Logic and Mechanized Reasoning

Satisfiability Modulo Theories

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Based on tutorials by Barrett, Griggio, Jovanović, de Moura, Oliveras, and Tinelli

SMT Overview

SMT Solving

SMT Theories

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SMT Theories

Overview: Satisfiability Modulo Theories

Problem

Check if a quantifier-free first-order formula can be satisfied with respect to some background theories.

Example (QF_UFLRA)

$$(z = 1 \vee z = 0) \wedge (x - y + z = 1) \wedge (f(x) > f(y))$$

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2. Uninterpreted functions (UF).

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1. Linear real arithmetic (LRA).
2. Uninterpreted functions (UF).
3. Satisfiable with $z \mapsto 0, x \mapsto 1, y \mapsto 0, f(1) \mapsto 1, f(0) \mapsto 0$

Overview: Many SMT Applications I

Schedule n jobs, each composed of m consecutive tasks, on m machines using at most t timeslots.

Schedule in 8 time slots.

$d_{i,j}$	Machine 1	Machine 2
Job 1	2	1
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$$
\begin{array}{l}
(t_{1,1} \geq 0) \wedge (t_{1,2} \geq t_{1,1} + 2) \wedge (t_{1,2} + 1 \leq 8) \\
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Example from [DMB11]

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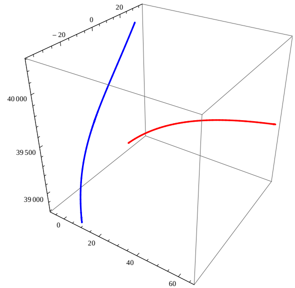
\end{array}
$$

Example from [DMB11]

Run SMT solver (QF_IDL)

$$t_{1,1} \mapsto 5,\ \ t_{1,2} \mapsto 7,\ \ t_{2,1} \mapsto 2,\ \ t_{2,2} \mapsto 6,\ \ t_{3,1} \mapsto 0,\ \ t_{3,2} \mapsto 3$$

Overview: Many SMT Applications II



$$T_1^x(t) = -3.2484 + 270.7t + 433.12t^2 - 324.83999t^3$$
$$T_1^y(t) = 15.1592 + 108.28t + 121.2736t^2 - 649.67999t^3$$
$$T_1^z(t) = 38980.8 + 5414t - 21656t^2 + 32484t^3$$

$$T_2^x(t) = 1.0828 - 135.35t + 234.9676t^2 2 + 3248.4t^3$$
$$T_2^y(t) = 18.40759 - 230.6364t - 121.2736t^2 - 649.67999t^3$$
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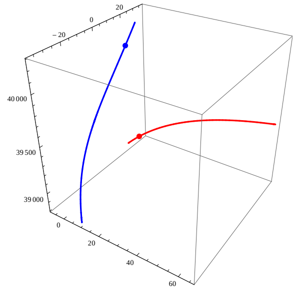
$D = 5$ $\qquad H = 1000$ $\qquad 0 \leq t \leq \frac{1}{20}$

$|T_1^z(t) - T_2^z(t)| \leq H \qquad (T_1^x(t) - T_2^x(t))^2 + (T_1^y(t) - T_2^y(t))^2 \leq D^2$

Example from [NM12]

Overview: Many SMT Applications II



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Example from [NM12]

Run SMT solver (QF_NRA)

$$t \mapsto \frac{319}{16384} \approx 0.019470215$$

Overview: Modeling and Solving

Modeling

- Depending on the problem domain, select a fitting theory.
- Consider expressivity vs solving complexity.

Solving

- Get an SMT solver that supports the theory.
- Hope for the best.

Overview: Uninterpreted Functions

Uninterpreted Functions (QF_UF)

Simplest first-order theory, with equality, applications of uninterpreted functions, and variables of uninterpreted types.

Reflexivity: $x = x$

Symmetry: $x = y \Rightarrow y = x$

Transitivity: $x = y \wedge y = z \Rightarrow x = z$

Congruence: $x = y \Rightarrow f(x) = f(y)$

Example

$$f(f(f(x))) = x \ \wedge \ f(f(f(f(f(x))))) = x \ \wedge \ f(x) \neq x$$

Overview: Arrays

Theory of Arrays [McC93]

Operates over types array, index, element and function symbols

$$\begin{aligned} \_[\_] &: \text{array} \times \text{index} \mapsto \text{element} \\ \text{store} &: \text{array} \times \text{index} \times \text{element} \mapsto \text{array} \quad . \end{aligned}$$

Read-Over-Write-1: $\text{store}(a, i, e)[i] = e$

Read-Over-Write-2: $i \neq j \Rightarrow \text{store}(a, i, e)[j] = a[j]$

Extensionality: $a \neq b \Rightarrow \exists i : a[i] \neq b[i]$

Example

$$\text{store}(\text{store}(a, i, a[j]), j, a[i]) = \text{store}(\text{store}(a, j, a[i]), i, a[j])$$

Overview: Arithmetic

Arithmetic

Arithmetic constraints (inequalities, equalities) over arithmetic (real or integer) variables.

- Difference logic (QF_RDL, QF_IDL):

$$x - y \leq 1 \ , \qquad\qquad x - y > 10 \ .$$

- Linear arithmetic (QF_LRA, QF_LIA):

$$2x - 3y + 4z \leq 5 \ .$$

- Non-linear arithmetic (QF_NRA, QF_NIA):

$$x^2 + 3xy + y^2 > 0 \ .$$

Overview: Bitvectors

Bitvectors (QF_BV)

Operates over fixed-size bit-vectors, with bit-vector operations:

- concatenation $a \circ b$, extraction $a[i:j]$
- bit-wise operators $\sim a$, $a|b$, $a \& b$, ...
- shifts $a \ll k$, $b \gg k$ (logical, arithmetic)
- arithmetic $a + b$, $a - b$, $a * b$, a/b, ...
- predicates $=$, $<$, $\leq$, ... (signed and unsigned)

Semantics similar to programming languages.

Example (a is 32-bits)

$$(\sim a \,\&\, (a+1)) >_u a$$

Overview: Other Interesting Theories

Some other theories

- Floating point [BDG+14, ZWR14]
- Inductive data-types [BST07]
- Strings and regular expressions [LRT+14, KGG+09]
- Quantifiers [DMB07, RTG+13]
- Differential Equations [GKC13]
- ...

SMT Overview

SMT Solving

SMT Theories

Solving: Introduction

Check T-satisfiability of a T-formula Γ

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1. Convert to DNF

$$\Gamma \Leftrightarrow \bigvee_{i=1}^{D} (L_1^i \wedge L_2^i \wedge \cdots \wedge L_{n_i}^i) \ .$$

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2. If any of disjuncts is T-satisfiable, return SAT, else UNSAT.

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2. If any of disjuncts is T-satisfiable, return SAT, else UNSAT.

Theory solver/Decision procedure for T

Procedure to decide satisfiability of a conjunction of T-literals.

Solving: Apply a SAT Solver

Use a SAT solver

- Instead of DNF: Apply a SAT solver.
- Check the literals selected by the SAT solver.
- If not T-satisfiable, add a blocking clause.

Solving: Very Lazy SMT Example

View

Theory

$$\neg a = b$$
$$(x = a \vee x = b)$$
$$(y = a \vee y = b)$$
$$(z = a \vee z = b)$$
$$\neg x = y$$

Solving: Very Lazy SMT Example

View
Boolean

$$
\begin{array}{l}
\neg p_1 \\
(\; p_2 \;\lor\; p_3 \;) \\
(\; p_4 \;\lor\; p_5 \;) \\
(\; p_6 \;\lor\; p_7 \;) \\
\neg p_8
\end{array}
$$

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Check with SAT solver

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$$(\ p_6 \lor p_7\ )$$
$$\neg p_8$$

Check with SAT solver

$$[\![\ \neg p_1\ ,\ \neg p_8\ ,\ \neg p_3\ ,\ p_2\ ,\ \neg p_5\ ,\ p_4\ ,\ \neg p_7\ ,\ p_6\ ]\!]$$

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$$[\![\neg a = b, \neg x = y, \neg x = b, x = a, \neg y = b, y = a, \neg z = b, z = a]\!]$$

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Check with SAT solver

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Check with T-solver

$$x = a \wedge y = a \Rightarrow x = y$$

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Check with SAT solver

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Check with T-solver

$$x = a \wedge y = a \Rightarrow x = y$$

Add blocking clause: $x = y \vee \neg x = a \vee \neg y = a$

Solving: Very Lazy SMT Example

View
Boolean

$$\begin{array}{l} \neg p_1 \\ (\ p_2 \ \vee \ p_3 \ ) \\ (\ p_4 \ \vee \ p_5 \ ) \\ (\ p_6 \ \vee \ p_7 \ ) \\ \neg p_8 \\ (\ p_8 \ \vee \ \neg p_2 \ \vee \ \neg p_4 \ ) \end{array}$$

Check with SAT solver

Solving: Very Lazy SMT Example

View
Boolean

$$\neg p_1$$
$$(\ p_2 \ \lor\ p_3\ )$$
$$(\ p_4 \ \lor\ p_5\ )$$
$$(\ p_6 \ \lor\ p_7\ )$$
$$\neg p_8$$
$$(\ p_8 \ \lor\ \neg p_2 \ \lor\ \neg p_4\ )$$

Check with SAT solver

$$[\![\ \neg p_1\ ,\ \neg p_8\ ,\ \neg p_3\ ,\ p_2\ ,\ \neg p_4\ ,\ p_5\ ,\ \neg p_7\ ,\ p_6\ ]\!]$$

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View
Theory

Check with SAT solver

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Check with T-solver

Satisfiable: $a, x, z \mapsto c_1, \quad b, y \mapsto c_2$

Solving: Very Lazy SMT

Properties

- SAT and T-solver only communicate via existing literals.
- Number of possible conflicts finite $\Rightarrow$ termination.
- Reuse the improvements in SAT solving.
- SAT solver is "blind" and can get lost :(.

Solving: Very Lazy SMT

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Integrate closely with SAT solver: DPLL(T) [DMR02, NOT05]

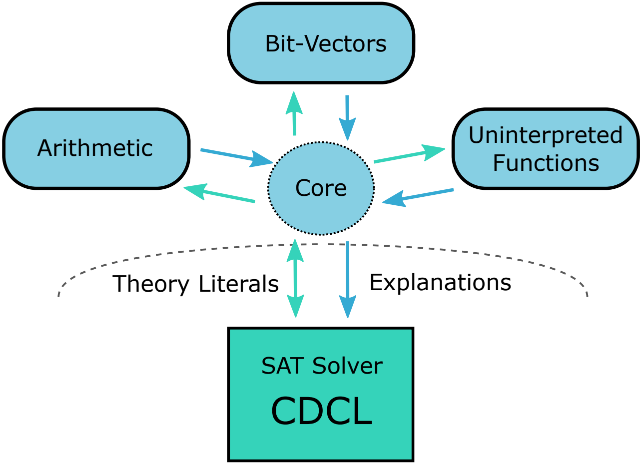
Incremental: Check T-satisfiability along the SAT solver.

Backtrack: Backtrack with SAT solver and keep context.

Propagation: If existing literals are implied tell SAT solver

Conflict: Small conflict explanations.

Solving: Typical Architecture



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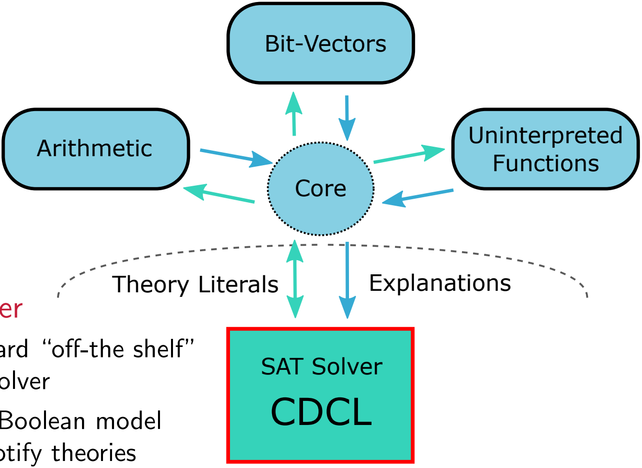


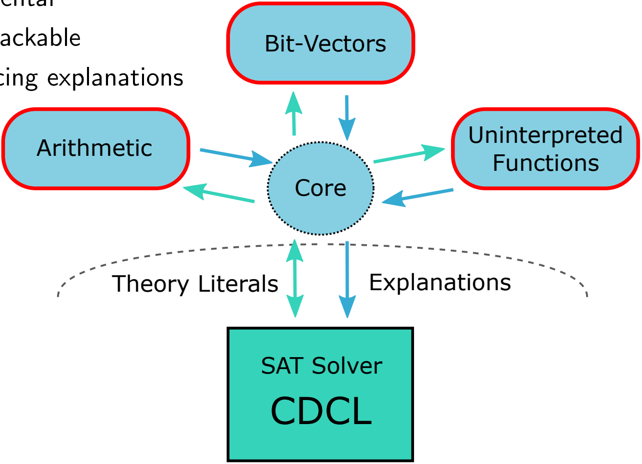
SAT Solver

- Standard "off-the shelf" SAT solver
- Build Boolean model and notify theories

Solving: Typical Architecture

Theory Decision Procedures

- Check conjunctions of literals
- Incremental
- Backtrackable
- Producing explanations



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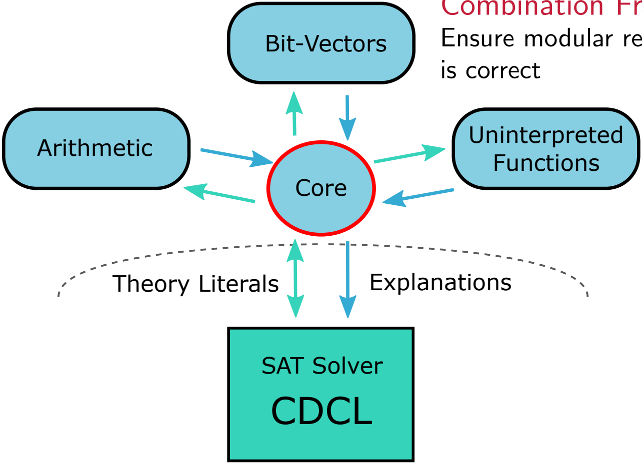
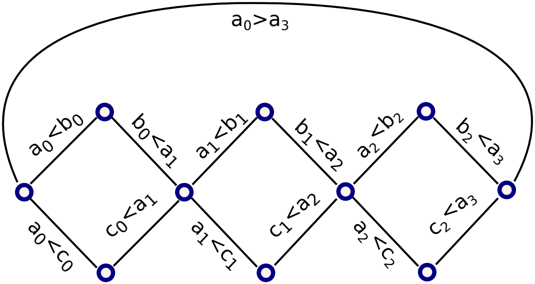


Combination Framework

Ensure modular reasoning is correct

Solving: Great but not Perfect



Example (Diamonds)

$$a_0 > a_n \wedge \bigwedge_{k=0}^{n-1} \left((a_k < b_k \wedge b_k < a_{k+1}) \vee (a_k < c_k \wedge c_k < a_{k+1}) \right)$$

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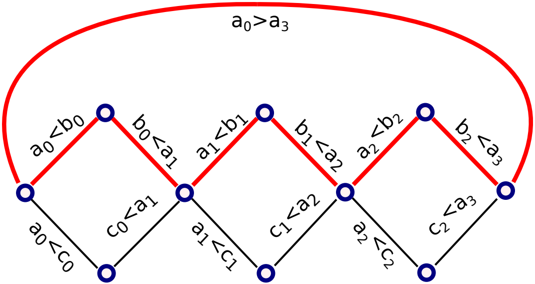

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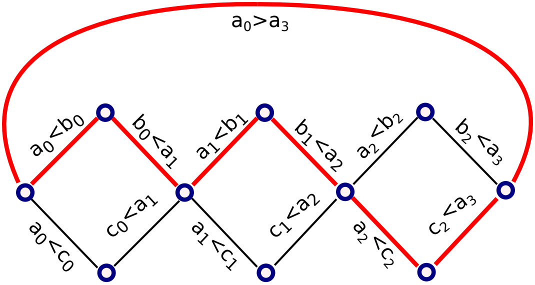

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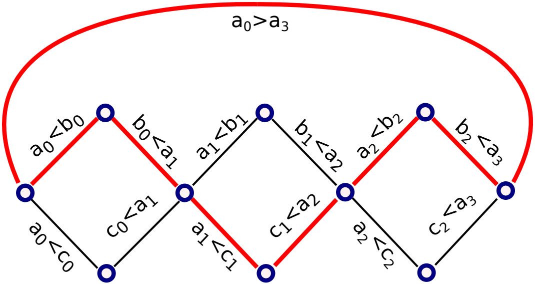

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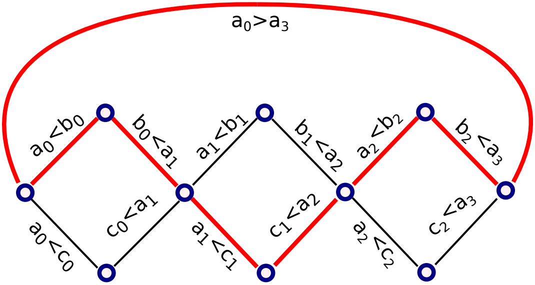

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And so on... Exponential enumeration of paths.

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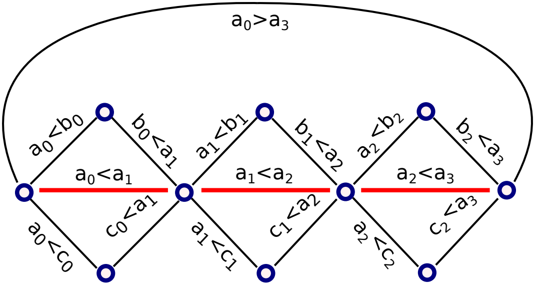

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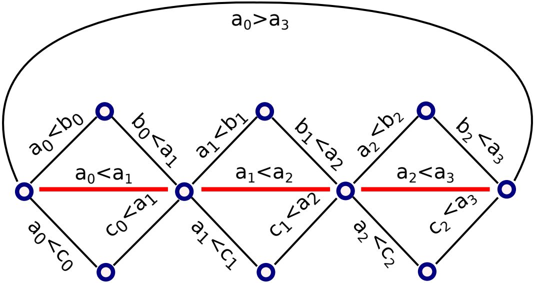

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Feature/Flaw: Can only use existing literals!

SMT Overview

SMT Solving

SMT Theories

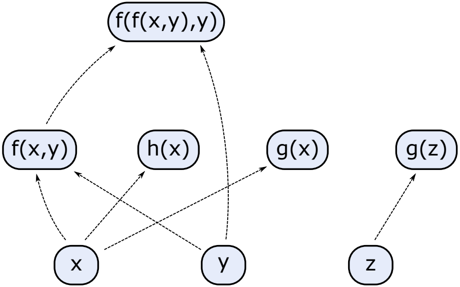
Theories: Uninterpreted Functions

- Literals are of the form $x = y$, $x \neq y$, $x = f(x,f(y,z))$.
- Can be decided in $O(n\log(n))$ based on congruence closure.
- Efficient theory propagation for equalities.
- Can generate:
  - small explanations [DNS05],
  - minimal explanations [NO07],
  - smallest explanations NP-hard [FFHP].
- Typically the core of the SMT solver and used in other theories.

Theories: Uninterpreted Functions and Congruence Closure

Example

$$\llbracket\, f(x,y) = x,\ h(x) = g(x),\ f(f(x,y),y) = z,\ g(x) \neq g(z)\ \rrbracket$$



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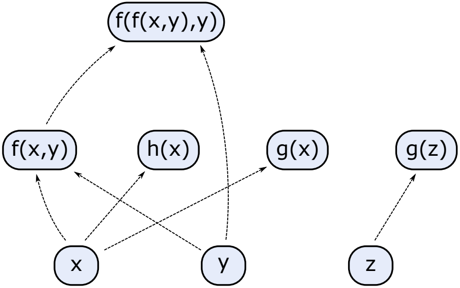

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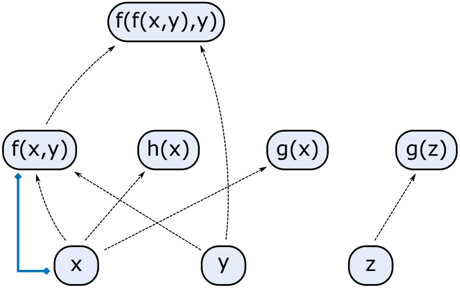

Theories: Uninterpreted Functions and Congruence Closure

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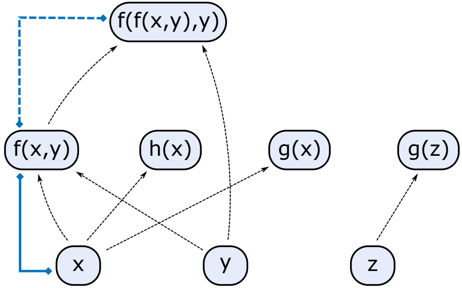

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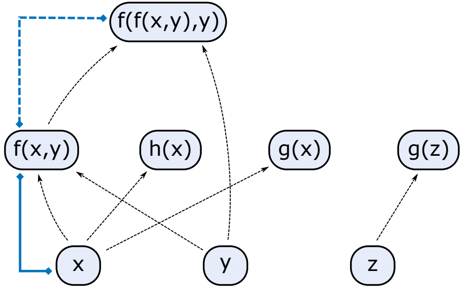

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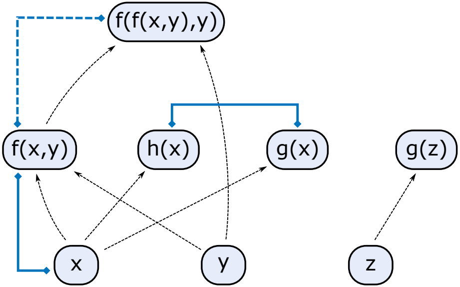

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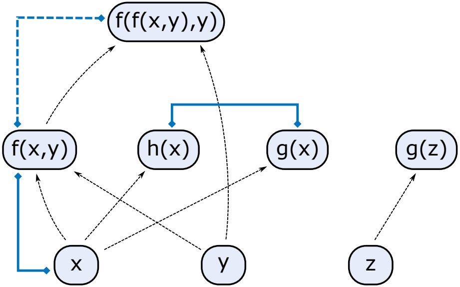

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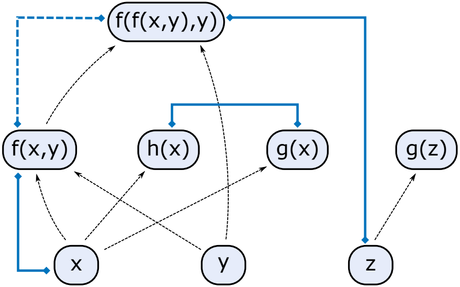

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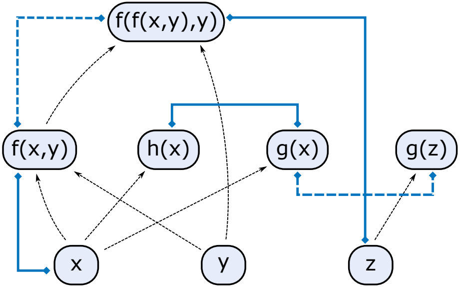

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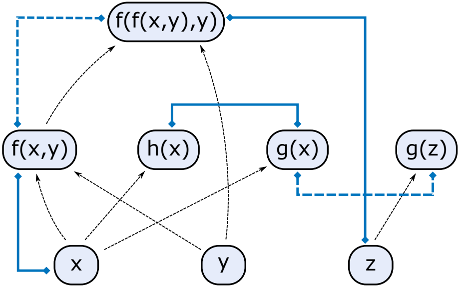

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$$[\![\, f(x,y) = x,\ h(x) = g(x),\ f(f(x,y),y) = z,\ g(x) \neq g(z) \,]\!]$$

Conflict:

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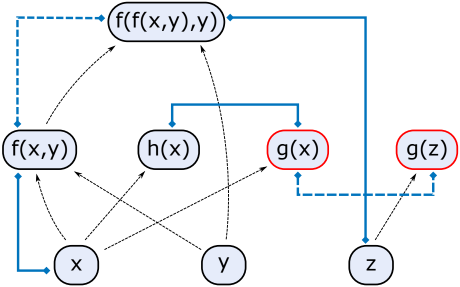

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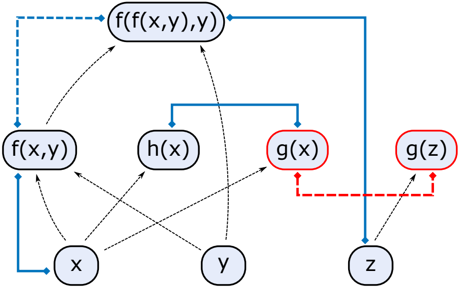

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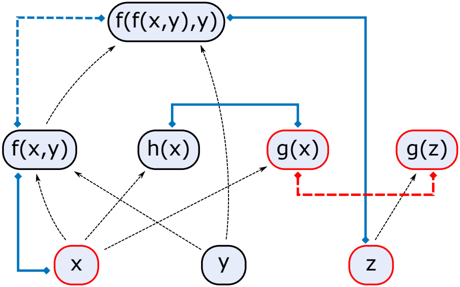

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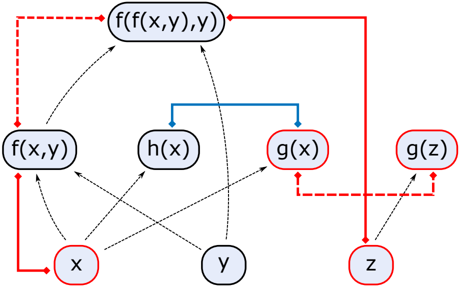

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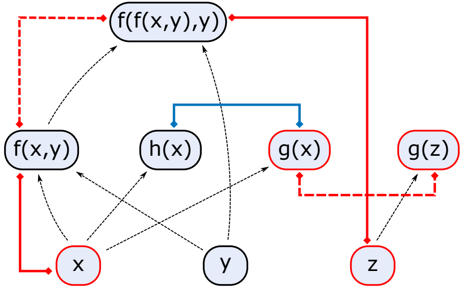
2. $f(f(x,y),y) = z$



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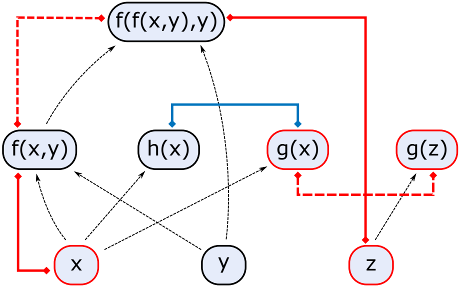
1. $g(x) \neq g(z)$
2. $f(f(x,y),y) = z$
3. $f(x,y) = x$



Theories: Difference Logic

- Literals are of the form $x - y \bowtie k$, where
  - $\bowtie \in \{\leq, <, =, \neq, >, \geq\}$,
  - x and y are arithmetic variables (integer or real),
  - k is a constant (integer or real).
- We can rewrite $x - y = k$ to $(x - y \leq k) \wedge (x - y \geq k)$.
- In integers, we can rewrite $x - y < k$ to $x - y \leq k - 1$.
- In reals, we can rewrite $x - y < k$ to $x - y \leq k - \delta$.
- Can assume: all literals of the form $x - y \leq k$.

Theories: Difference Logic Theory

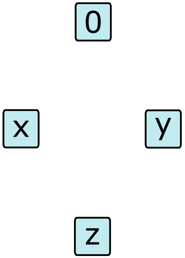
- Any solution to a set of literals can be shifted:
  - if v is a satisfying assignment, so is $v'(x) = v(x) + k$.
- We can use this to also process simple bounds $x \leq k$:
  - introduce fresh variable z (for zero),
  - rewrite each $x \leq k$ to $x - z \leq k$,
  - given a solution v, shift it so that $v(z) = 0$.
- If we allow (dis)equalities as literals, then:
  - in reals, satisfiability is polynomial;
  - in integers, satisfiability is NP-hard.

Theories: Difference Logic Example

Example

$$\llbracket\, x \leq 1,\ x - y \leq 2,\ y - z \leq 3,\ z - x \leq -6 \,\rrbracket$$

- Construct a graph from literals.
- Check if there is a negative path.



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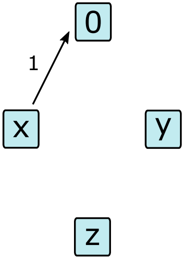

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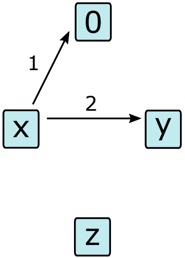

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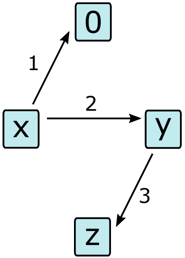

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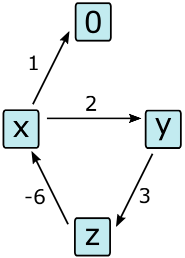

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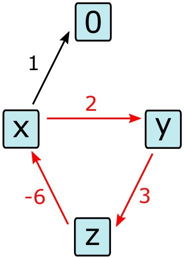

Theories: Difference Logic Example

Example

$$\llbracket\, x \leq 1,\; x - y \leq 2,\; y - z \leq 3,\; z - x \leq -6 \,\rrbracket$$

- Construct a graph from literals.
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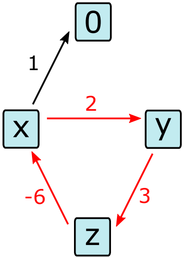
Theorem

literals unsatisfiable $\Leftrightarrow \exists$ *negative path.*



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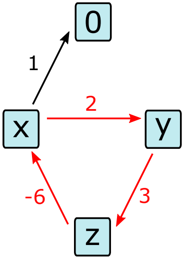


Theorem
literals unsatisfiable $\Leftrightarrow \exists$ *negative path.*

- Conflict:
  $(x - y \leq 2)$,
  $(y - z \leq 3)$,
  $(z - x \leq -6)$.

Theories: Arrays

$$\forall a, i, e \ : \ \mathsf{store}(a, i, e)[i] = e$$
$$\forall a, i, j, e \ : \ i \neq j \Rightarrow \mathsf{store}(a, i, e)[j] = a[j]$$
$$\forall a, b \ : \ a \neq b \Rightarrow \exists i : a[i] \neq b[i]$$

Common approach:

- UF + lemmas on demand [BB09, DMB09].
- Use UF as if store and $\_[\_]$ were uninterpreted.
- If UNSAT in UF, then UNSAT in arrays too.
- If SAT and solution respects array axioms, then SAT (lucky).
- If not, then refine by instantiating violated axioms.

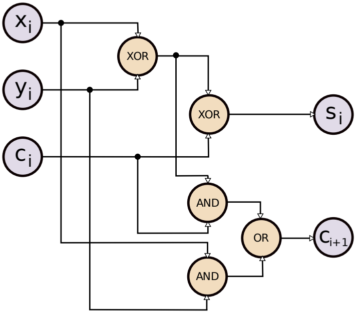
Theories: Bit-Vectors

Common approach:

1. Heavy preprocessing
2. Encode into SAT (bit-blasting)
3. Run a SAT solver

Alternatives [HBJ$^+$14, ZWR16] not yet mature.

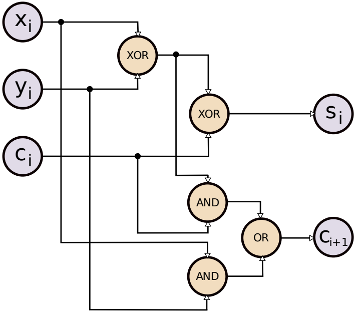
Theories: Bit-Vectors and Bit-Blasting



Translation to CNF

- Each node a new variables
- XOR introduces 4 clauses
- AND introduces 3 clauses
- OR introduces 3 clauses
- **17 new clauses**
- **5 new variables**

Theories: Bit-Vectors and Bit-Blasting



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Bit-Blasting Addition/Multiplication

$x_{[32]} + y_{[32]}$ 544 new clauses, 160 new variables

$x_{[32]} \times y_{[32]}$ 10016 new clauses, 3008 new variables

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