) is tuned for a specific **x**
- **Key idea:** pick a single q(**z**) from some family Q that best approximates p(**z** | **x**)

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Variational Inference

$z = x_{1:T}$ $x = x_0$

Terminology:

- $q(\mathbf{z})$: the **variational approximation**
- Q: the **variational family**
- Usually $q_\theta(\mathbf{z})$ is parameterized by some θ called **variational parameters**
- Usually $p_\alpha(\mathbf{z} \mid \mathbf{x})$ is parameterized by some fixed α – we'll call them the parameters

Example Algorithms:

- mean-field variational inference
- loopy belief propagation
- tree-reweighted belief propagation
- expectation propagation

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Variational Inference

Is this trivial?

- Note: We are not defining a new distribution simple $q_\theta(\mathbf{z} \mid \mathbf{x})$, there is one simple $q_\theta(\mathbf{z})$ for each $p_\alpha(\mathbf{z} \mid \mathbf{x})$
- Consider the MCMC equivalent of this:
  - you could draw samples $z^{(i)} \sim p(\mathbf{z} \mid \mathbf{x})$
  - then train some simple $q_\theta(\mathbf{z})$ on $z^{(1)}, z^{(2)}, \ldots, z^{(N)}$
  - hope that the sample adequately represents the posterior for the given $\mathbf{x}$
- How is VI different from this?
  - VI doesn't require sampling
  - VI is fast and deterministic
  - Why? b/c we choose an objective function (KL divergence) that defines which q_θ best approximates p_α, and exploit the special structure of q_θ to optimize it

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Variational Inference

V.I. offers a new design decision

- Choose the distribution $p_\alpha(\mathbf{z} \mid \mathbf{x})$ that you really want, i.e. don't just simpify it to make it computationally convenient
- Then design a the structure of another distribution $q_\theta(\mathbf{z})$ such that V.I. is efficient

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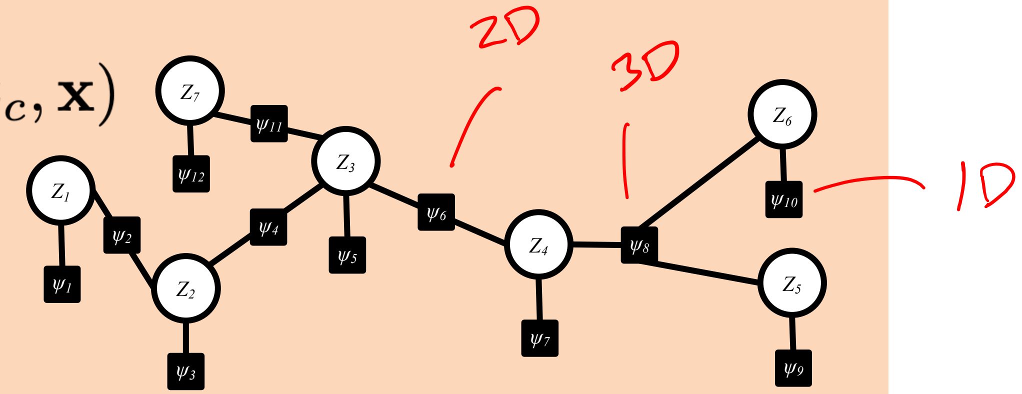
THE MEAN FIELD APPROXIMATION

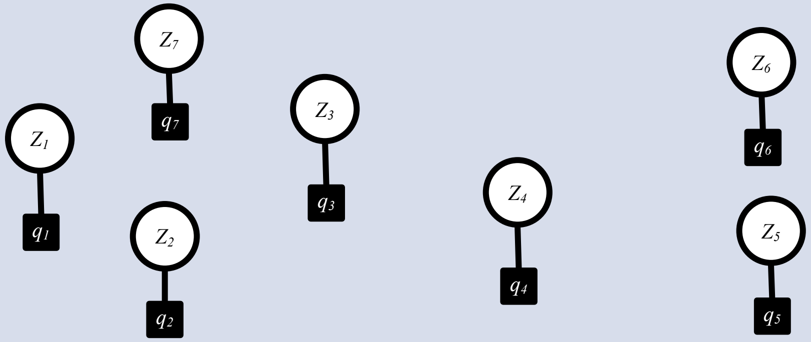
Mean Field Approximation

The **mean field approximation** assumes our variational approximation $q_\theta(\mathbf{z})$ treats each variable as independent

$$p_\alpha(\mathbf{z} \mid \mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{c \in \mathcal{C}} \psi_c(\mathbf{z}_c, \mathbf{x})$$



$$q_\theta(\mathbf{z}) = \prod_{t=1}^{T} q_t(z_t)$$



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Ising Model

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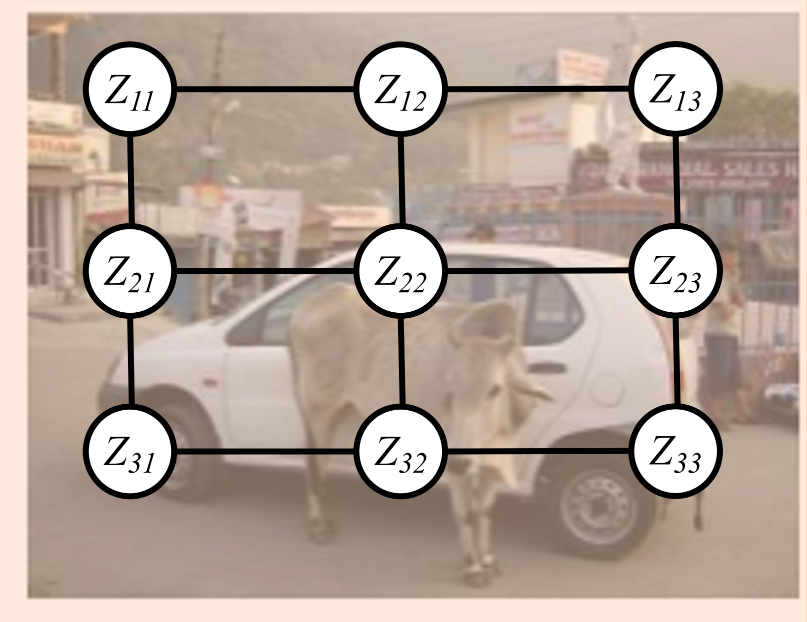

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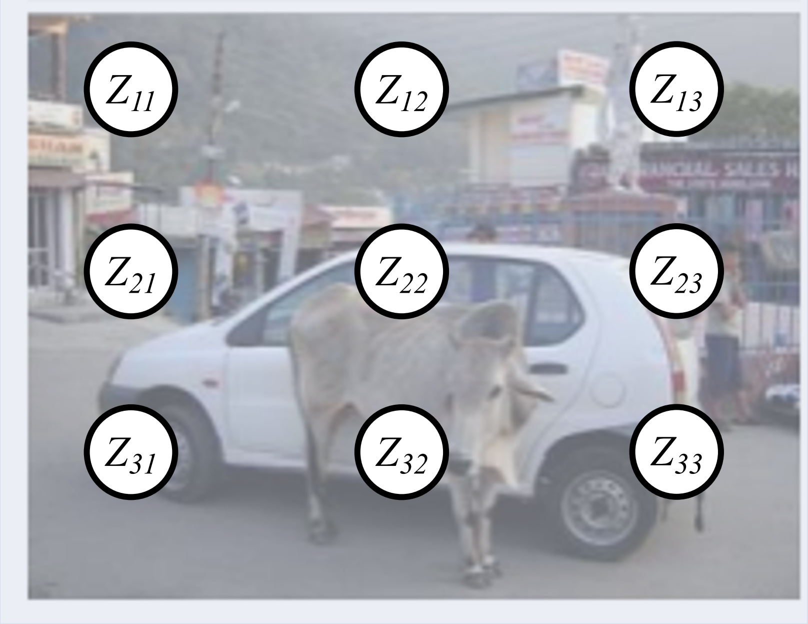
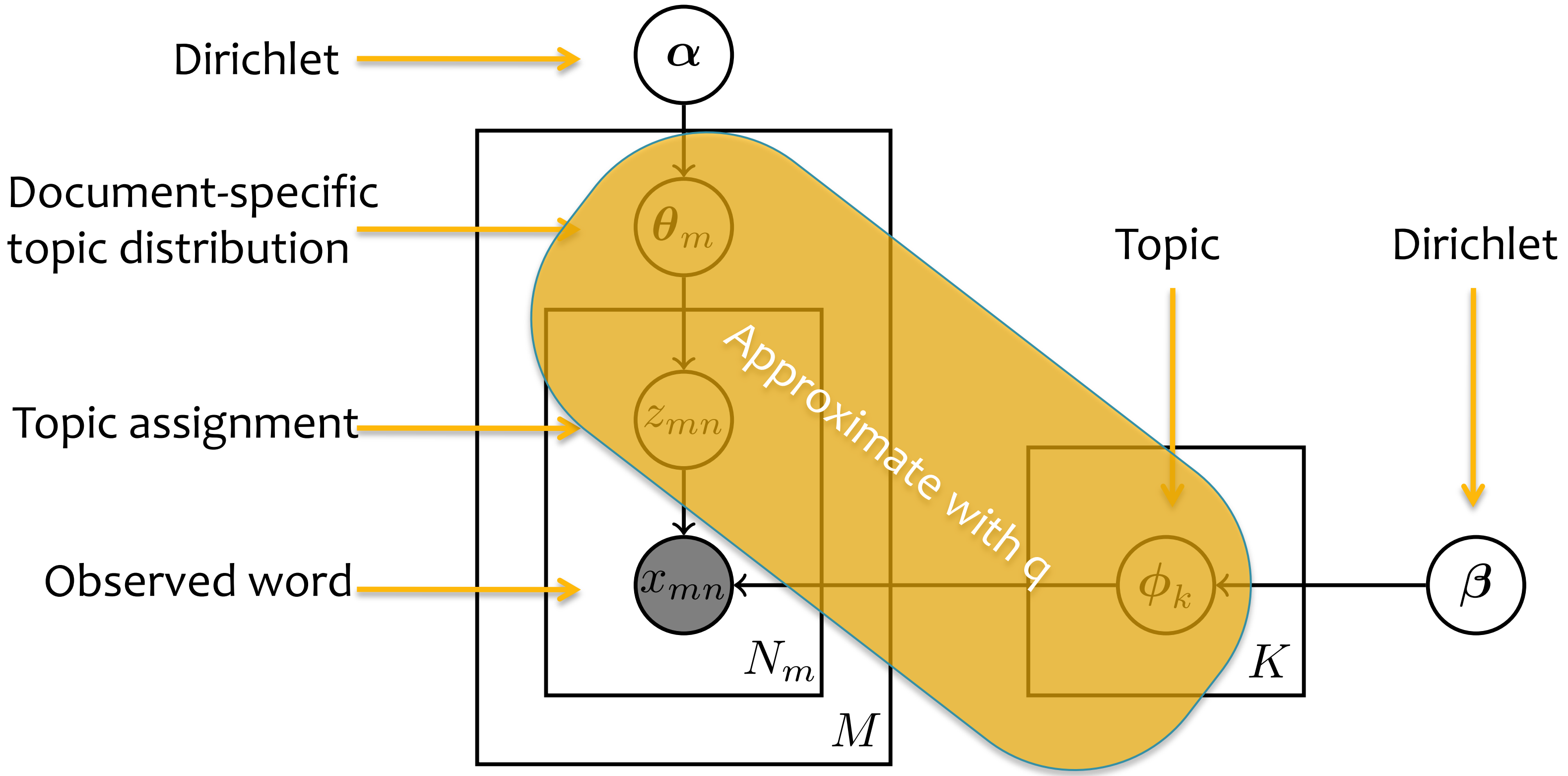


Mean Field Approximation

Latent Dirichlet Allocation (LDA)

- Uncollapsed Variational Inference, aka. Explicit V.I. (original distribution)



Mean Field Approximation

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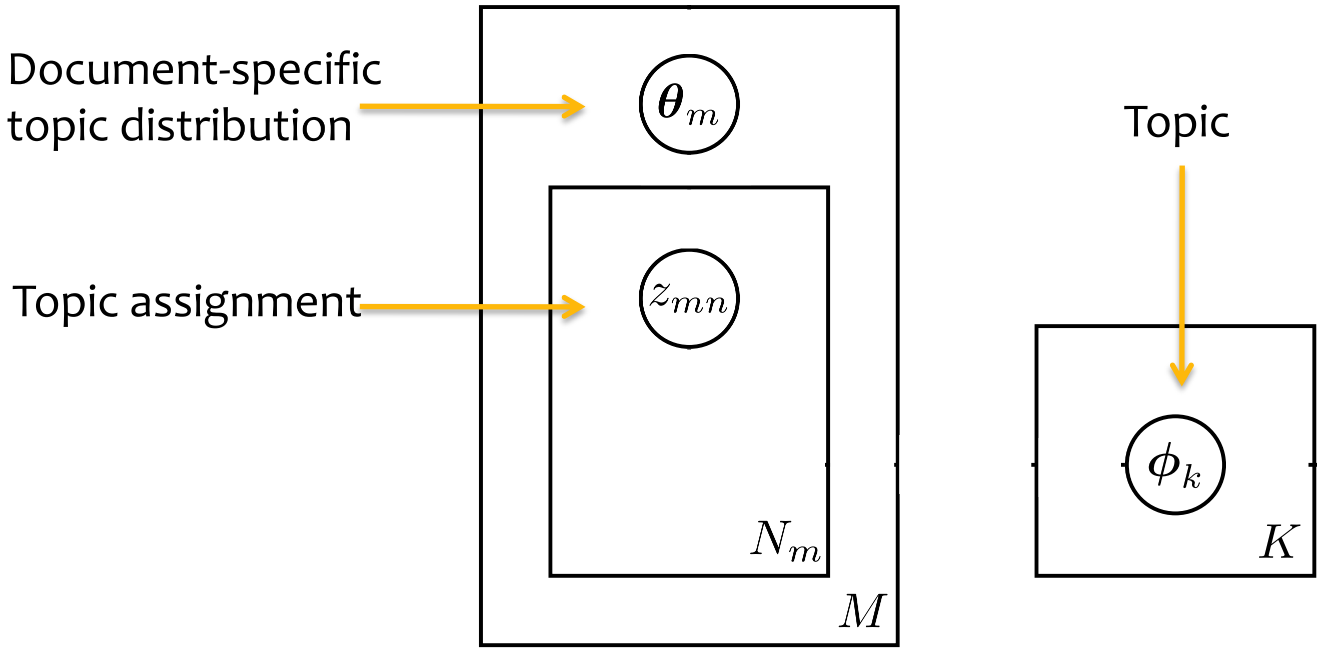


MEAN FIELD VARIATIONAL INFERENCE

Two Cases for Intractability

Suppose we want to work with p(z|x)

- *Case 1*:

  given a **joint distribution** p(x, z)

$$\Rightarrow p(z \mid x) = \frac{p(x, z)}{p(x)}$$

we assume p(x) is intractable

- *Case 2*:

  give **factor graph** and potentials

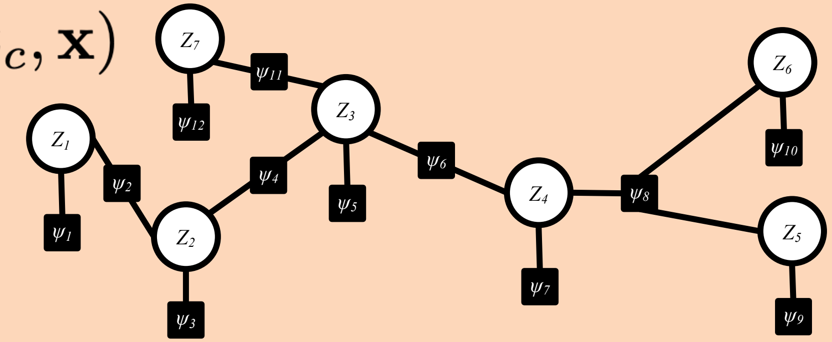
$$\Rightarrow p(z \mid x) = \frac{\tilde{p}(x, z)}{Z(x)}$$

we assume Z(x) is intractable

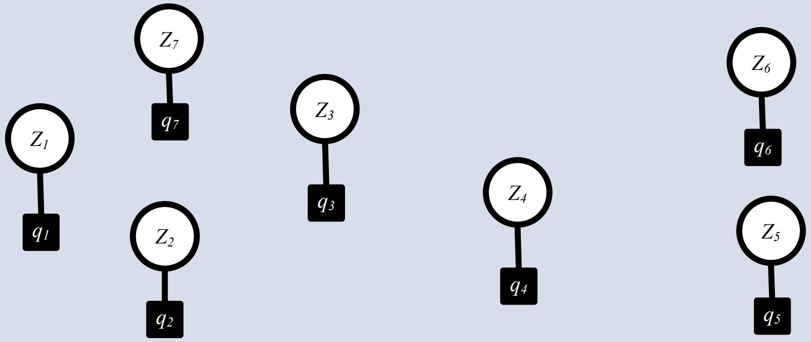
Mean Field Approximation

The **mean field approximation** assumes our variational approximation $q_\theta(\mathbf{z})$ treats each variable as independent

$$p_\alpha(\mathbf{z} \mid \mathbf{x}) = \frac{1}{Z(\mathbf{x})} \prod_{c \in \mathcal{C}} \psi_c(\mathbf{z}_c, \mathbf{x})$$

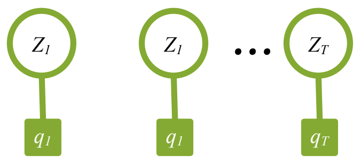


$$q_\theta(\mathbf{z}) = \prod_{t=1}^{T} q_t(z_t)$$



Mean Field V.I. Overview

1. *Goal*: estimate $p_\alpha(\mathbf{z} \mid \mathbf{x})$
   we assume this is intractable to compute exactly
2. *Idea*: approximate with another distribution $q_\theta(\mathbf{z}) \approx p_\alpha(\mathbf{z} \mid \mathbf{x})$ for each $\mathbf{x}$
3. *Mean Field*: assume $q_\theta(\mathbf{z}) = \prod_t q_t(z_t; \theta)$
   i.e., we decompose over variables
   other choices for the decomposition of $q_\theta(\mathbf{z})$ give rise to "structured mean field"



4. *Optimization Problem*: pick the q that minimizes KL(q || p)

$$\hat{q}(\mathbf{z}) = \underset{q(\mathbf{z}) \in \mathcal{Q}}{\operatorname{argmin}} \, \mathsf{KL}(q(\mathbf{z}) || p(\mathbf{z} \mid \mathbf{x}))$$

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \, \mathsf{KL}(q_\theta(\mathbf{z}) || p_\alpha(\mathbf{z} \mid \mathbf{x}))$$

equivalent

5. *Optimization Algorithm*: coordinate descent
   i.e. pick the best $q_t(z_t)$ based on the other $\{ q_s(z_s) \}_{s \neq t}$ being fixed

Optimizing KL Divergence

- *Question*: How do we minimize KL?

$$\hat{\theta} = \operatorname*{argmin}_{\theta \in \Theta} \underbrace{\mathsf{KL}(q_\theta(\mathbf{z}) \parallel p_\alpha(\mathbf{z} \mid \mathbf{x}))}$$

- *Answer #1*: Oh no! We can't even compute this KL.

Why we can't compute KL…

$$\begin{aligned}
\mathsf{KL}(q(\mathbf{z}) \parallel p(\mathbf{z} \mid \mathbf{x})) &= E_{q(\mathbf{z})}\left[\log\left(\frac{q(\mathbf{z})}{p(\mathbf{z} \mid \mathbf{x})}\right)\right] \\
&= E_{q(\mathbf{z})}[\log q(\mathbf{z})] - E_{q(\mathbf{z})}[\log p(\mathbf{z} \mid \mathbf{x})] \\
&= E_{q(\mathbf{z})}[\log q(\mathbf{z})] - E_{q(\mathbf{z})}\boxed{[\log p(\mathbf{x}, \mathbf{z})] + E_{q(\mathbf{z})}[\log p(\mathbf{x})]} \\
&= E_{q(\mathbf{z})}[\log q(\mathbf{z})] - E_{q(\mathbf{z})}[\log p(\mathbf{x}, \mathbf{z})] + \underbrace{\log p(\mathbf{x})}
\end{aligned}$$

we have the same problem with an intractable data likelihood p(x) or an intractable partition function Z(x)

we assumed this is intractable to compute!

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\end{aligned}$$

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Optimizing KL Divergence

- *Question*: How do we minimize KL?

$$\hat{\theta} = \operatorname*{argmin}_{\theta \in \Theta} \underbrace{\mathsf{KL}(q_\theta(\mathbf{z}) \parallel p_\alpha(\mathbf{z} \mid \mathbf{x}))}$$

- *Answer #2*: We don't need to compute this KL
  We can instead maximize the ELBO (i.e. **E**vidence **L**ower **BO**und)

$$\mathsf{ELBO}(q_\theta) = E_{q_\theta(\mathbf{z})}\left[\log p_\alpha(\mathbf{x}, \mathbf{z})\right] - E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right]$$

The ELBO for a DGM

Here is why…

$$\begin{aligned}
\theta &= \operatorname*{argmin}_{\theta} \mathsf{KL}(q_\theta(\mathbf{z}) \parallel p_\alpha(\mathbf{z} \mid \mathbf{x})) \\
&= \operatorname*{argmin}_{\theta} E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right] - E_{q_\theta(\mathbf{z})}\left[\log p_\alpha(\mathbf{x}, \mathbf{z})\right] + \underbrace{\log p_\alpha(\mathbf{x})} \\
&= \operatorname*{argmin}_{\theta} E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right] - E_{q_\theta(\mathbf{z})}\left[\log p_\alpha(\mathbf{x}, \mathbf{z})\right] \\
&= \operatorname*{argmax}_{\theta} \mathsf{ELBO}(q_\theta)
\end{aligned}$$

dropping the
intractable term
gives the ELBO

Optimizing KL Divergence

- *Question*: How do we minimize KL?

$$\hat{\theta} = \operatorname*{argmin}_{\theta \in \Theta} \underbrace{\mathrm{KL}(q_\theta(\mathbf{z}) \parallel p_\alpha(\mathbf{z} \mid \mathbf{x}))}$$

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$$\mathrm{ELBO}(q_\theta) = E_{q_\theta(\mathbf{z})}\left[\log \tilde{p}_\alpha(\mathbf{z} \mid \mathbf{x})\right] - E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right]$$

The ELBO for a UGM

Here is why…

$$\begin{aligned}
\theta &= \operatorname*{argmin}_{\theta} \mathrm{KL}(q_\theta(\mathbf{z}) \parallel p_\alpha(\mathbf{z} \mid \mathbf{x})) \\
&= \operatorname*{argmin}_{\theta} E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right] - E_{q_\theta(\mathbf{z})}\left[\log \tilde{p}_\alpha(\mathbf{z} \mid \mathbf{x})\right] + \underbrace{\log Z_\alpha(\mathbf{x})} \\
&= \operatorname*{argmin}_{\theta} E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right] - E_{q_\theta(\mathbf{z})}\left[\log \tilde{p}_\alpha(\mathbf{z} \mid \mathbf{x})\right] \\
&= \operatorname*{argmax}_{\theta} \mathrm{ELBO}(q_\theta)
\end{aligned}$$

dropping the
intractable term
gives the ELBO

ELBO as Objective Function

What does maximizing ELBO(q_θ) accomplish?

$$\text{ELBO}(q_\theta) = \boxed{E_{q_\theta(\mathbf{z})}\left[\log p_\alpha(\mathbf{x}, \mathbf{z})\right]} \boxed{- E_{q_\theta(\mathbf{z})}\left[\log q_\theta(\mathbf{z})\right]}$$

1. The first expectation is high if q_θ puts probability mass on the same values of **z** that p_α puts probability mass

2. The second term is the entropy of q_θ and the entropy will be high if q_θ spreads its probability mass evenly

ELBO as lower bound

- *For a DGM*:
  - ELBO(q) is a lower bound for log p(x)
- *For a UGM*:
  - ELBO(q) is a lower bound for log Z(x)

Takeaway: in variational inference, we find the q that gives the **tightest bound** on the normalization constant for p(z | x)

ELBO's relation to log p(x)

Theorem:

for any q, $\log p(x) \geq \text{ELBO}(q)$

i.e. $\text{ELBO}(q)$ is a lower bound on $\log p(x)$

Proof #1:

Recall Jensen's Inequality: $f(E[x]) \geq E[f(x)]$, for concave f

$$\log p(x) = \log \int_z p(x,z)\, dz \quad \text{(marginal)}$$

$$= \log \int_z p(x,z) \frac{q(z)}{q(z)}\, dz \quad \text{(mult. by 1)}$$

$$= \log E_{q(z)}\left[p(x,z) / q(z) \right] \quad \text{(def. of expectation)}$$

$$\geq E_{q(z)}\left[\log \left(\frac{p(x,z)}{q(z)} \right) \right] \quad \text{(by Jensen's Ineq.)}$$

$$= E_{q(z)}[\log p(x,z)] - E_{q(z)}[\log q(z)] = \text{ELBO}(q)$$

$$\Rightarrow \log p(x) \geq \text{ELBO}(q)$$

Proof #2:

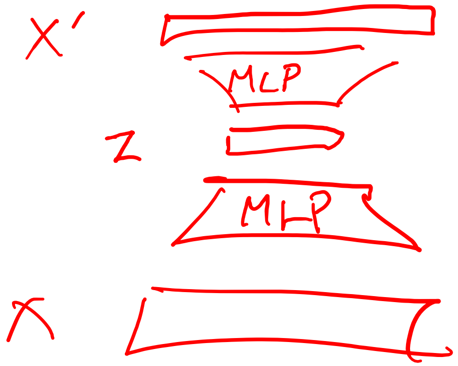
1. $\log p(x) = KL(q \| p) + \text{ELBO}(q)$
2. $KL(q \| p) \geq 0$ (without proof)
3. $\Rightarrow \log p(x) \geq \text{ELBO}(q)$

Key Takeaway:

minimizing KL is the same as finding a tight ELBO(q) lower bound for log p(x)

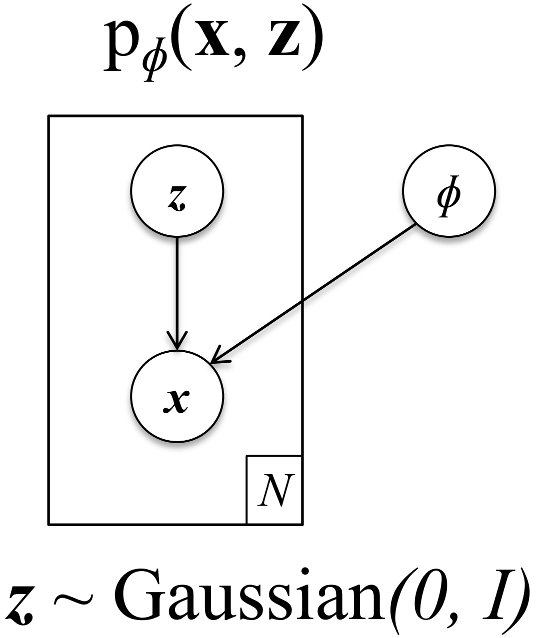
VARIATIONAL AUTOENCODERS

Why VAEs?



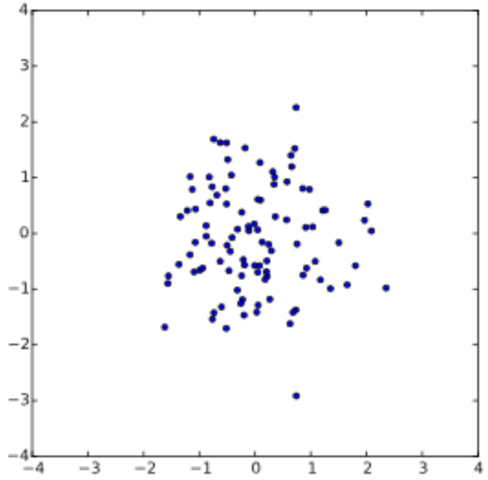
- **Autoencoders:**
  - learn a low dimensional representation of the input, but hard to work with as a generative model
  - one of the key limitations of autoencoders is that we have no way of **sampling** from them!
- **Variational autoencoders (VAEs)**
  - by contrast learn a continuous latent space that is **easy to sample from!**
  - can **generate** new data (e.g. images) by sampling from the learned generative model

Variational Autoencoders



Graphical Model Perspective

- The DGM diagram shows that the VAE model is quite simple as a graphical model (ignoring the neural net details that give rise to **x**)
- Sampling from the model is easy:
  - Consider a DGM where $\mathbf{x} = g_\phi(\mathbf{z}/10 + \mathbf{z}/||\mathbf{z}||)$ (i.e. we don't use parameters ϕ)
  - Then we can draw samples of **z** and directly convert them to values **x**
- **Key idea of VAE:** define $g_\phi(z)$ as a neural net and learn ϕ from data



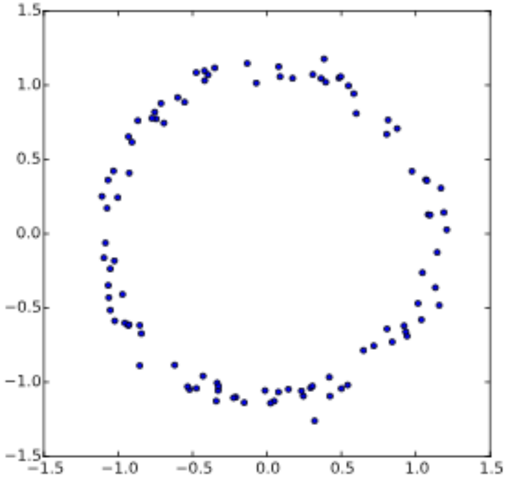


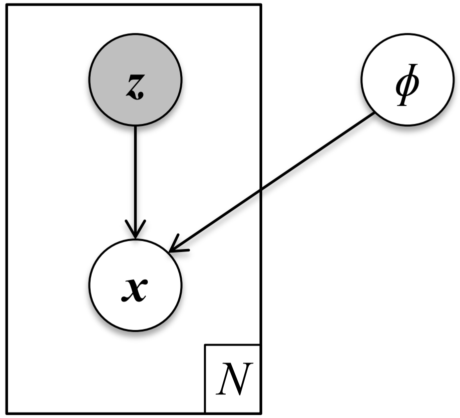
Figure from Doersch (2016)

Variational Autoencoders

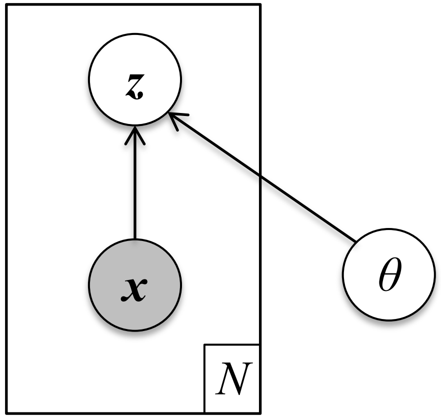
Neural Network Perspective

- We can view a variational autoencoder (VAE) as an autoencoder consisting of two neural networks
- VAEs (as encoders) define two distributions:
  - **encoder:** $q_\theta(\mathbf{z} \mid \mathbf{x})$
  - **decoder:** $p_\phi(\mathbf{x} \mid \mathbf{z})$
- Parameters θ and ϕ are neural network parameters (i.e. θ are not the variational parameters)

$p_\phi(\mathbf{x} \mid \mathbf{z})$



$q_\theta(\mathbf{z} \mid \mathbf{x})$



Variational Autoencoders

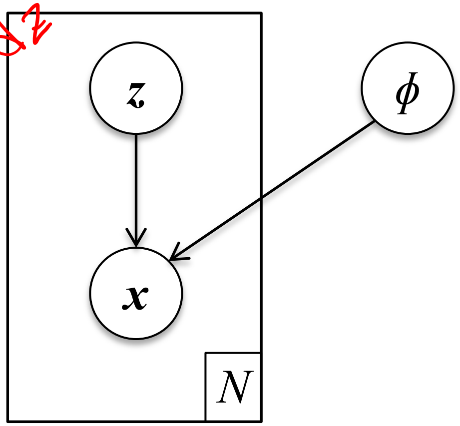
Graphical Model Perspective

- We can also view the VAE from the perspective of variational inference
- In this case we have two distributions:
  - **model:** $p_\phi(\mathbf{z} \mid \mathbf{x})$
  - **variational approximation:** $q_{\lambda=f(x;\,\theta)}(\mathbf{z} \mid \mathbf{x})$
- We have the same model parameters ϕ
- The variational parameters λ are a function of NN parameters θ

$$\sum_{i=1}^{N} \log p(x^{(i)})$$

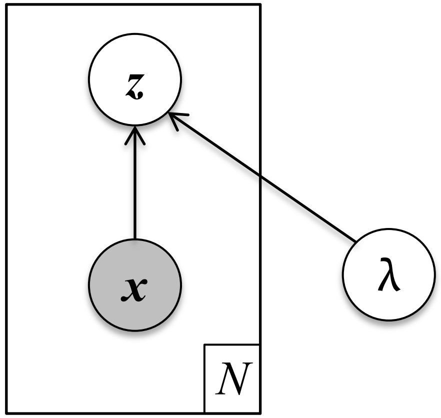
$$p(x^{(i)}) = \int_z p(x^{(i)}, z)\,dz$$

$p_\phi(\mathbf{x}, \mathbf{z})$



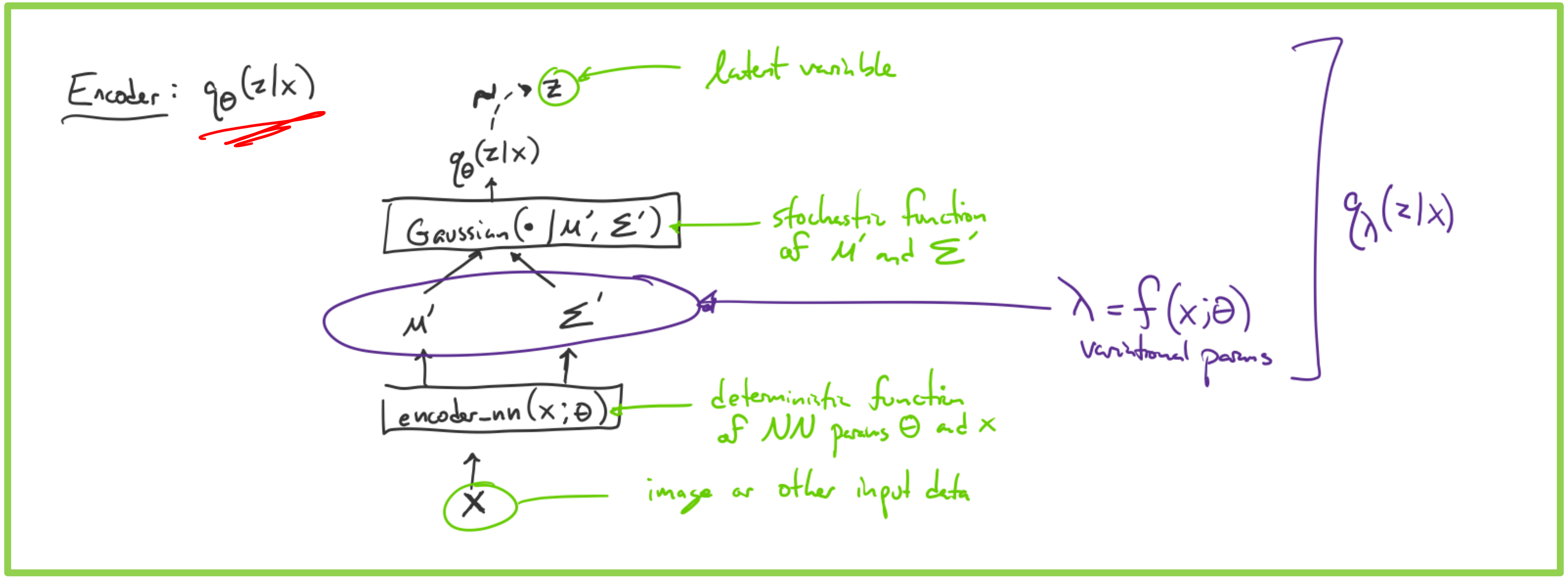
$z \sim \text{Gaussian}(0, I)$

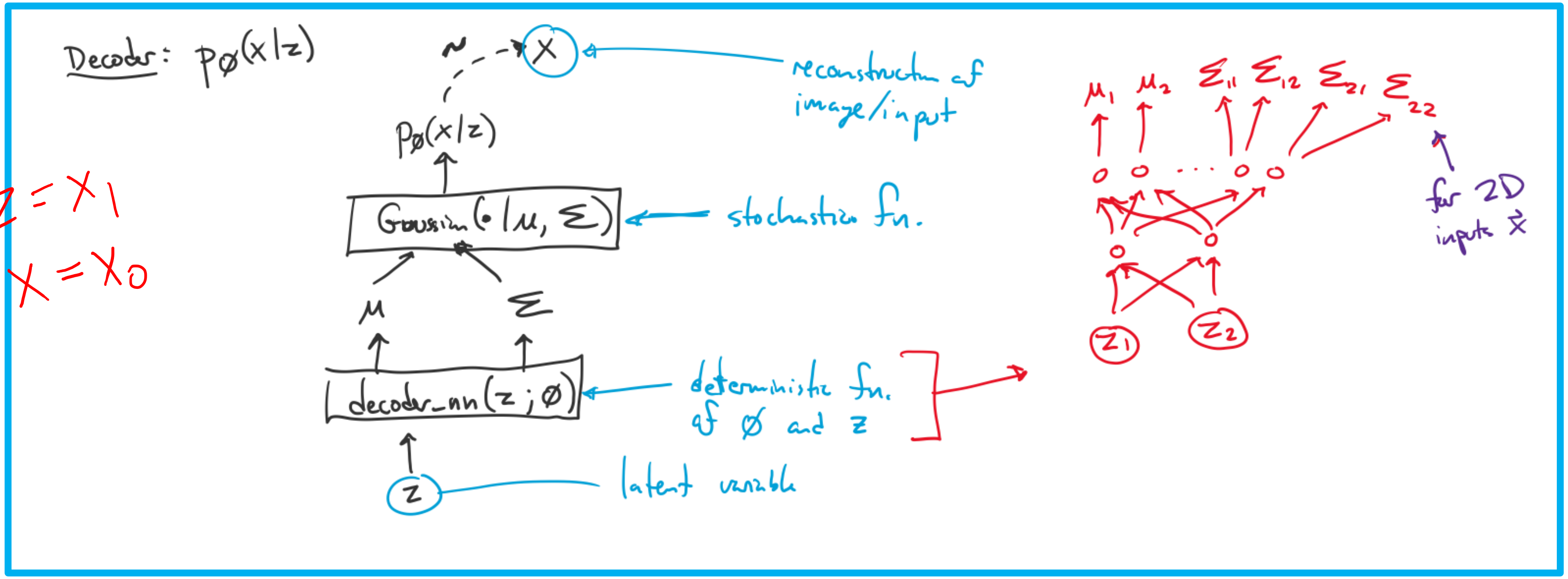
$q_\lambda(\mathbf{z} \mid \mathbf{x})$

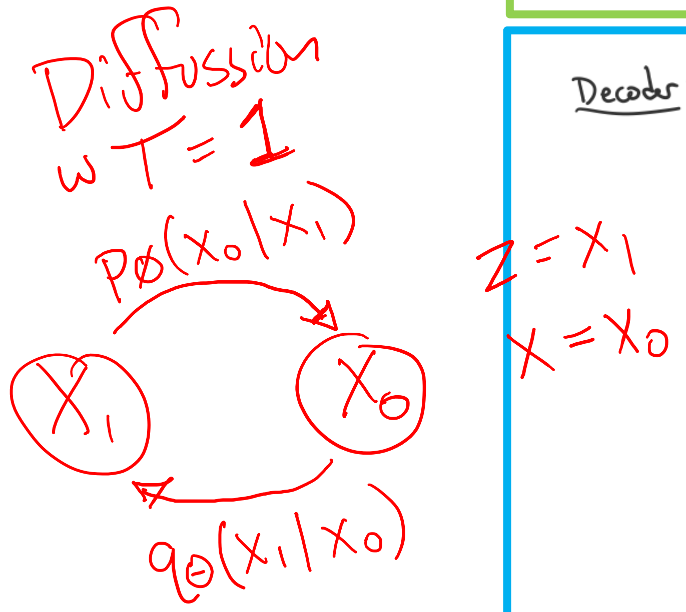


$\lambda = f(x;\, \theta)$

VAEs: Neural Network View







VAEs: Neural Network View

Training VAE

Dataset : $D = \{\vec{x}^{(i)}\}_{i=1}^{N}$ unlabeled data (eg. images)

Loss Fn :

ELBO instead of $\log p(x^{(i)})$

$$\ell(\theta,\phi) = \sum_{i=1}^{N} \ell_i(\theta,\phi)$$

$$\ell_i(\theta,\phi) = \underbrace{-\mathbb{E}_{q_\theta(z|x^{(i)})}\left[\log p_\phi(x^{(i)}|z)\right]}_{\text{reconstruction loss (just like an autoencoder)}} + \underbrace{KL\left(q_\theta(z|x^{(i)}) \,\|\, p(z)\right)}_{\text{Q1: what does this term accomplish? A: regularizer}}$$

$$= \underbrace{-\sum_{s=1}^{S} q_\theta(z^{(s)}|x^{(i)}) \log p_\phi(x^{(i)}|z^{(s)})}_{\text{where } z^{(s)} \sim q_\theta(\cdot|x^{(i)})\ \forall s} + KL\left(q_\theta(z|x^{(i)}) \,\|\, p(z)\right)$$

Monte Carlo Approx.

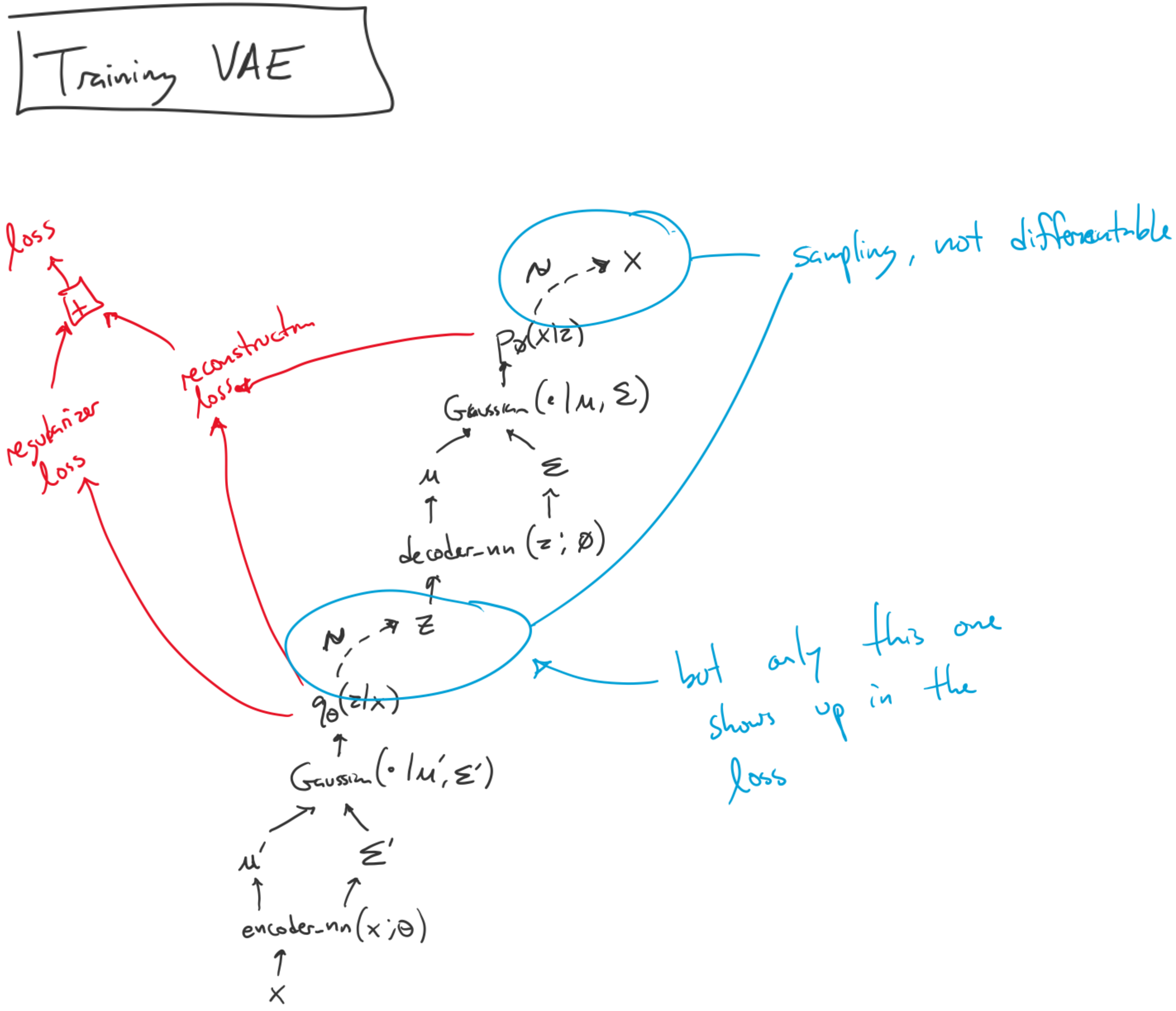
NN Perspective :

just backpropagation with a loss consisting of two terms

a) reconstruction loss

b) regularizer

VAEs: Neural Network View



Reparameterization Trick

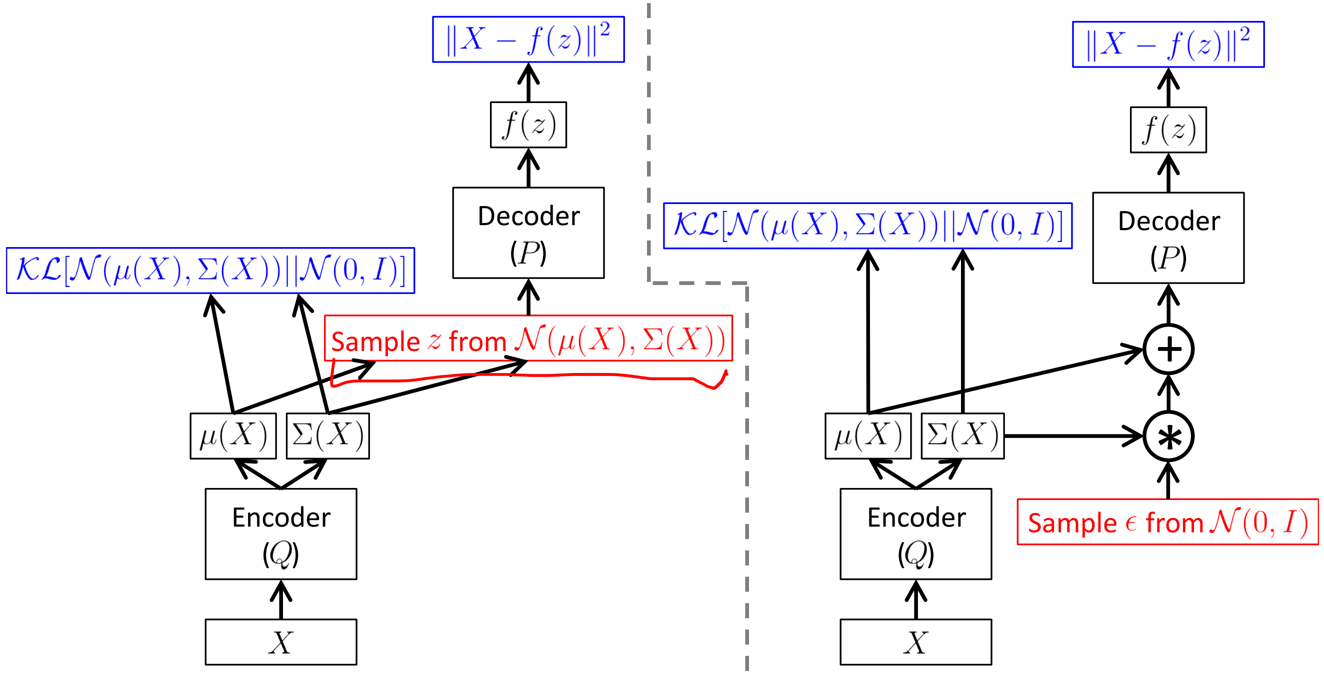


Figure 4: A training-time variational autoencoder implemented as a feedforward neural network, where $P(X|z)$ is Gaussian. Left is without the "reparameterization trick", and right is with it. Red shows sampling operations that are non-differentiable. Blue shows loss layers. The feedforward behavior of these networks is identical, but backpropagation can be applied only to the right network.

Figure from Doersch (2016)

VAE RESULTS

VAEs for Image Generation

Kingma & Welling (2014)
- introduced VAEs
- applied to image generation

Model
- $p_\phi(\mathbf{z}) \sim N(\mathbf{z}; 0, \mathbf{I})$
- $p_\phi(\mathbf{x} \mid \mathbf{z})$ is a multivariate Gaussian with mean and variance computed by an MLP, fully connected neural network with a single hidden layer with parameters ϕ
- $q_\theta(\mathbf{z} \mid \mathbf{x})$ is a multivariate Gaussian with diagonal covariance structure and with mean and variance computed by an MLP with parameters θ

Auto-Encoding Variational Bayes

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Abstract

How can we perform efficient inference and learning in directed probabilistic models, in the presence of continuous latent variables with intractable posterior distributions, and large datasets? We introduce a stochastic variational inference and learning algorithm that scales to large datasets and, under some mild differentiability conditions, even works in the intractable case. Our contributions is two-fold. First, we show that a reparameterization of the variational lower bound yields a lower bound estimator that can be straightforwardly optimized using standard stochastic gradient methods. Second, we show that for i.i.d. datasets with continuous latent variables per datapoint, posterior inference can be made especially efficient by fitting an approximate inference model (also called a recognition model) to the intractable posterior using the proposed lower bound estimator. Theoretical advantages are reflected in experimental results.

Figure from Kingma & Welling (2014)

VAEs for Image Generation

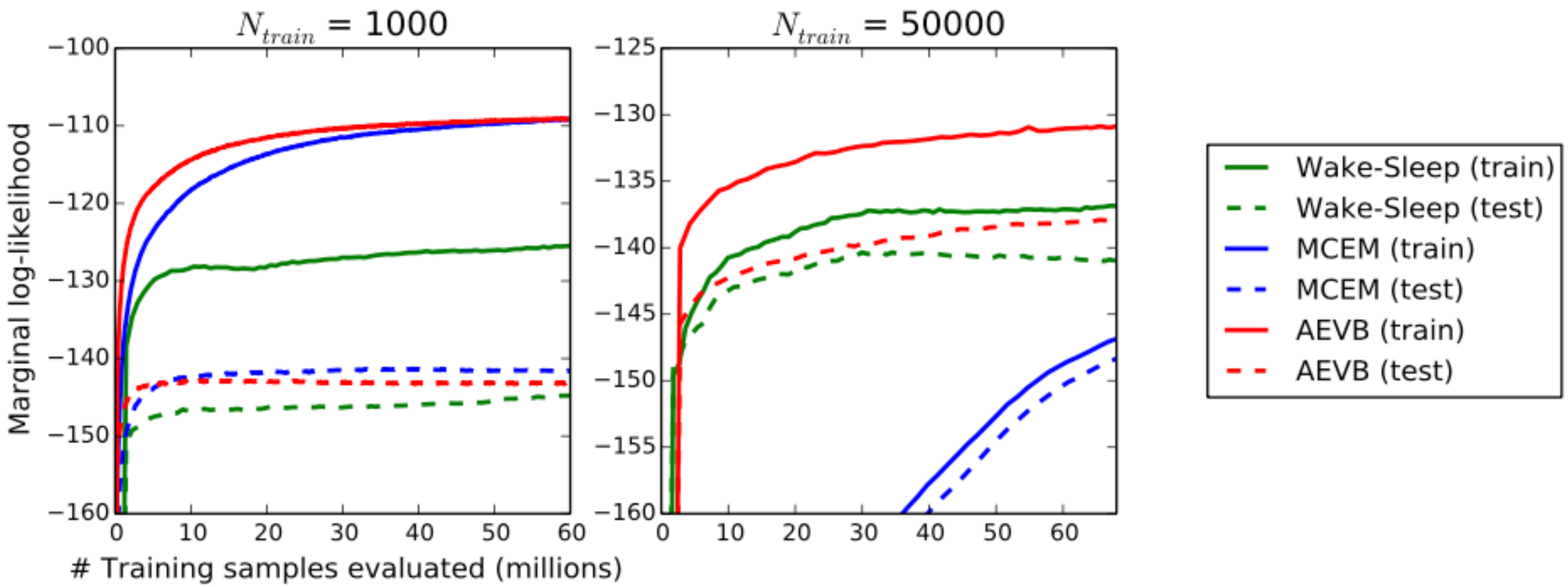
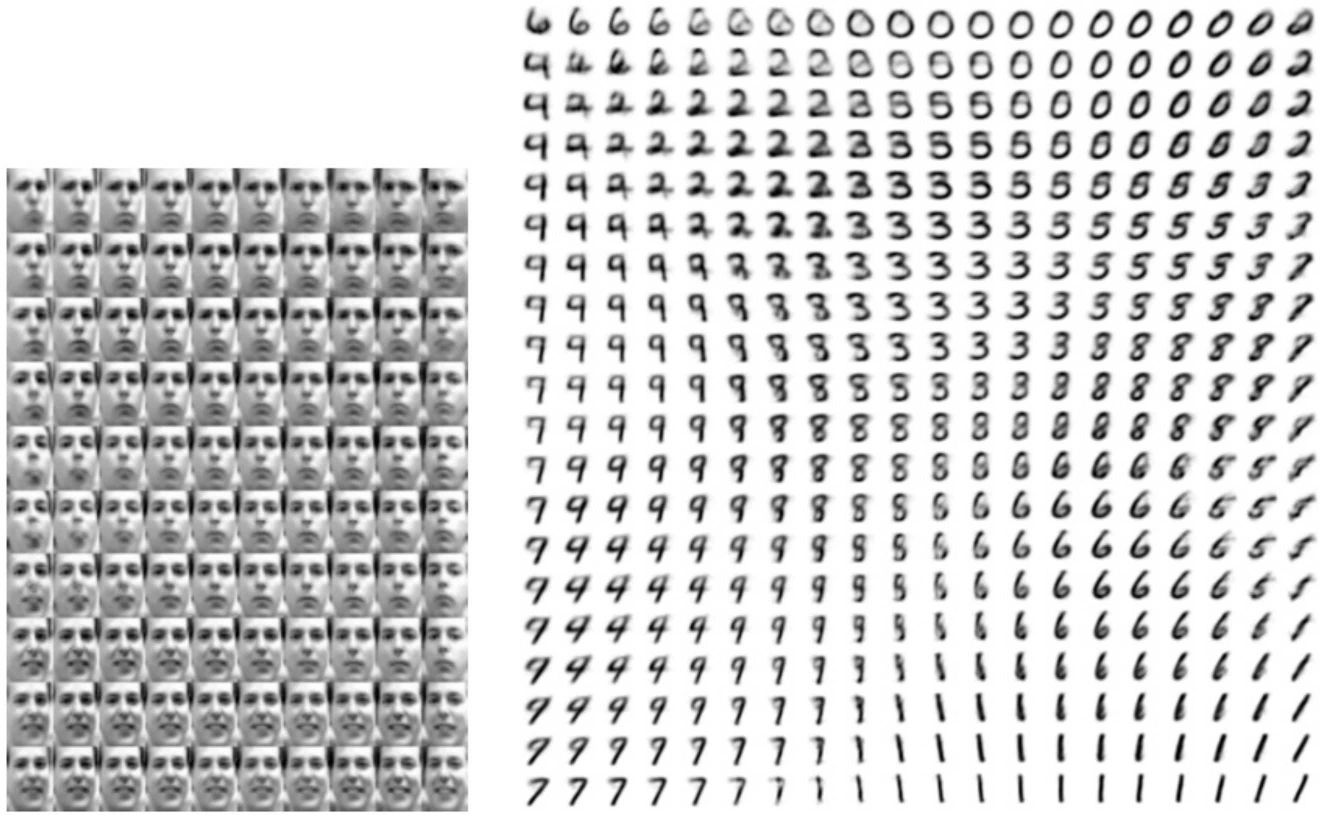


Figure 3: Comparison of AEVB to the wake-sleep algorithm and Monte Carlo EM, in terms of the estimated marginal likelihood, for a different number of training points. Monte Carlo EM is not an on-line algorithm, and (unlike AEVB and the wake-sleep method) can't be applied efficiently for the full MNIST dataset.

Figure from Kingma & Welling (2014)

VAEs for Image Generation

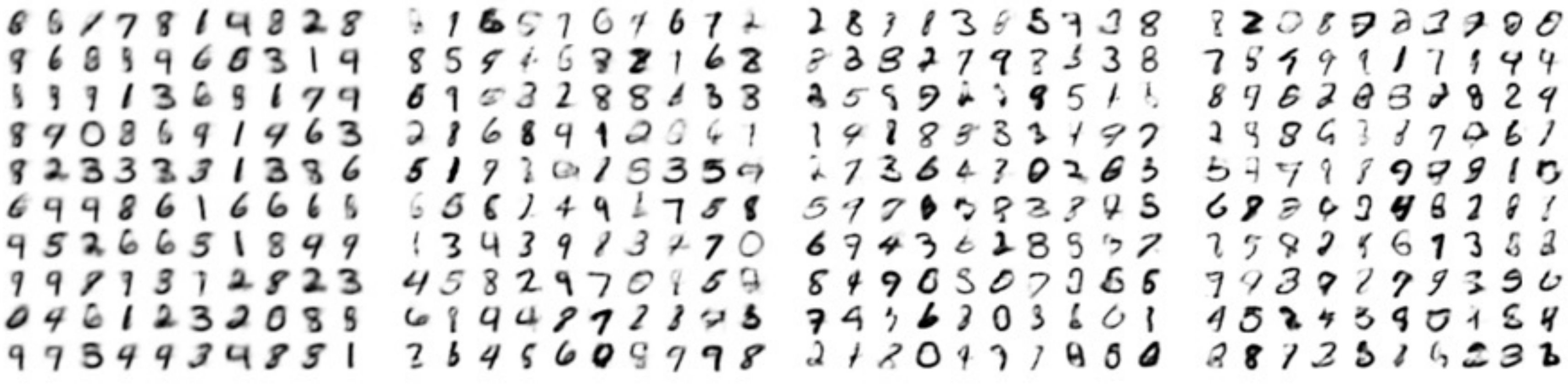


(a) Learned Frey Face manifold (b) Learned MNIST manifold

Figure 4: Visualisations of learned data manifold for generative models with two-dimensional latent space, learned with AEVB. Since the prior of the latent space is Gaussian, linearly spaced coordinates on the unit square were transformed through the inverse CDF of the Gaussian to produce values of the latent variables $\mathbf{z}$. For each of these values $\mathbf{z}$, we plotted the corresponding generative $p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z})$ with the learned parameters $\boldsymbol{\theta}$.

Figure from Kingma & Welling (2014)

VAEs for Image Generation



(a) 2-D latent space (b) 5-D latent space (c) 10-D latent space (d) 20-D latent space

Figure 5: Random samples from learned generative models of MNIST for different dimensionalities of latent space.

Figure from Kingma & Welling (2014)

VAEs for Text Generation

Bowman et al. (2015)

- example of an application of VAEs to discrete data
- built on the sequence-to-sequence framework:
  - input is read in by an LSTM
  - output is generated by an LSTM-LM

Model

- $p_\phi(\mathbf{z}) \sim N(\mathbf{z}; 0, \mathbf{I})$
- $p_\phi(\mathbf{x} \mid \mathbf{z})$ is an LSTM Language Model with parameters ϕ
- $q_\theta(\mathbf{z} \mid \mathbf{x})$ is a multivariate Gaussian with mean and variance computed by an LSTM with parameters θ

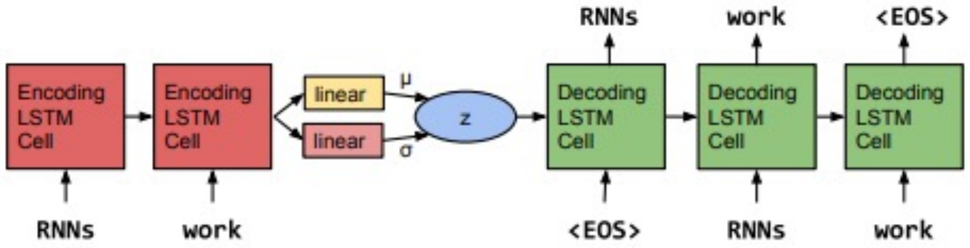


Figure 1: The core structure of our variational autoencoder language model. Words are represented using a learned dictionary of embedding vectors.

Figure from Bowman et al. (2015)

VAEs for Text Generation

INPUT	**we looked out at the setting sun .**	**i went to the kitchen .**	**how are you doing ?**
MEAN	*they were laughing at the same time .*	*i went to the kitchen .*	*what are you doing ?*
SAMP. 1	*ill see you in the early morning .*	*i went to my apartment .*	*“ are you sure ?*
SAMP. 2	*i looked up at the blue sky .*	*i looked around the room .*	*what are you doing ?*
SAMP. 3	*it was down on the dance floor .*	*i turned back to the table .*	*what are you doing ?*

Table 7: Three sentences which were used as inputs to the VAE, presented with greedy decodes from the mean of the posterior distribution, and from three samples from that distribution.

“ i want to talk to you . ”
“i want to be with you . ”
“i do n’t want to be with you . ”
i do n’t want to be with you .
she did n’t want to be with him .
he was silent for a long moment .
he was silent for a moment .
it was quiet for a moment .
it was dark and cold .
there was a pause .
it was my turn .

Table 8: Paths between pairs of random points in VAE space: Note that intermediate sentences are grammatical, and that topic and syntactic structure are usually locally consistent.

Figure from Bowman et al. (2015)

VQ-VAE

- Vector Quantized VAE (VQ-VAE) learns a continuous codebook, but the encoder outputs discrete codes
- Decoder takes a code and generates a sample conditioned on it

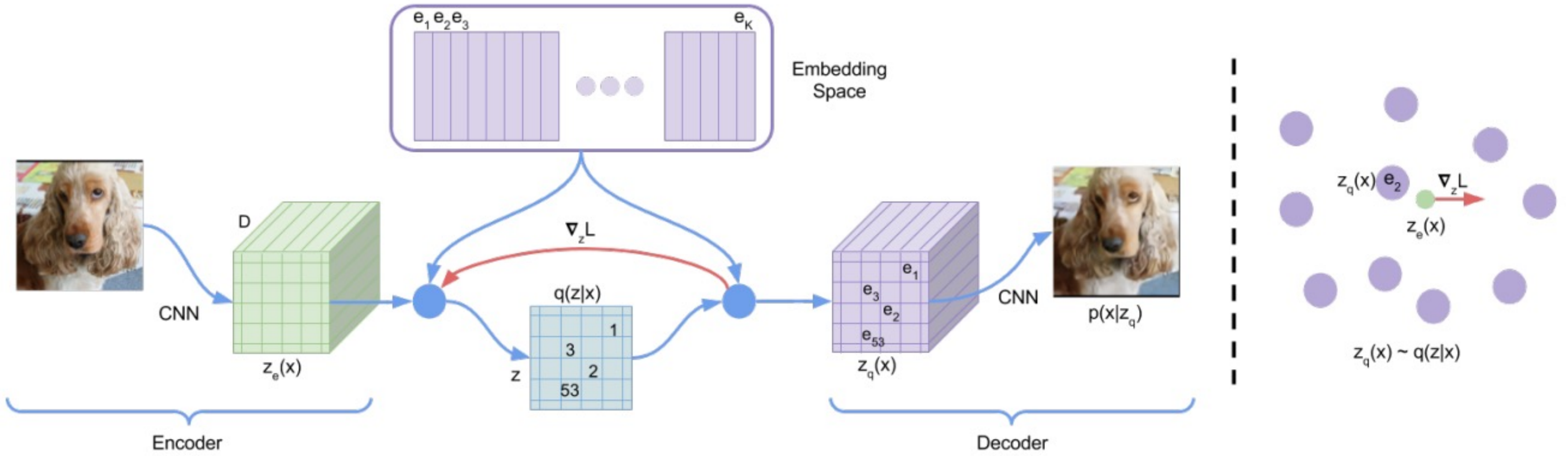


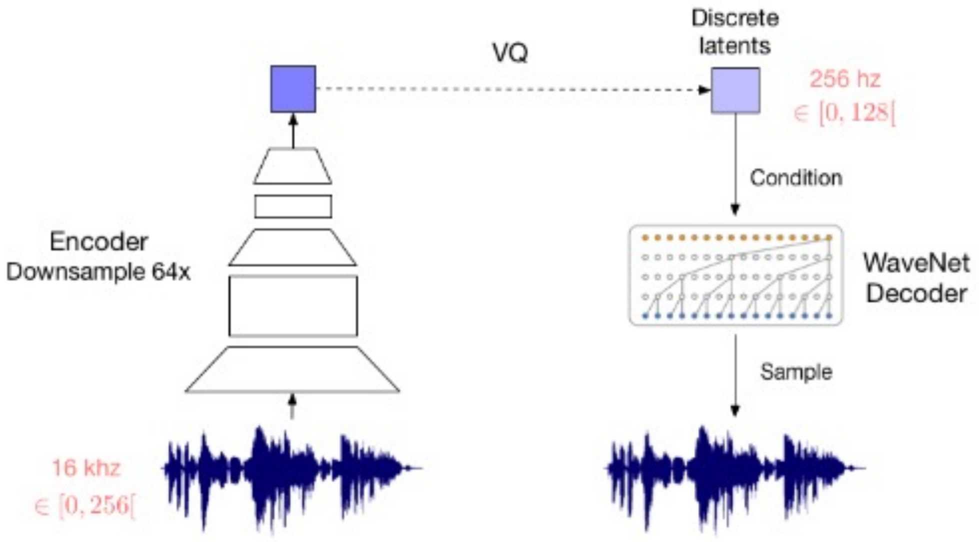
Figure 1: Left: A figure describing the VQ-VAE. Right: Visualisation of the embedding space. The output of the encoder $z(x)$ is mapped to the nearest point e_2. The gradient $\nabla_z L$ (in red) will push the encoder to change its output, which could alter the configuration in the next forward pass.

Figure from van den Oord et al. (2018)

VQ-VAE

- Vector Quantized VAE (VQ-VAE) learns a continuous codebook, but the encoder outputs discrete codes
- Decoder takes a code and generates a sample conditioned on it

Example: Generating Audio

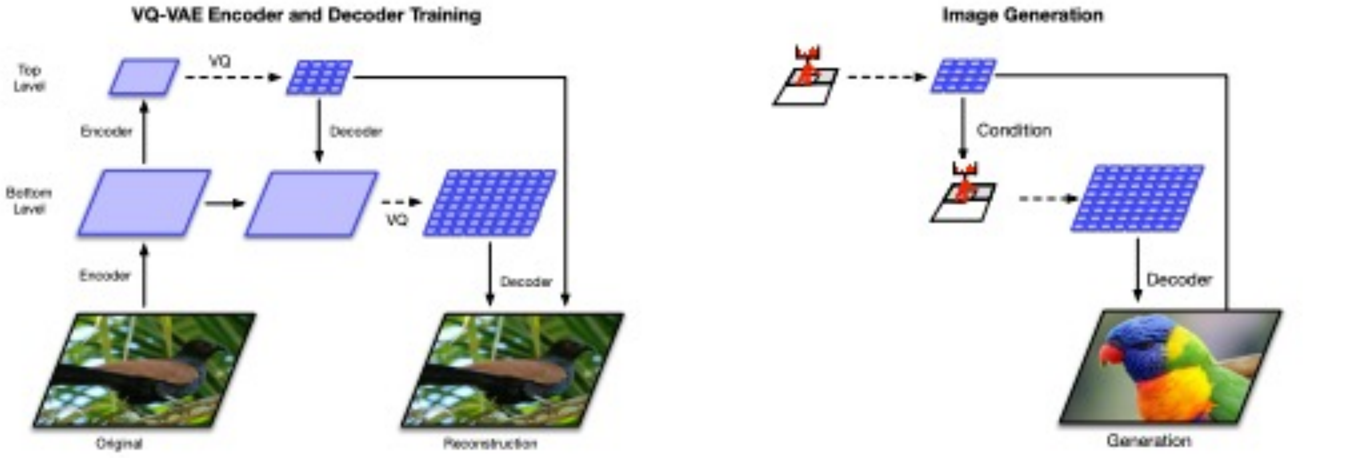


https://avdnoord.github.io/homepage/vqvae

Figure from van den Oord et al. (2018)

VQ-VAE

- VQ-VAE-2 extended the original idea by learning two levels (bottom and top) and a strong prior over the latent space
- Samples from this new model can be convincing even at high-fidelity



(a) Overview of the architecture of our hierarchical VQ-VAE. The encoders and decoders consist of deep neural networks. The input to the model is a 256×256 image that is compressed to quantized latent maps of size 64×64 and 32×32 for the *bottom* and *top* levels, respectively. The decoder reconstructs the image from the two latent maps.

(b) Multi-stage image generation. The top-level PixelCNN prior is conditioned on the class label, the bottom level PixelCNN is conditioned on the class label as well as the first level code. Thanks to the feed-forward decoder, the mapping between latents to pixels is fast. (The example image with a parrot is generated with this model).

Figure from Razavi et al. (2019)

- VQ-VAE-2 extended the original idea by learning two levels (bottom and top) and a strong prior over the latent space
- Samples from this new model can be convincing even at high-fidelity

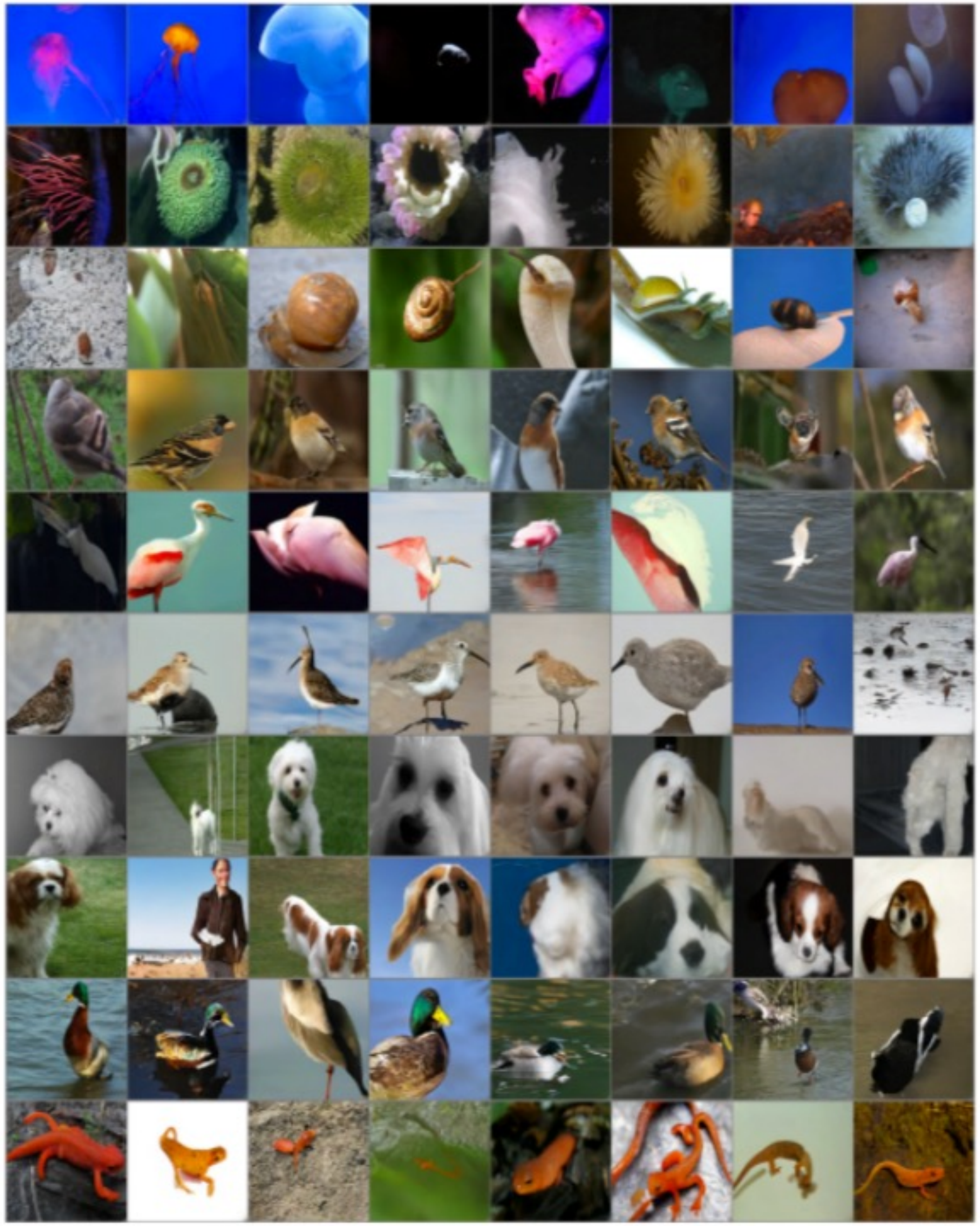


Figure 4: Class conditional random samples. Classes from the top row are: 108 sea anemone, 109 brain coral, 114 slug, 11 goldfinch, 130 flamingo, 141 redshank, 154 Pekinese, 157 papillon, 97 drake, and 28 spotted salamander.

Figure from Razavi et al. (2019)

- VQ-VAE-2 extended the original idea by learning two levels (bottom and top) and a strong prior over the latent space
- Samples from this new model can be convincing even at high-fidelity

Figure from Razavi et al. (2019)

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- VQ-VAE-2 extended the original idea by learning two levels (bottom and top) and a strong prior over the latent space
- Samples from this new model can be convincing even at high-fidelity

Figure from Razavi et al. (2019)

PROMPTING

Prompting

- Language models are trained to maximize the likelihood of sequences in the training data
- Most condition on the previous tokens to generate the next tokens

Definition: An *autoregressive language model* defines a probability distribution over sequences $\mathbf{x}_{1:T}$ of the form:

$$p(\mathbf{x}_{1:T}) = \prod_{t=1}^{T} p(x_t \mid x_1, \ldots, x_{t-1})$$

- Key idea behind prompting: provide a prefix string to the such that its likely completion is the answer you want

GPT-3 Example Output

Context

```
The City
BY C. P. CAVAFY
TRANSLATED BY EDMUND KEELEY
[Poem text omitted]


SOME TREES
John Ashbery
[Poem text omitted]


Shadows on the Way
Wallace Stevens
```

Generated Poem

```
I must have shadows on the way
If I am to walk I must have
Each step taken slowly and alone
To have it ready made

And I must think in lines of grey
To have dim thoughts to be my guide
Must look on blue and green
And never let my eye forget
That color is my friend
And purple must surround me too

The yellow of the sun is no more
Intrusive than the bluish snow
That falls on all of us.  I must have
Grey thoughts and blue thoughts walk with me
If I am to go away at all.
```

Conditional Sampling:
To draw a sample, feed in a prompt (i.e. context) and then sample from the model conditioned on that prompt

Examples from GPT-3

What else can a large LM (attempt to) do?

Using the idea of **prompts**, we can apply LMs to a **variety of different problems** in natural language processing.

In the **zero-shot setting**, we simply feed the context to the model and observe how it completes the sequence. (i.e. there is no additional training)

Answer fact-based questions:

Context →	Organisms require energy in order to do what?
Correct Answer →	mature and develop.
Incorrect Answer →	rest soundly.
Incorrect Answer →	absorb light.
Incorrect Answer →	take in nutrients.

Complete sentences logically:

Context →	My body cast a shadow over the grass because
Correct Answer →	the sun was rising.
Incorrect Answer →	the grass was cut.

Complete analogies:

Context →	lull is to trust as
Correct Answer →	cajole is to compliance
Incorrect Answer →	balk is to fortitude
Incorrect Answer →	betray is to loyalty
Incorrect Answer →	hinder is to destination
Incorrect Answer →	soothe is to passion

Reading comprehension:

Context →	anli 1: anli 1: Fulton James MacGregor MSP is a Scottish politician who is a Scottish National Party (SNP) Member of Scottish Parliament for the constituency of Coatbridge and Chryston. MacGregor is currently Parliamentary Liaison Officer to Shona Robison, Cabinet Secretary for Health & Sport. He also serves on the Justice and Education & Skills committees in the Scottish Parliament. Question: Fulton James MacGregor is a Scottish politican who is a Liaison officer to Shona Robison who he swears is his best friend. True, False, or Neither?
Correct Answer →	Neither
Incorrect Answer →	True
Incorrect Answer →	False

Examples from GPT-3

Prompting for Instruction Fine-tuned Models

- Models like ChatGPT, Llama-2 Chat, etc. have been fine-tuned as chat assistants
- These (often) were trained with specific prompt templates that segment the prompt into different parts: (1) system (2) assistant (3) user

Llama-2 Chat	sys:	[INST] <<SYS>> You are a helpful AI assistant… <</SYS>> [/INST]
	asst:	[INST] Organisms require energy in order to do what? [/INST]
	user:	mature and develop
Alpaca	sys:	### Instruction:
	asst:	### Instruction: Organisms require energy in order to do what?
	user:	### Response: mature and develop

Prompt formats from https://github.com/BerriAI/litellm/blob/main/litellm/llms/prompt_templates/factory.py

Zero-shot LLMs

- GPT-2 (1.5B parameters) for unsupervised prediction on various tasks
- GPT-2 models p(output | input, task)
  – translation: *(translate to french, english text, french text)*
  – reading comprehension: *(answer the question, document, question, answer)*
- Why does this work?

”I’m not the cleverest man in the world, but like they say in French: **Je ne suis pas un imbecile [I’m not a fool].**

In a now-deleted post from Aug. 16, Soheil Eid, Tory candidate in the riding of Joliette, wrote in French: ”**Mentez mentez, il en restera toujours quelque chose**,” which translates as, ”**Lie lie and something will always remain.**”

“I hate the word ‘**perfume**,”’ Burr says. ‘It’s somewhat better in French: ‘**parfum**.’

If listened carefully at 29:55, a conversation can be heard between two guys in French: “**-Comment on fait pour aller de l’autre coté? -Quel autre coté?**”, which means “**- How do you get to the other side? - What side?**”.

If this sounds like a bit of a stretch, consider this question in French: **As-tu aller au cinéma?**, or **Did you go to the movies?**, which literally translates as Have-you to go to movies/theater?

“**Brevet Sans Garantie Du Gouvernement**”, translated to English: “**Patented without government warranty**”.

Table 1. Examples of naturally occurring demonstrations of English to French and French to English translation found throughout the WebText training set.

Figures from Radford et al. (2019)

Zero-shot LLMs

- GPT-2 (1.5B parameters) for unsupervised prediction on various tasks
- GPT-2 models p(output | input, task)
  - translation: *(translate to french, english text, french text)*
  - reading comprehension: *(answer the question, document, question, answer)*
- Why does this work?

Language Models are Unsupervised Multitask Learners

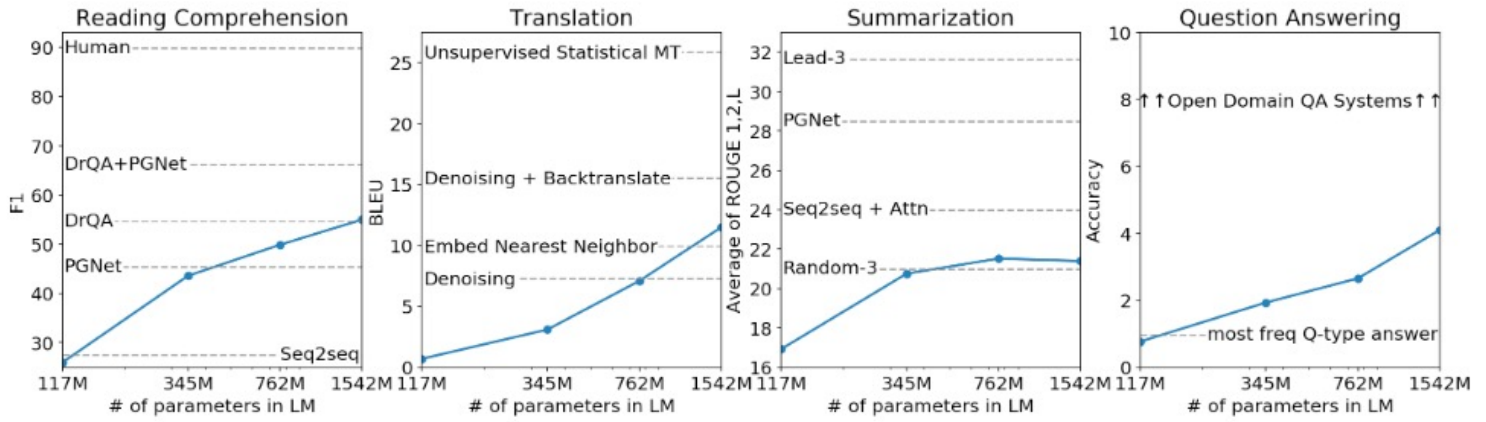

Figure 1. Zero-shot task performance of WebText LMs as a function of model size on many NLP tasks. Reading Comprehension results are on CoQA (Reddy et al., 2018), translation on WMT-14 Fr-En (Artetxe et al., 2017), summarization on CNN and Daily Mail (See et al., 2017), and Question Answering on Natural Questions (Kwiatkowski et al., 2019). Section 3 contains detailed descriptions of each result.

	LAMBADA (PPL)	LAMBADA (ACC)	CBT-CN (ACC)	CBT-NE (ACC)	WikiText2 (PPL)	PTB (PPL)	enwik8 (BPB)	text8 (BPC)	WikiText103 (PPL)	1BW (PPL)
SOTA	99.8	59.23	85.7	82.3	39.14	46.54	0.99	1.08	18.3	**21.8**
117M	**35.13**	45.99	**87.65**	**83.4**	**29.41**	65.85	1.16	1.17	37.50	75.20
345M	**15.60**	55.48	**92.35**	**87.1**	**22.76**	47.33	1.01	**1.06**	26.37	55.72
762M	**10.87**	**60.12**	**93.45**	**88.0**	**19.93**	**40.31**	**0.97**	**1.02**	22.05	44.575
1542M	**8.63**	**63.24**	**93.30**	**89.05**	**18.34**	**35.76**	**0.93**	**0.98**	**17.48**	42.16

Table 3. Zero-shot results on many datasets. No training or fine-tuning was performed for any of these results. PTB and WikiText-2 results are from (Gong et al., 2018). CBT results are from (Bajgar et al., 2016). LAMBADA accuracy result is from (Hoang et al., 2018) and LAMBADA perplexity result is from (Grave et al., 2016). Other results are from (Dai et al., 2019).

Figures from Radford et al. (2019)

IN-CONTEXT LEARNING

Few-shot Learning

- Few-shot learning can be done via in-context learning
- Typically, a task description is presented first
- Then a sequence of input/output pairs from a training dataset are presented in sequence

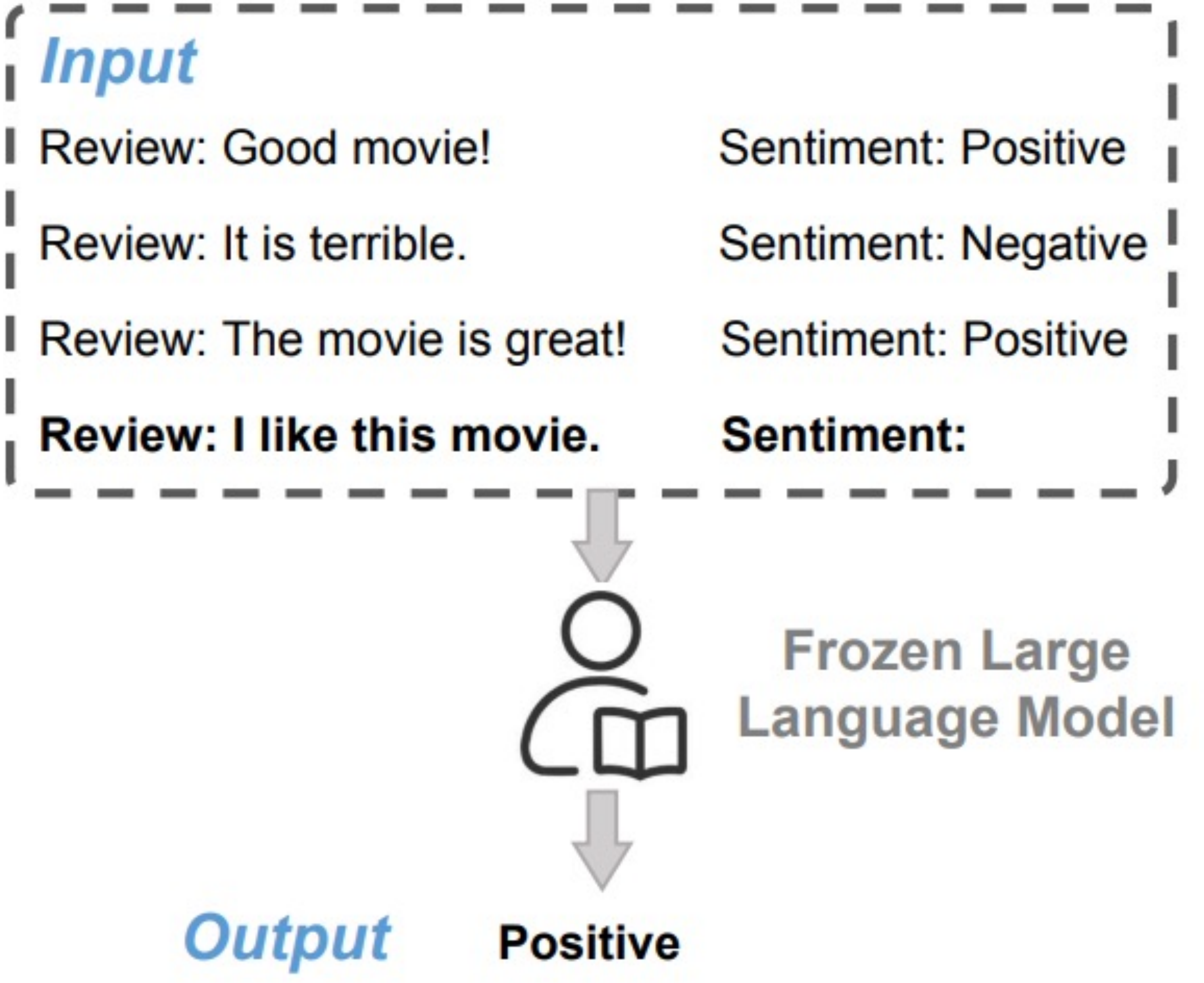

Figure from https://arxiv.org/pdf/2310.09881.pdf

Few-shot In-context Learning

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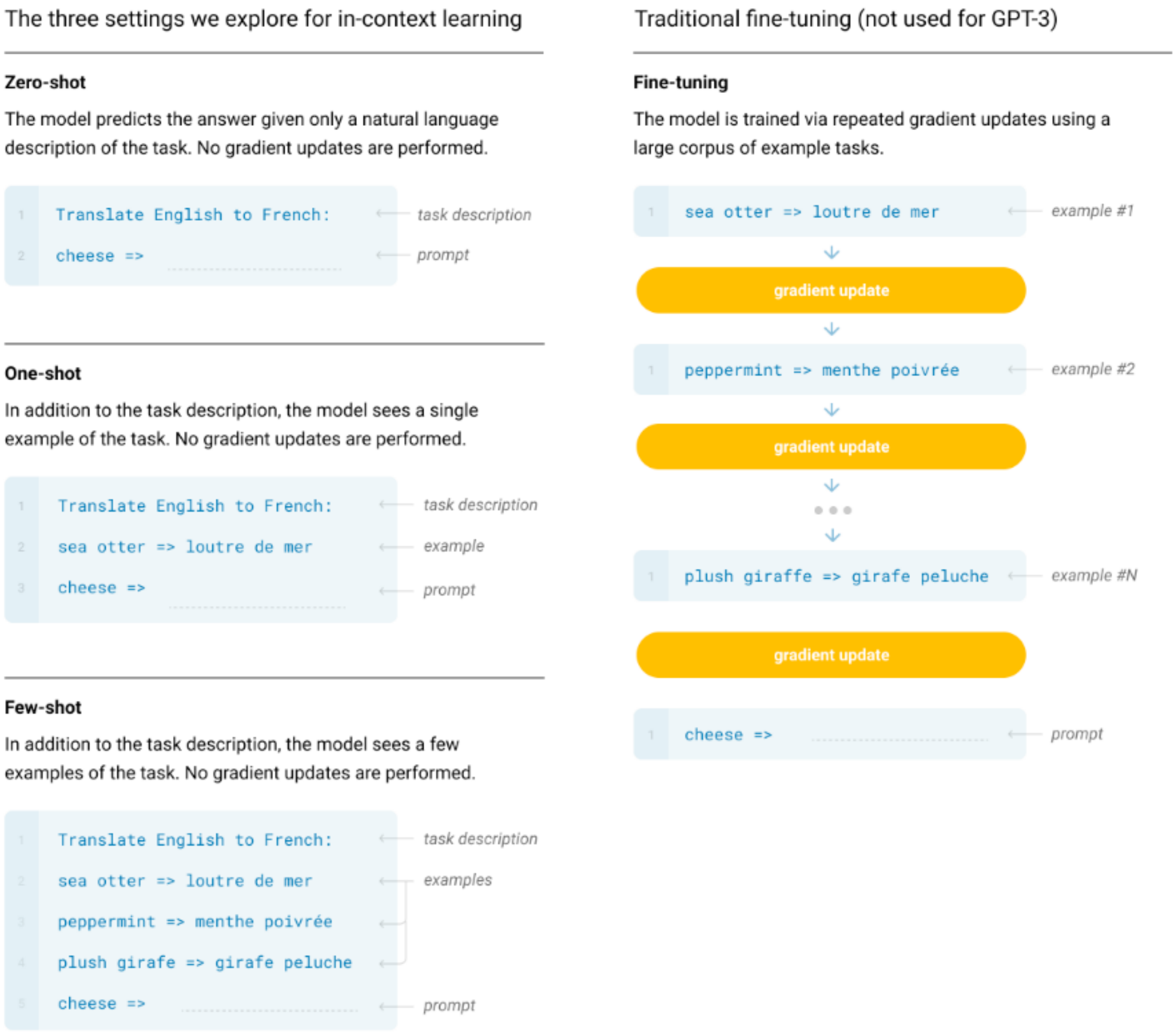


Figure from http://arxiv.org/abs/2005.14165

Few-shot In-context Learning

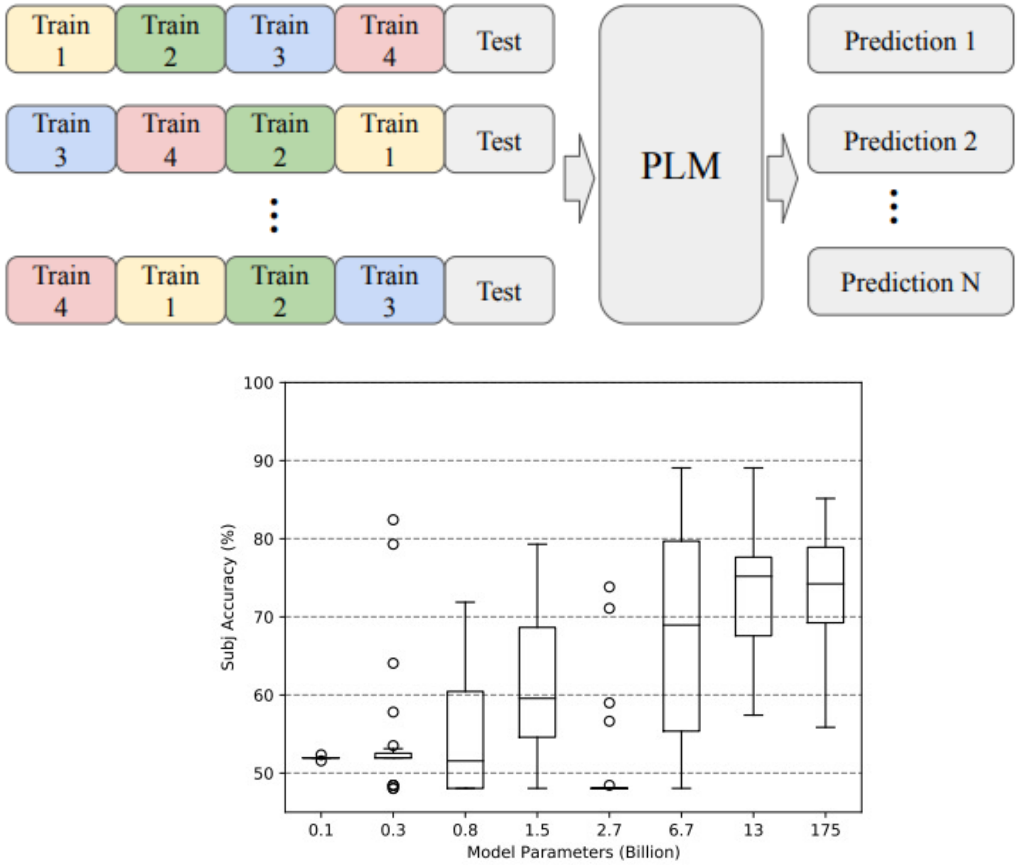


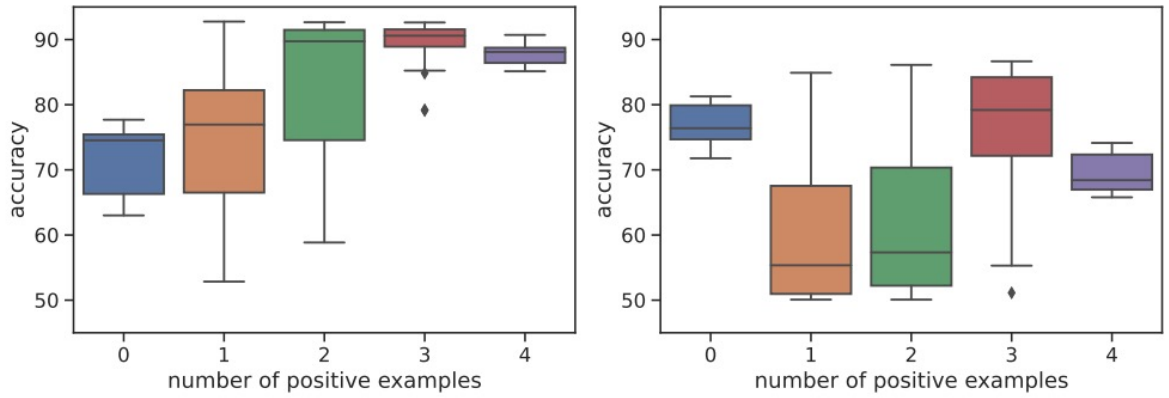
Figure 1: Four-shot performance for 24 different sample orders across different sizes of GPT-family models (GPT-2 and GPT-3) for the SST-2 and Subj datasets.

In-context learning can be sensitive to…

1. **the order the training examples are presented**
2. the balance of labels (e.g. positive vs. negative)
3. the number of unique labels covered

Figure from http://arxiv.org/abs/2104.08786

Few-shot In-context Learning



(a) Amazon (b) SST-2

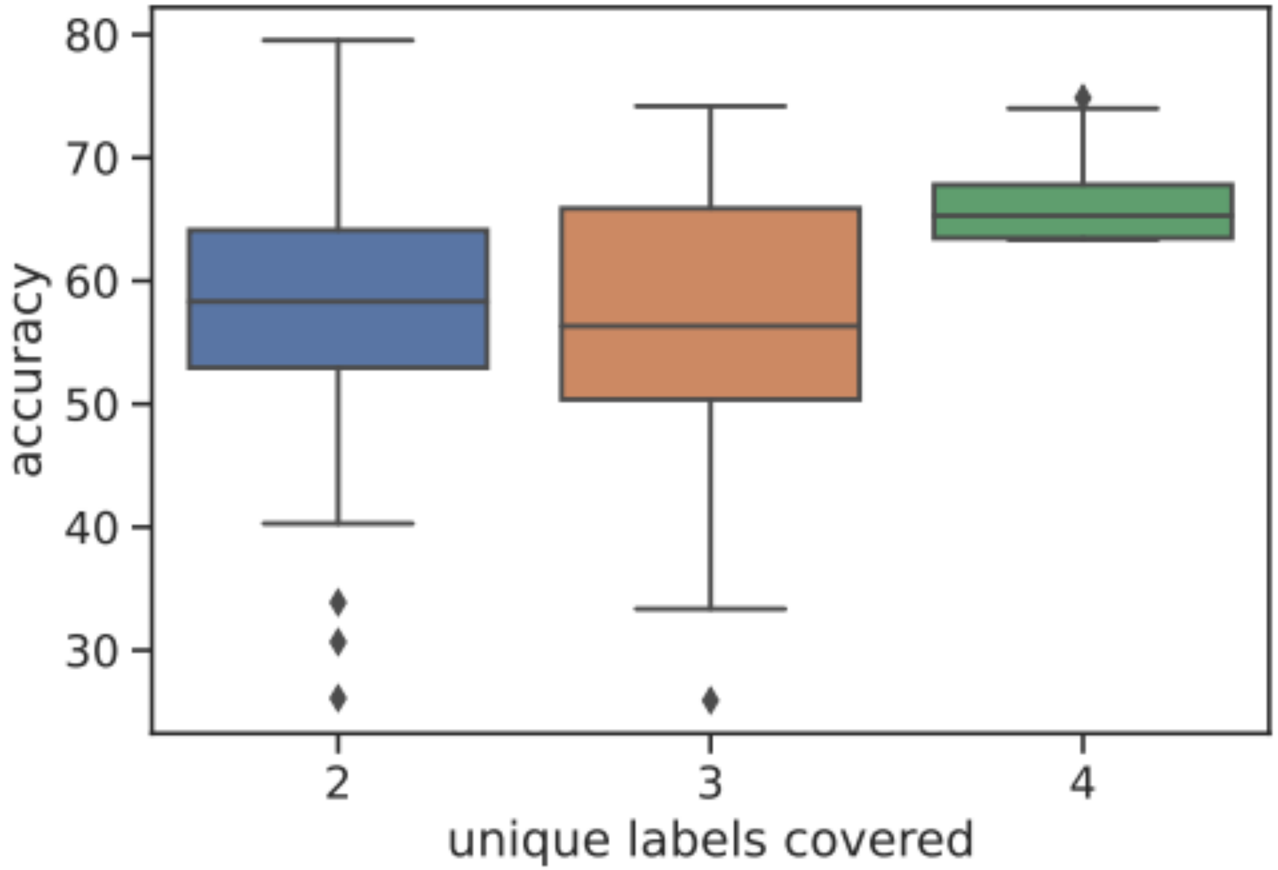
Figure 3: Accuracies of Amazon and SST-2 with varying **label balance** (number of positive examples in demonstration), across 100 total random samples of 4 demonstration examples.

In-context learning can be sensitive to…

1. the order the training examples are presented
2. **the balance of labels (e.g. positive vs. negative)**
3. the number of unique labels covered

Figure from https://aclanthology.org/2022.emnlp-main.622.pdf

Few-shot In-context Learning



In-context learning can be sensitive to…

1. the order the training examples are presented
2. the balance of labels (e.g. positive vs. negative)
3. **the number of unique labels covered**

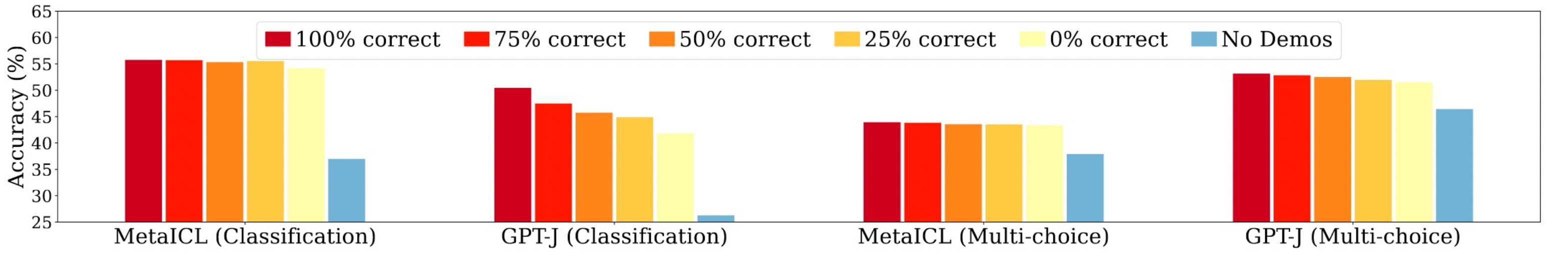
Figure from https://aclanthology.org/2022.emnlp-main.622.pdf

Few-shot In-context Learning

You would expect these to be important…

1. whether or not the training examples have the true label (as opposed to a random one)
2. having more in-context training examples

…but it's not always the case



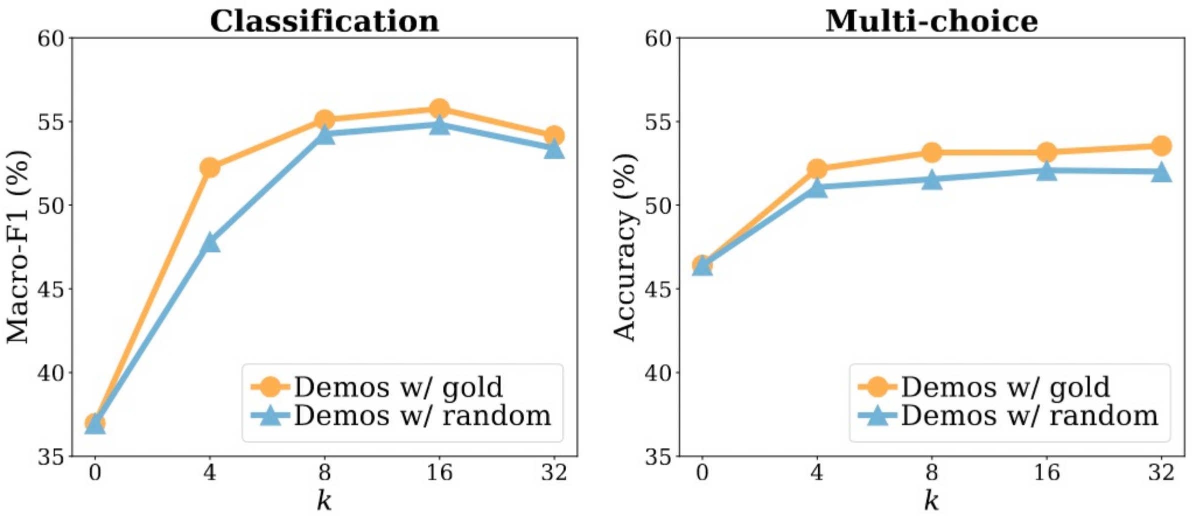
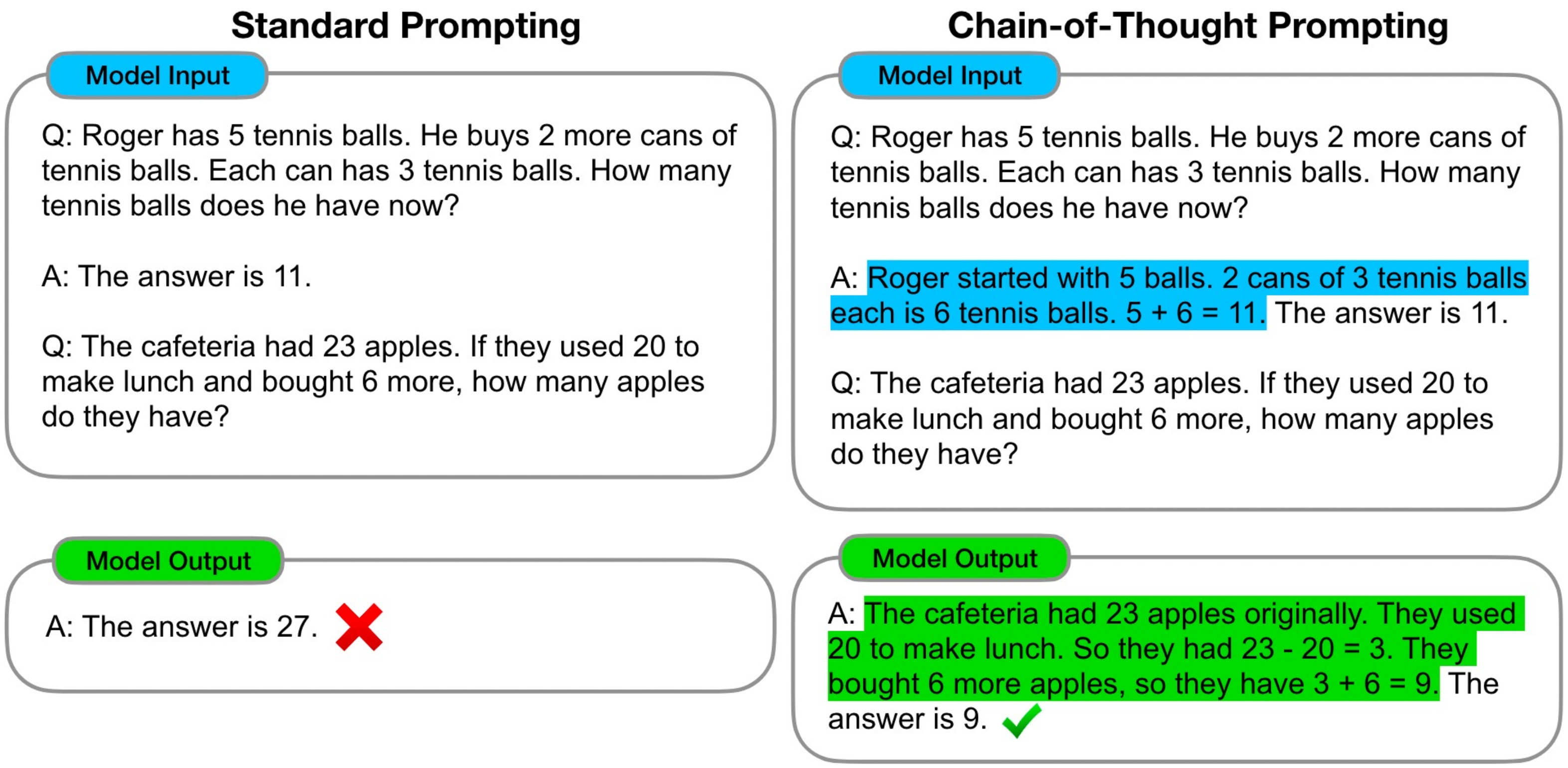


Figure from http://arxiv.org/abs/2202.12837

CHAIN-OF-THOUGHT PROMPTING

Chain-of-Thought Prompting

- Asking the model to reason about its answer can improve its performance for few-shot in-context learning
- **Chain-of-thought prompting** provides such reasoning in the in-context examples



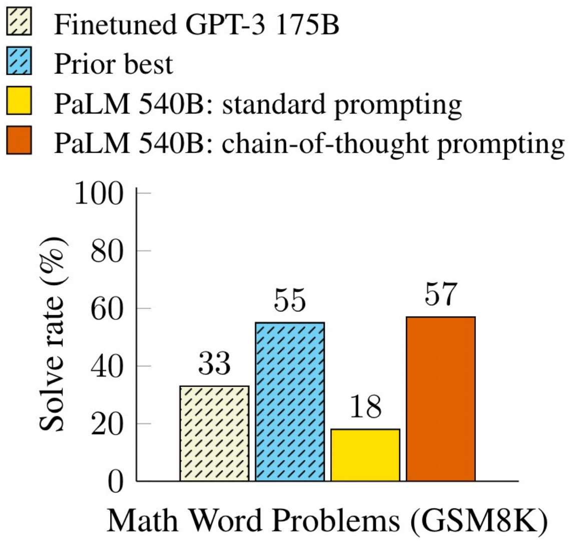


Figure 2: PaLM 540B uses chain-of-thought prompting to achieve new state-of-the-art performance on the GSM8K benchmark of math word problems. Finetuned GPT-3 and prior best are from Cobbe et al. (2021).

Figure from http://arxiv.org/abs/2201.11903

Chain-of-Thought Prompting

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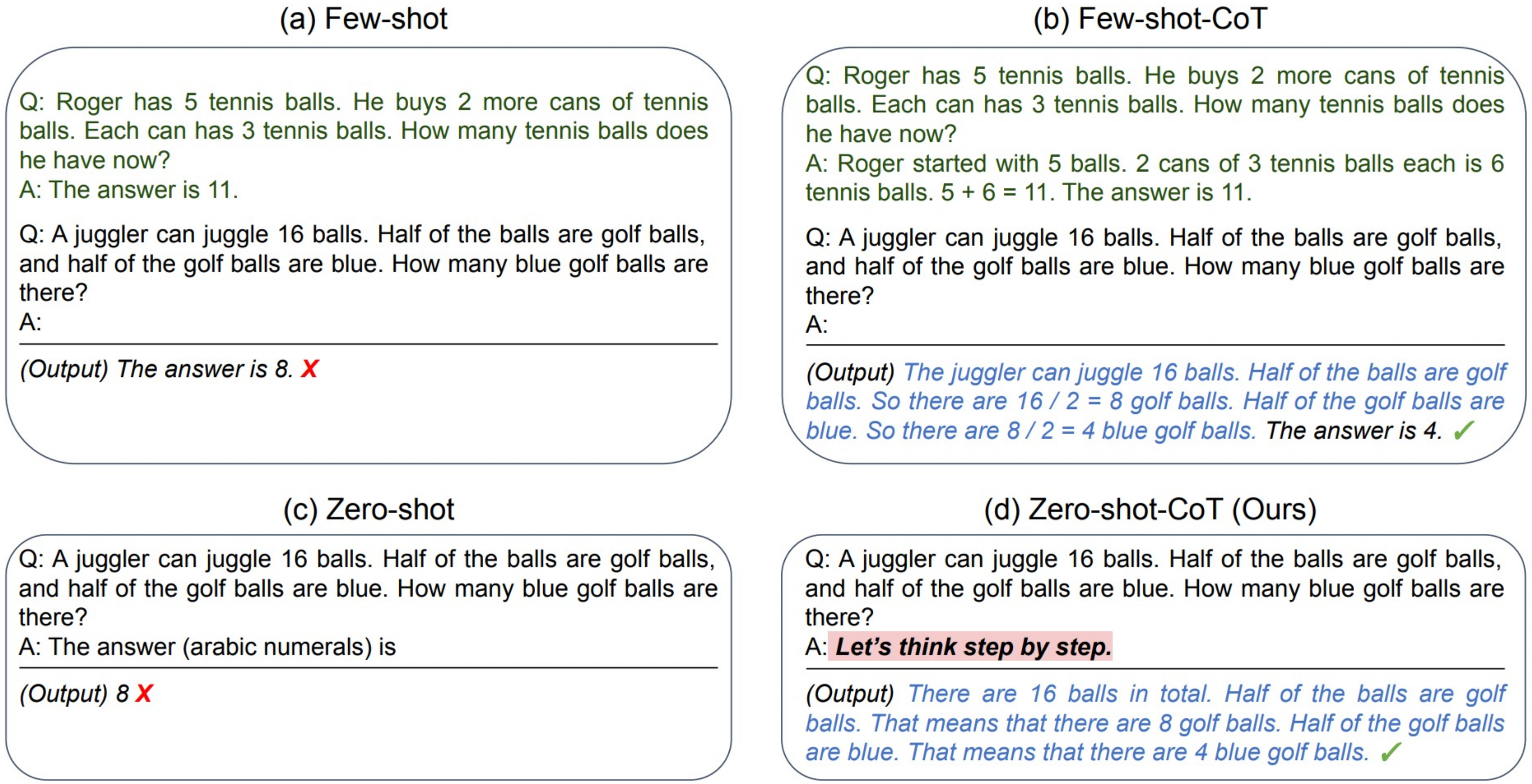

- But the model does better even if you just prompt it to reason step-by-step

Figure from https://arxiv.org/pdf/2205.11916.pdf

Chain-of-Thought Prompting

- Asking the model to reason about its answer can improve its performance for few-shot in-context learning
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	MultiArith	GSM8K
Zero-Shot	**17.7**	**10.4**
Few-Shot (2 samples)	33.7	15.6
Few-Shot (8 samples)	33.8	15.6
Zero-Shot-CoT	**78.7**	**40.7**
Few-Shot-CoT (2 samples)	84.8	41.3

- But the model does better even if you just prompt it to reason step-by-step

Figure from https://arxiv.org/pdf/2205.11916.pdf

(a) Few-shot

Q: Roger has 5 tennis balls. He buys 2 more cans of tennis balls. Each can has 3 tennis balls. How many tennis balls does he have now?
A: The answer is 11.

Q: A juggler can juggle 16 balls. Half of the balls are golf balls, and half of the golf balls are blue. How many blue golf balls are there?
A:

(Output) The answer is 8. ✗

(b) Few-shot-CoT

Q: Roger has 5 tennis balls. He buys 2 more cans of tennis balls. Each can has 3 tennis balls. How many tennis balls does he have now?
A: Roger started with 5 balls. 2 cans of 3 tennis balls each is 6 tennis balls. 5 + 6 = 11. The answer is 11.

Q: A juggler can juggle 16 balls. Half of the balls are golf balls, and half of the golf balls are blue. How many blue golf balls are there?
A:

(Output) The juggler can juggle 16 balls. Half of the balls are golf balls. So there are 16 / 2 = 8 golf balls. Half of the golf balls are blue. So there are 8 / 2 = 4 blue golf balls. The answer is 4. ✓

(c) Zero-shot

Q: A juggler can juggle 16 balls. Half of the balls are golf balls, and half of the golf balls are blue. How many blue golf balls are there?
A: The answer (arabic numerals) is

(Output) 8 ✗

(d) Zero-shot-CoT (Ours)

Q: A juggler can juggle 16 balls. Half of the balls are golf balls, and half of the golf balls are blue. How many blue golf balls are there?
A: ***Let's think step by step.***

(Output) There are 16 balls in total. Half of the balls are golf balls. That means that there are 8 golf balls. Half of the golf balls are blue. That means that there are 4 blue golf balls. ✓