Assignment 2

due Wednesday, September 7, 2022

Remember that homework is due at 6pm on the due date.

Problem 1 (3 points)

Express the following using summation notation, and prove it by induction:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

(You can use notation like $\sum_{i \le n}$ or $\sum_{i=1}^{n}$, as you prefer.)

Problem 2 (4 points)

In class, we described an algorithm to solve the tower of Hanoi problem and proved that n disks can be moved from one peg to another with $2^n - 1$ steps. Prove, as clearly as you can, that this is optimal: it is impossible to move n disks from one peg to another with a smaller number of steps.

Problem 3 (3 points)

Prove that for $n \ge 3$ a convex n-gon has $n(n-3)/2$ diagonals. (If it helps to draw a picture, feel free to take a picture with your phone and submit it with your homework.)

Problem 4 (3 points)

Recall that the Fibonacci numbers are defined recursively as follows:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n. \end{aligned}$$

Show $\sum_{i<n} F_i = F_{n+1} - 1$.

Problem 5 (3 points)

Let α and β be the two roots of $x^2 = x + 1$. Show that for every n, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$.

Problem 6 (4 points)

Remember the recursive definition of the greatest common divisor function:

$$\gcd(x, y) = \begin{cases} x & \text{if } y = 0 \\ \gcd(y, \operatorname{mod}(x, y)) & \text{otherwise} \end{cases}$$

Notice that the easiest way to show that the recursion is well founded is to notice that the second argument decreases with each recursive call.

Show that for every nonnegative x and y, there are integers a and b such that $\gcd(x, y) = ax + by$. You can use the fact that for nonzero y, we have $x = \operatorname{div}(x, y) \cdot y + \operatorname{mod}(x, y)$, where $\operatorname{div}(x, y)$ denotes integer division.