10-301/601: Introduction to Machine Learning Lecture 19: Clustering & Bagging

Henry Chai & Matt Gormley

11/27/23

Front Matter

Announcements

• HW8 released 11/20, due 12/1 (Friday) at 11:59 PM

Clustering

- Goal: split an *unlabeled* data set into groups or clusters of "similar" data points
- Use cases:
 - Organizing data
 - Discovering patterns or structure
 - Preprocessing for downstream machine learning tasks
- Applications:

3

Recall: Similarity for *k*NN

- Intuition: predict the label of a data point to be the label of the "most similar" training point two points are "similar" if the distance between them is small
- Euclidean distance: $d(\mathbf{x}, \mathbf{x}') = \|\mathbf{x} \mathbf{x}'\|_2$

Partition-Based Clustering

- Given a desired number of clusters, K, return a partition of the data set into K groups or clusters, $\{C_1, \dots, C_K\}$, that optimize some objective function
- 1. What objective function should we optimize?

2. How can we perform optimization in this setting?



Example Clusterings

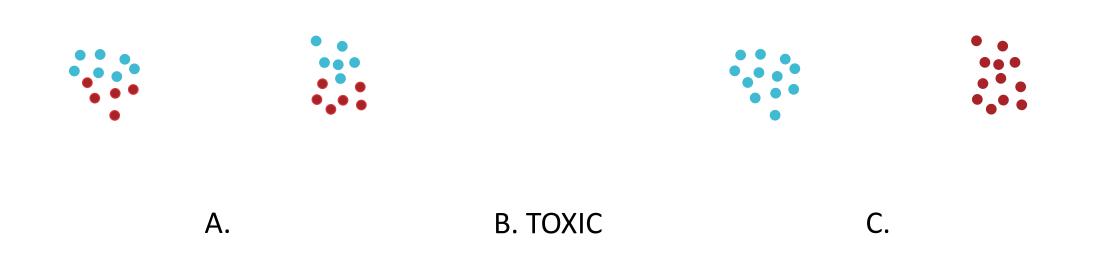




Option A



Example Clusterings



Poll Question 1: Which option do you prefer? Poll Question 2: Justify your response to the previous question

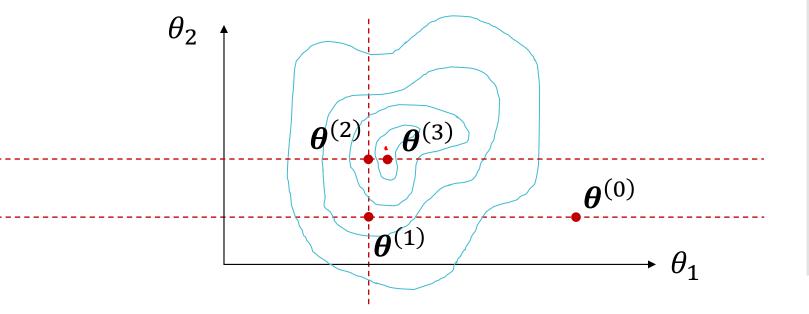
Recipe for *K*-means

• Define a model and model parameters - Assume K clusters al use the Euclidean distance - Parameters: N1,..., NK (centers) Z⁽¹⁾, Z⁽²⁾,..., Z^(N) (assignments) Write down an objective function $\frac{1}{2} \min \sum_{i=1}^{N} \|\mathbf{x}^{(i)} - \boldsymbol{y}_{z^{(i)}}\|_{z}$ • Optimize the objective w.r.t. the model parameters - Block coordinate descent

Coordinate Descent

• Goal: minimize some objective $\widehat{\boldsymbol{\theta}} = \operatorname{argmin} J(\boldsymbol{\theta})$

• Idea: iteratively pick one variable and minimize the objective w.r.t. just that variable, *keeping all others fixed*.



Block Coordinate Descent

- Goal: minimize some objective $\widehat{\alpha}, \widehat{\beta} = \operatorname{argmin} J(\alpha, \beta)$
- Idea: iteratively pick one *block* of variables (*α* or *β*) and minimize the objective w.r.t. that block, keeping the other(s) fixed.
 - Ideally, blocks should be the largest possible set of variables that can be efficiently optimized simultaneously

Optimizing the *K*-means objective

$$\begin{aligned} \hat{\mu}_{1}, \dots, \hat{\mu}_{K}, z^{(1)}, \dots, z^{(N)} &= \operatorname{argmin} \sum_{n=1}^{N} \left\| x^{(n)} - \mu_{z^{(n)}} \right\|_{2} \\ \cdot & \text{If } \mu_{1}, \dots, \mu_{K} \text{ are fixed} \\ \hat{z}^{(i)} &= \left\| \operatorname{argmin}_{k \in \{1, \dots, K\}} \right\| \left\| x^{(i)} - \mu_{k} \right\|_{2} \\ \cdot & \text{If } z^{(1)}, \dots, z^{(N)} \text{ are fixed} \\ \hat{\mu}_{k} &= \frac{1}{N_{k}} \sum_{i:z^{(i)}=k} x^{(i)} \\ & \text{wher } N_{k} \quad \text{is the $\#$ of determined} \\ & \text{wher } N_{k} \quad \text{is the $\#$ of determined} \end{aligned}$$

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11/27/23

K-means Algorithm

- Input: $\mathcal{D} = \{ (\boldsymbol{x}^{(n)}) \}_{n=1}^{N}, K$
- 1. Initialize cluster centers μ_1, \dots, μ_K
- 2. While NOT CONVERGED
 - a. Assign each data point to the cluster with the nearest cluster center:

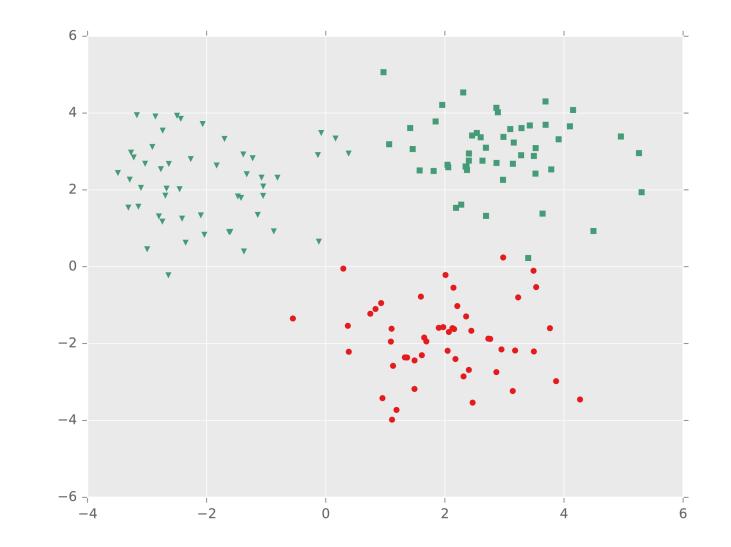
 $z^{(n)} = \underset{k}{\operatorname{argmin}} \left\| \boldsymbol{x}^{(n)} - \boldsymbol{\mu}_{k} \right\|_{2}$

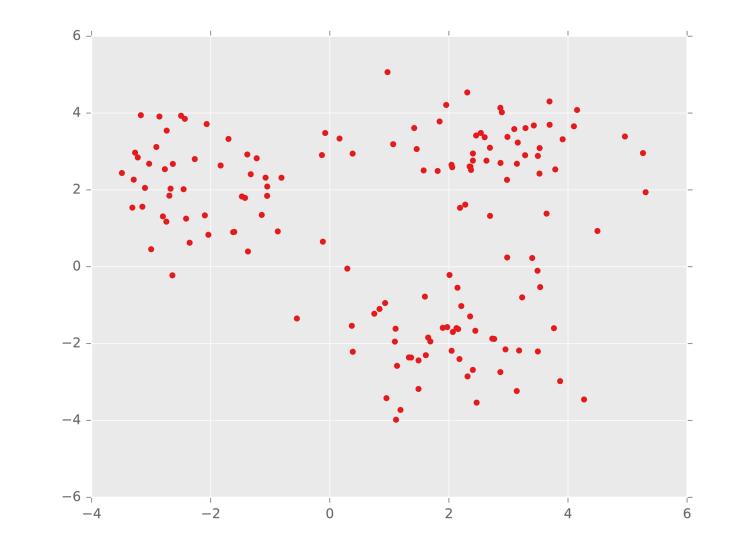
b. Recompute the cluster centers:

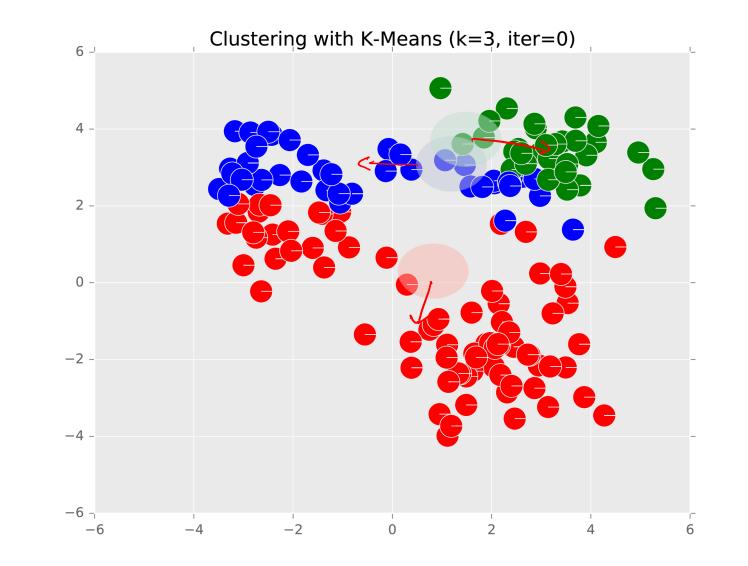
$$\boldsymbol{\mu}_k = \frac{1}{N_k} \sum_{\substack{n:z^{(n)}=k}} \boldsymbol{x}^{(n)}$$

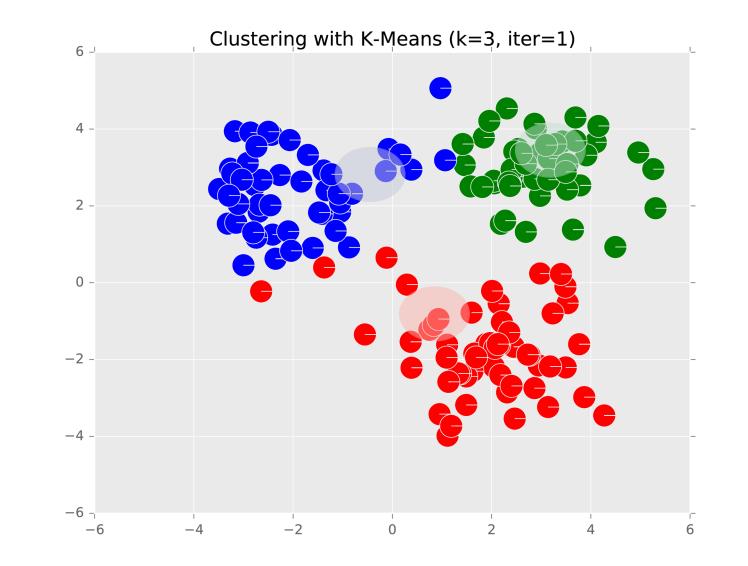
where N_k is the number of data points in cluster k

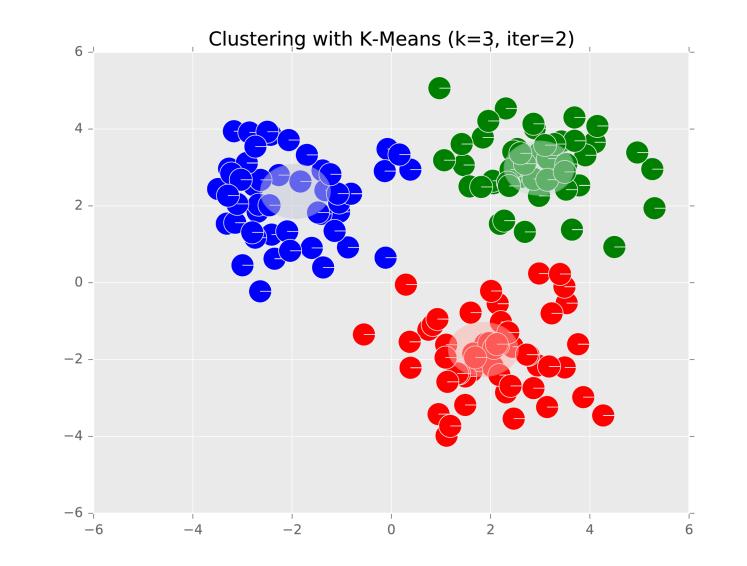
• Output: cluster centers $\mu_1, ..., \mu_K$ and cluster assignments $z^{(1)}, ..., z^{(N)}$

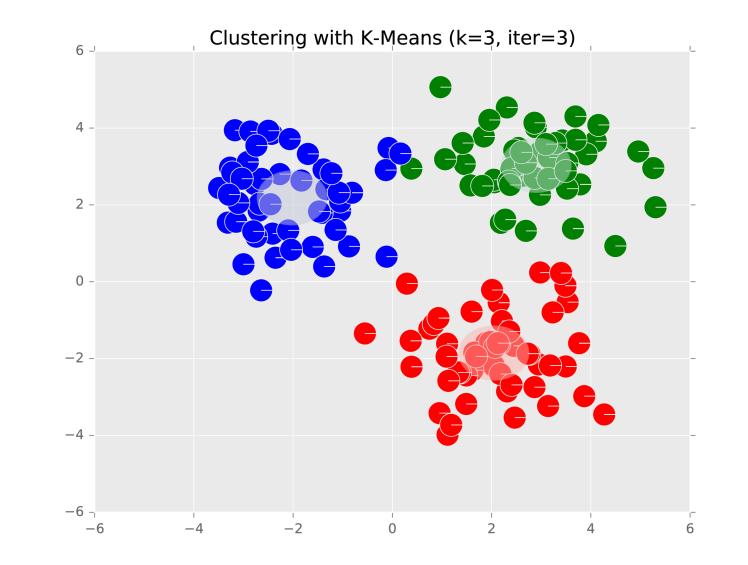


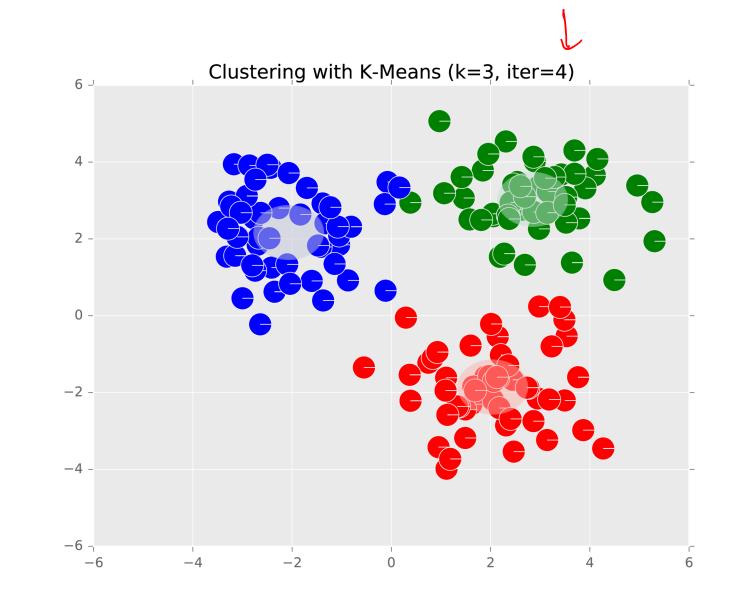


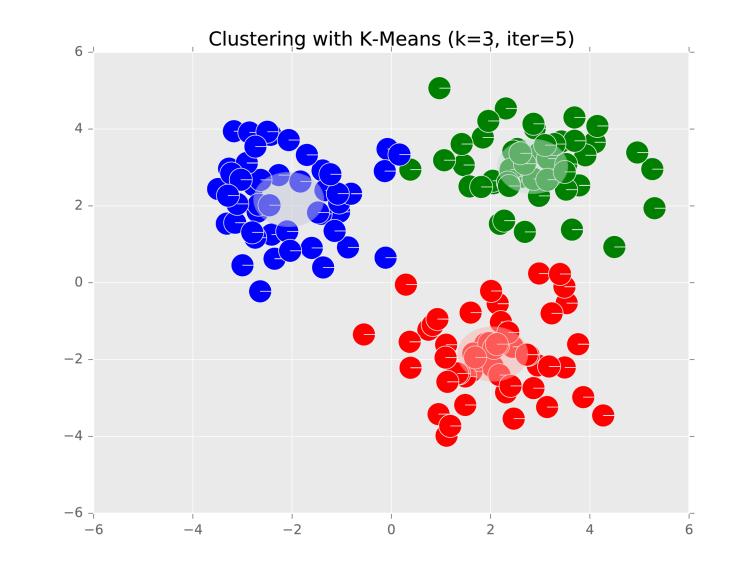


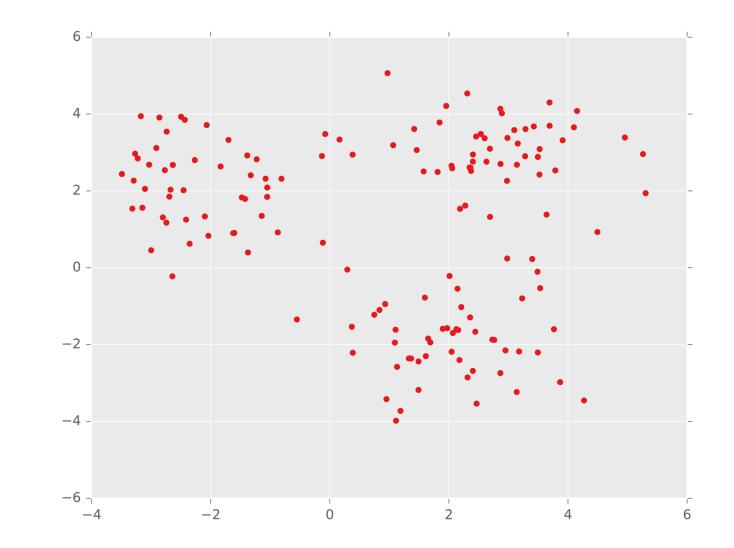


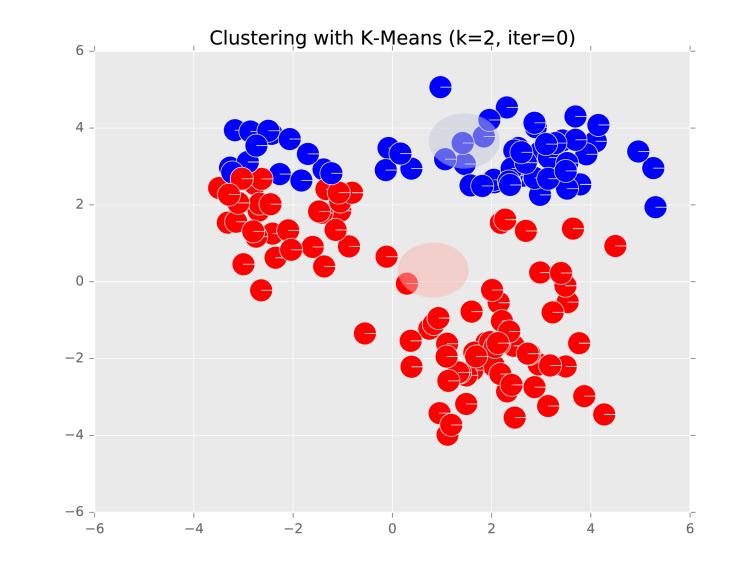


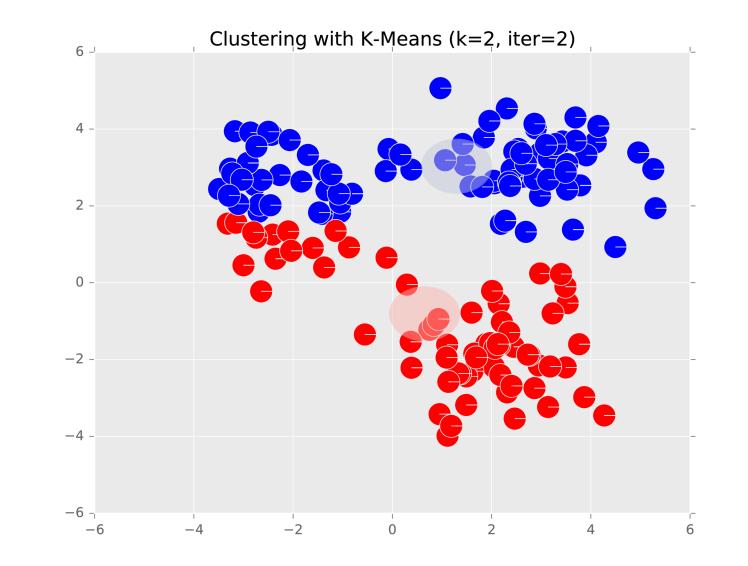


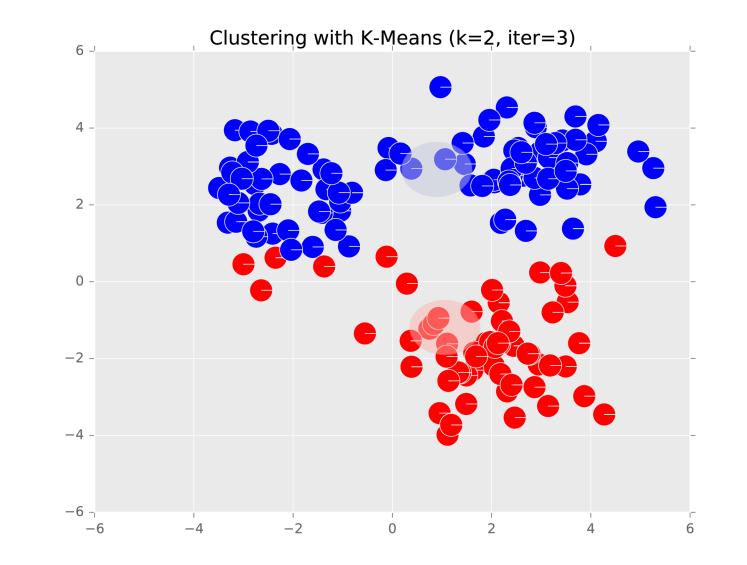


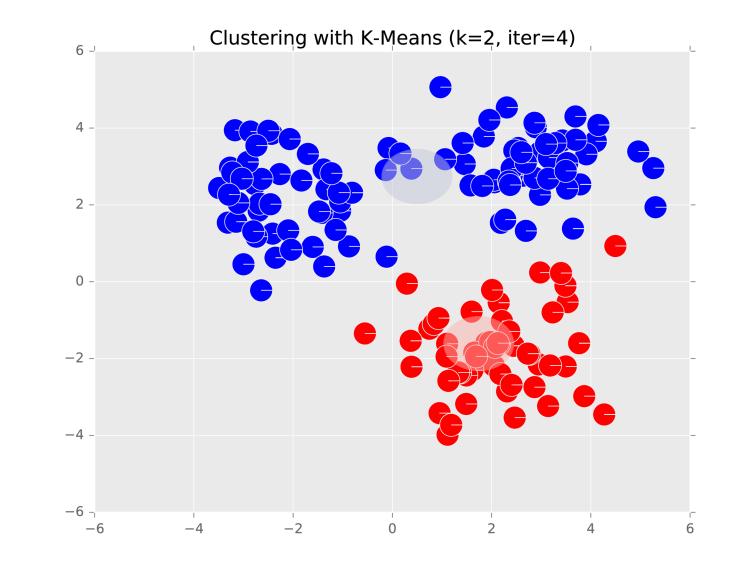


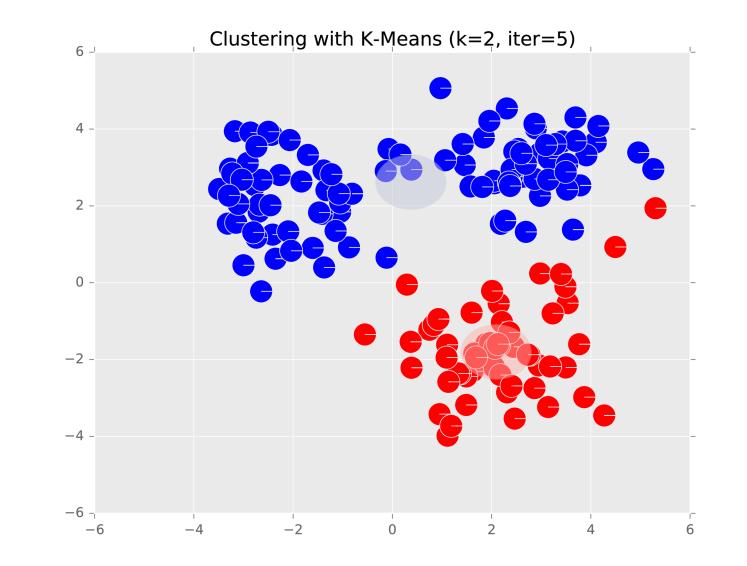


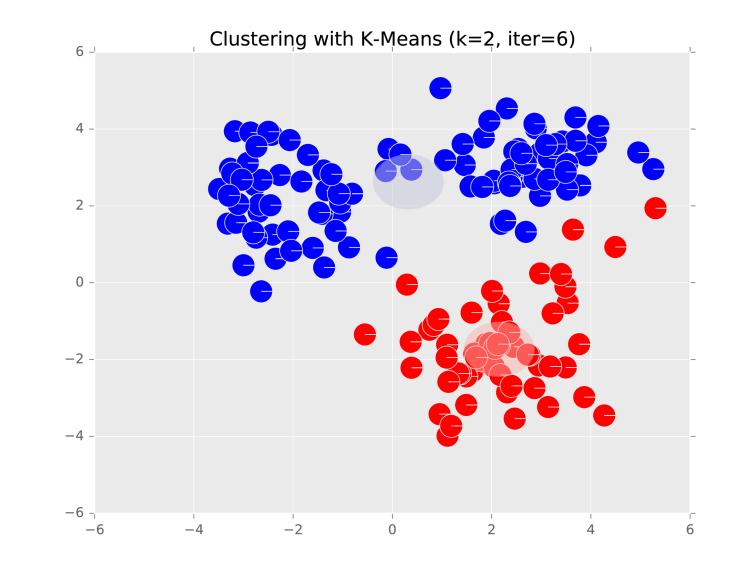


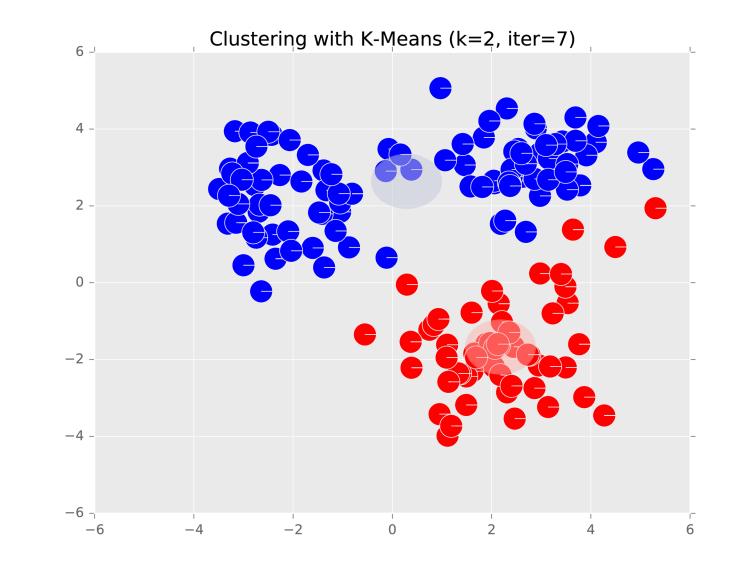






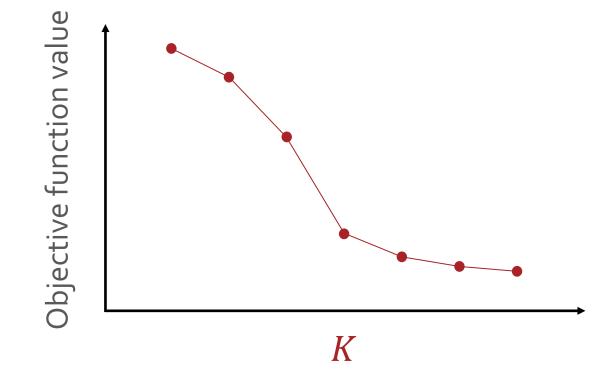




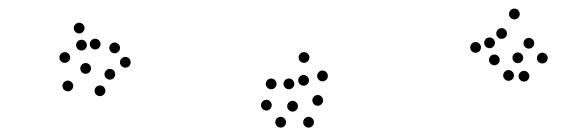


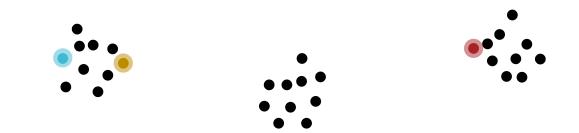
Setting K

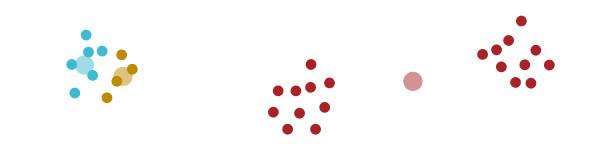
• Idea: choose the value of *K* that minimizes the objective function



• Better Idea: look for the characteristic "elbow" or largest decrease when going from K - 1 to K

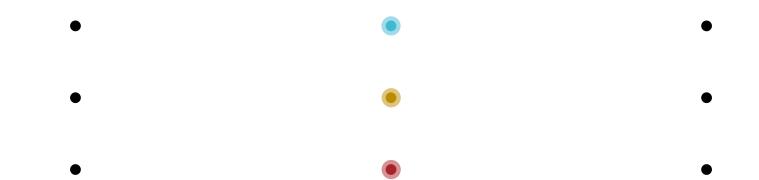






- Common choice: choose *K* data points at random to be the initial cluster centers (Lloyd's method)

 - • •



- Lloyd's method converges to a local minimum and that local minimum can be arbitrarily bad (relative to the optimal clusters)
 - This is because the *K*-means objective is nonconvex!
- Intuition: want initial cluster centers to be far apart from one another

K-means++ (Arthur and Vassilvitskii, 2007)

- 1. Choose the first cluster center randomly from the data points.
- 2. For each other data point x, compute D(x), the distance between x and the nearest cluster center.
- 3. Select the next cluster center proportional to $D(x)^2$.
- 4. Repeat 2 and 3 K 1 times.
- *K*-means++ achieves a O(log K) approximation to the optimal clustering in expectation
 - Both Lloyd's method and *K*-means++ can benefit from multiple random restarts.

K-means Learning Objectives

- You should be able to...
- Distinguish between coordinate descent and block coordinate descent
- 2. Define an objective function that gives rise to a "good" clustering
- 3. Apply block coordinate descent to an objective function preferring each point to be close to its nearest objective function to obtain the K-Means algorithm
- 4. Implement the K-Means algorithm
- 5. Connect the non-convexity of the K-Means objective function with the (possibly) poor performance of random initialization

The Netflix Prize

Netflix Prize

• 500,000 users

Leaderboard

Home

Rules

- 20,000 movies
- 100 million ratings
- Goal: To obtain lower error than Netflix's existing system on 3 million held out ratings

Update

Congratulations!

The Netflix Prize sought to substantially improve the accuracy of predictions about how much someone is going to enjoy a movie based on their movie preferences.

On September 21, 2009 we awarded the \$1M Grand Prize to team "BellKor's Pragmatic Chaos". Read about <u>their</u> <u>algorithm</u>, checkout team scores on the <u>Leaderboard</u>, and join the discussions on the <u>Forum</u>.

We applaud all the contributors to this quest, which improves our ability to connect people to the movies they love.

The Netflix Prize

Neti	XII	Pri	ze

Home Rules Leaderboard Update Download

Leaderboard

Showing Test Score. Click here to show quiz score

Display top $20 \sim$ leaders.

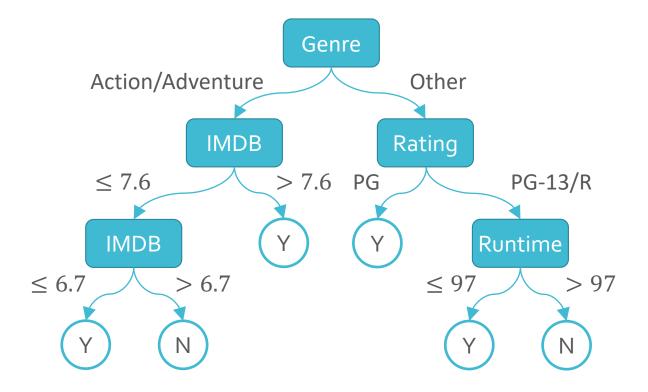
	Rank	Team Name	Best Test Score	% Improvement	Best Submit Time
	Grand	Prize - RMSE = 0.8567 - Winning Te	am: BellKor's Pragn	natic Chaos	
~)	1	BellKor's Pragmatic Chaos	0.8567	10.06	2009-07-26 18:18:28
(2	The Ensemble	0.8567	10.06	2009-07-26 18:38:22
	3	Grand Prize Team	0.8582	9.90	2009-07-10 21:24:40
	4	Opera Solutions and Vandelay United	0.8588	9.84	2009-07-10 01:12:31
	5	Vandelay Industries !	0.8591	9.81	2009-07-10 00:32:20
	6	PragmaticTheory	0.8594	9.77	2009-06-24 12:06:56
	7	BellKor in BigChaos	0.8601	9.70	2009-05-13 08:14:09
5	8	Dace_	0.8612	9.59	2009-07-24 17:18:43
ę	9	Feeds2	0.8622	9.48	2009-07-12 13:11:51
9 ·	10	BigChaos	0.8623	9.47	2009-04-07 12:33:59
الاسب	11	Opera Solutions	0.8623	9.47	2009-07-24 00:34:07
	12	BellKor	0.8624	9.46	2009-07-26 17:19:11

COMPLETED

MovielD	Runtime	Genre	Budget	Year	IMDB	Rating	Liked?
1	124	Action	18M	1980	8.7	PG	Y
2	105	Action	30M	1984	7.8	PG	Y
3	103	Comedy	6M	1986	7.8	PG-13	Ν
4	98	Adventure	16M	1987	8.1	PG	Y
5	128	Comedy	16.4M	1989	8.1	PG	Y
6	120	Comedy	11M	1992	7.6	R	Ν
7	120	Drama	14.5M	1996	6.7	PG-13	Ν
8	136	Action	115M	1999	6.5	PG	Y
9	90	Action	90M	2001	6.6	PG-13	Y
10	161	Adventure	100M	2002	7.4	PG	Ν
11	201	Action	94M	2003	8.9	PG-13	Y
12	94	Comedy	26M	2004	7.2	PG-13	Y
13	157	Biography	100M	2007	7.8	R	Ν
14	128	Action	110M	2007	7.1	PG-13	Ν
15	107	Drama	39M	2009	7.1	PG-13	Ν
16	158	Drama	61M	2012	7.6	PG-13	Ν
17	169	Adventure	165M	2014	8.6	PG-13	Y
18	100	Biography	9M	2016	6.7	R	Ν
19	130	Action	180M	2017	7.9	PG-13	Y
20	141	Action	275M	2019	6.5	PG-13	Y

Movie Recommendations

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Recall: Decision Tree Pros & Cons

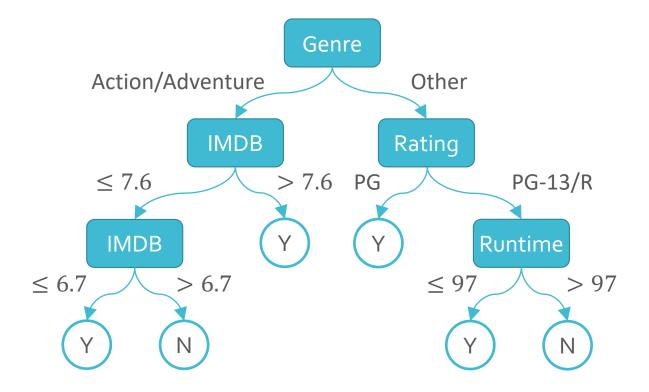
• Pros

- Interpretable
- Efficient (computational cost and storage)
- Can be used for classification and regression tasks
- Compatible with categorical and real-valued features

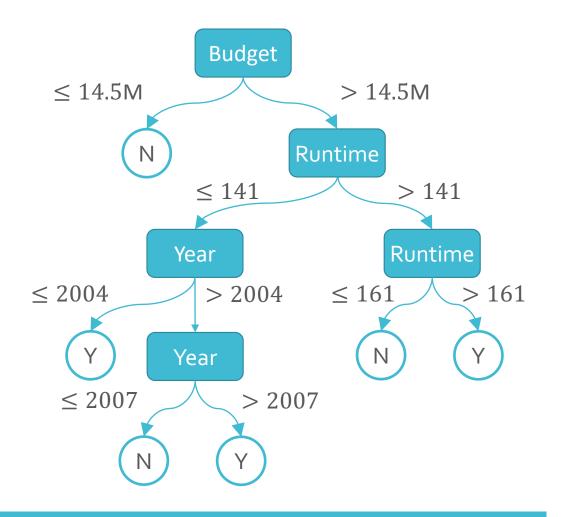
• Cons

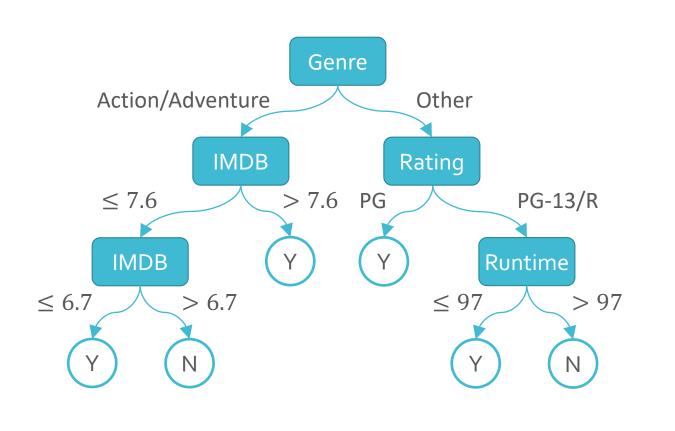
- Learned greedily: each split only considers the immediate impact on the splitting criterion
 - Not guaranteed to find the smallest (fewest number of splits) tree that achieves a training error rate of 0.
- Prone to overfit
- High variance

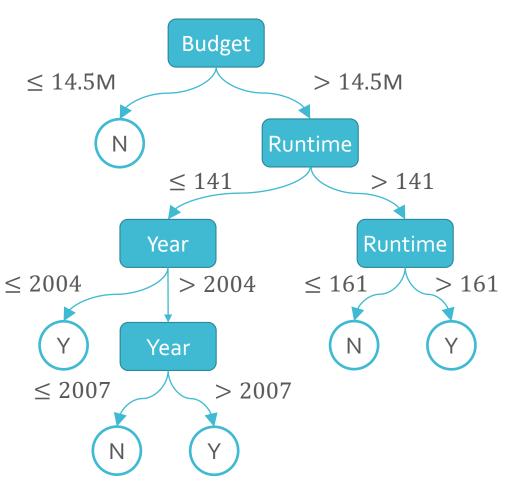
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Decision Trees: Pros & Cons

• Pros

- Interpretable
- Efficient (computational cost and storage)
- Can be used for classification and regression tasks
- Compatible with categorical and real-valued features

• Cons

- Learned greedily: each split only considers the immediate impact on the splitting criterion
 - Not guaranteed to find the smallest (fewest number of splits) tree that achieves a training error rate of 0.
- Prone to overfit
- High variance
 - Can be addressed via ensembles \rightarrow random forests

Random Forests Combines the prediction of many diverse decision trees to reduce their variability

• If *B* independent random variables $x^{(1)}, x^{(2)}, ..., x^{(B)}$ all have variance σ^2 , then the variance of $\frac{1}{B} \sum_{b=1}^{B} x^{(b)}$ is $\frac{\sigma^2}{B}$

• Random forests = sample bagging + feature bagging

= **b**ootstrap **<u>agg</u>**regat**<u>ing</u>** + split-feature randomization

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= bootstrap aggregating + split-feature randomization

Aggregating

- How can we combine multiple decision trees, $\{t_1, t_2, \dots, t_B\}$, to arrive at a single prediction?
- Regression average the predictions:

$$\bar{t}(\boldsymbol{x}) = \frac{1}{B} \sum_{b=1}^{B} t_b(\boldsymbol{x})$$

 Classification - plurality (or majority) vote; for binary labels encoded as {-1, +1}:

$$\bar{t}(\boldsymbol{x}) = \operatorname{sign}\left(\frac{1}{B}\sum_{b=1}^{B}t_{b}(\boldsymbol{x})\right)$$

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Bootstrapping

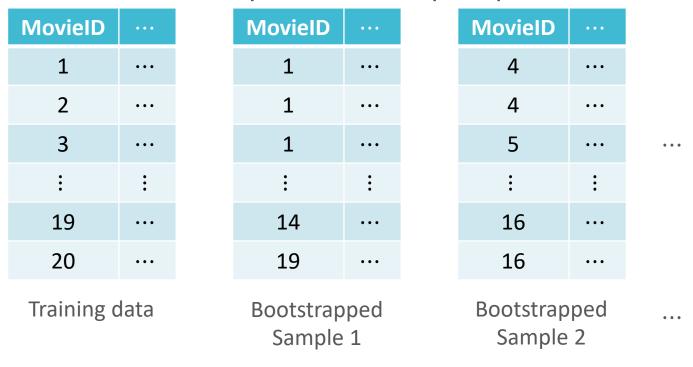
- Insight: one way of generating different decision trees is by changing the training data set
- Issue: often, we only have one fixed set of training data
- Idea: resample the data multiple times *with replacement*

ovielD		MovielD		MovielD	
1	•••	1		4	•••
2	•••	1	•••	4	•••
3	•••	1		5	•••
:	÷	:	÷	:	:
19	••••	14		16	
20	•••	19	•••	16	•••
Training data		Bootstrap Sample		Bootstrap Sample	-

Bootstrapping

Idea: resample the data multiple times with replacement
Each bootstrapped sample has the same number of data points as the original data set

• Duplicated points cause different decision trees to focus on different parts of the input space



- Issue: decision trees trained on bootstrapped samples still behave similarly
- Idea: in addition to sampling the data points (i.e., the rows), also sample the features (i.e., the columns)
- Each time a split is being considered, limit the possible features to a randomly sampled subset

Runtime Genre Budget Year IMDB Rating

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Random Forests

- Input: $\mathcal{D} = \{ (\mathbf{x}^{(n)}, y^{(n)}) \}_{n=1}^{N}, B, \rho$
- For b = 1, 2, ..., B
 - Create a dataset, \mathcal{D}_b , by sampling N points from the original training data \mathcal{D} with replacement
 - Learn a decision tree, t_b , using \mathcal{D}_b and the ID3 algorithm with split-feature randomization, sampling ρ features for each split

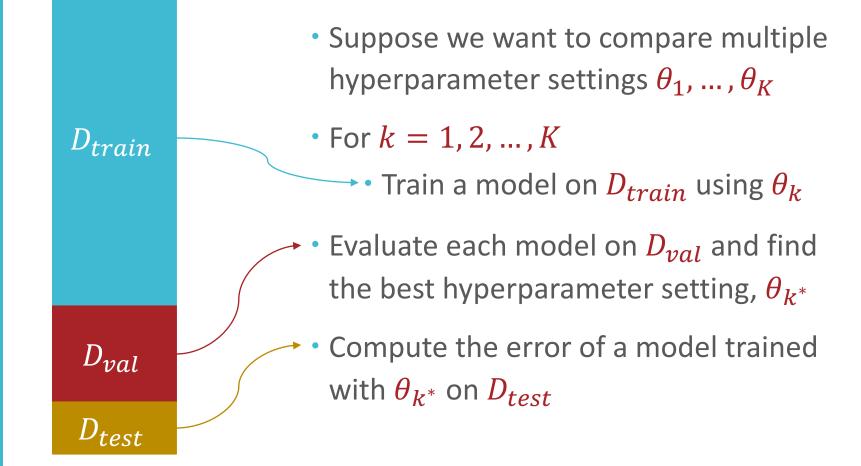
• Output: $\overline{t} = f(t_1, \dots, t_B)$, the aggregated hypothesis

How can we set *B* and ρ ?

- Input: $\mathcal{D} = \{ (x^{(n)}, y^{(n)}) \}_{n=1}^{N}, B, \rho$
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Recall: Validation Sets



• For each training point, $x^{(n)}$, there are some decision trees which $x^{(n)}$ was not used to train (roughly B/e trees or 37%)

• Let these be
$$t^{(-n)} = \left\{ t_1^{(-n)}, t_2^{(-n)}, \dots, t_{N_{-n}}^{(-n)} \right\}$$

• Compute an aggregated prediction for each $x^{(n)}$ using the trees in $t^{(-n)}$, $\overline{t}^{(-n)}(x^{(n)})$

• Compute the out-of-bag (OOB) error, e.g., for regression $E_{OOB} = \frac{1}{N} \sum_{n=1}^{N} (\bar{t}^{(-n)} (\boldsymbol{x}^{(n)}) - y^{(n)})^2$

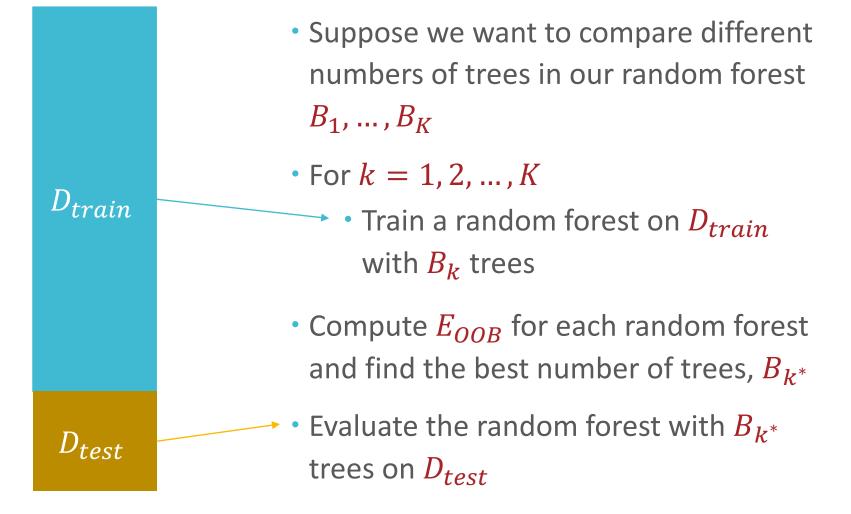
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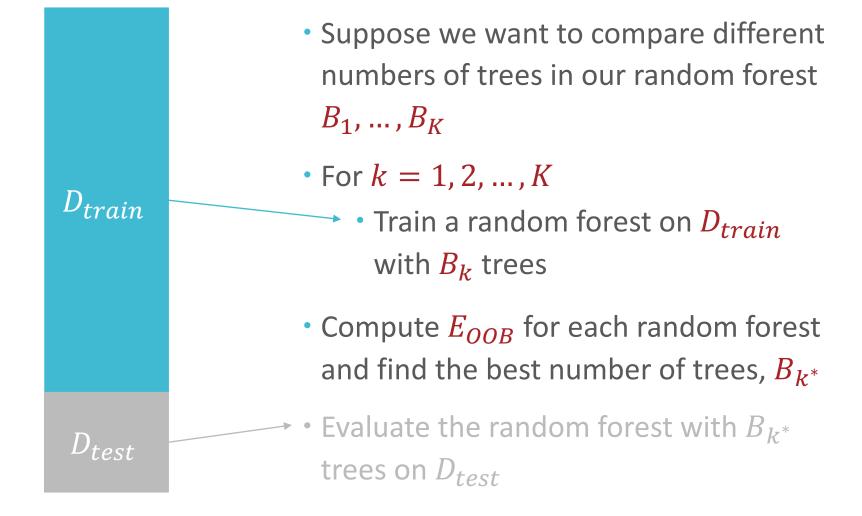
• Let these be
$$t^{(-n)} = \left\{ t_1^{(-n)}, t_2^{(-n)}, \dots, t_{N_{-n}}^{(-n)} \right\}$$

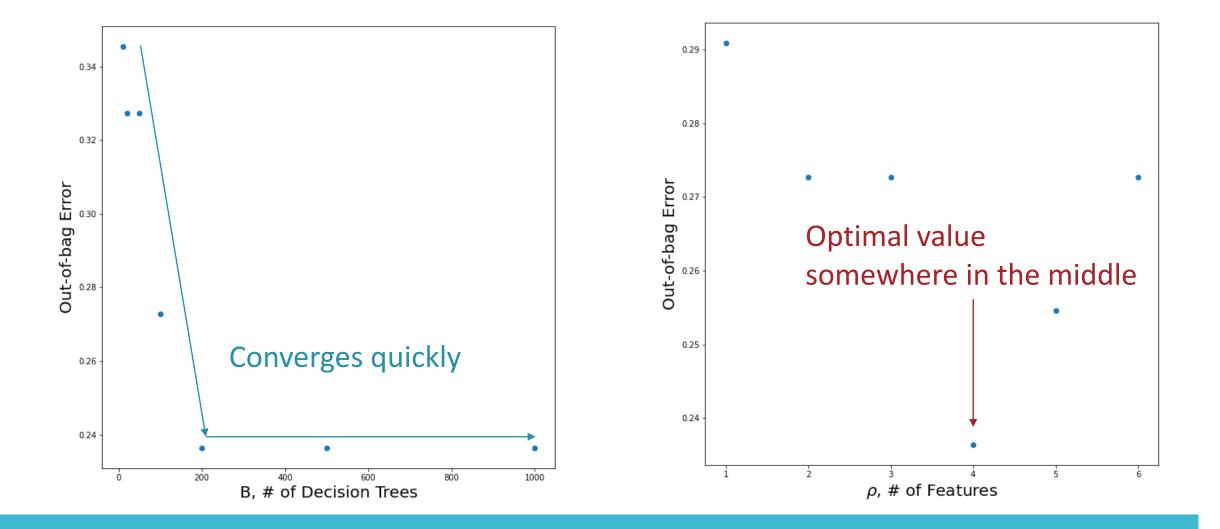
• Compute an aggregated prediction for each $x^{(n)}$ using the trees in $t^{(-n)}$, $\overline{t}^{(-n)}(x^{(n)})$

• Compute the out-of-bag (OOB) error, e.g., for classification $E_{OOB} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}(\bar{t}^{(-n)}(\boldsymbol{x}^{(n)}) \neq y^{(n)})$

• *E*_{00B} can be used for hyperparameter optimization!







Setting Hyperparameters

Feature Importance

- Some of the interpretability of decision trees gets lost when switching to random forests
- Random forests allow for the computation of "feature importance", a way of ranking features based on how useful they are at predicting the target
- Initialize each feature's importance to zero
- Each time a feature is chosen to be split on, add the reduction in entropy (weighted by the number of data points in the split) to its importance

Feature Importance

