Assignment 6

due Wednesday, October 12, 2022

Problem 1 (6 points)

Recall from the class and the textbook that an *autarky* is an assignment τ that satisfies all clauses in a CNF formula Γ that it touches. A literal l in a CNF formula Γ is called *pure* if $\neg l$ does not occur in Γ. An assignment that sets a pure literal to *true* and leaves all the other variables unassigned is an instance of an autarky.

A) (3 points) Write in Lean a predicate `isAutarky` that takes an assignment τ : `PropAssignment` and a CNF formula Γ : `CnfForm` and returns a Boolean whether τ is an autarky for Γ.

B) (3 points) Write in Lean a function `getPure` that a CNF formula Γ : `CnfForm` and returns a `List Lit` of all pure literals in Γ. The function does not need to find all pure literals until fixpoint, only the literals the are pure in Γ.

Problem 2 (14 points)

In this problem we focus on coloring a $n \times m$ grid with k colors. Consider all possible rectangles within the grid whose length and width are at least 2. The goal is to color the grid using k colors so that no such rectangle has the same color for its four corners. When this is possible, we say that the $n \times m$ grid is *k-colorable while avoiding monochromatic rectangles.* When using k colors, it is relatively easy to construct a valid k-coloring of a $k^2 \times k^2$ grid. However, only few valid k-colorings are known for grids that are larger than $k^2 \times k^2$. An example of a valid 3-coloring of the 9×9 grid is shown below.

```
0 0 1 1 2 2 0 1 2
2 0 0 1 1 2 2 0 1
1 2 0 0 1 1 2 2 0
0 1 2 0 0 1 1 2 2
2 0 1 2 0 0 1 1 2
2 2 0 1 2 0 0 1 1
1 2 2 0 1 2 0 0 1
1 1 2 2 0 1 2 0 0
0 1 1 2 2 0 1 2 0
```

A) (6 points) Write a Lean function that takes as input three natural numbers n, m, and k, which returns a CNF formula of Lean data type `CnfForm` which is satisfiable if an only if there exists a valid k-coloring of the $n \times m$ grid, i.e., a coloring without monochromatic rectangles. (Hint: The encoding requires two types of clauses. First, each square needs to have one color. Second, if four squares form the corners of a rectangle, then they cannot have the same color.)

B) (4 points) Use the Lean interface to CaDiCaL to solve the formula with $n = 10$, $m = 10$, and $k = 3$ and the formula with $n = 9$, $m = 12$, and $k = 3$. Both formulas should be satisfiable. The answer should consist of two lists (one for each formula) using Lean data type `List Lit` containing only all positive literals assigned to true ($\top$).

C) (4 points) Given a `List Lit` containing only all positive literals assigned to true for a grid-coloring problem, decode it into a grid of numbers similar to the 9×9 grid shown above. Use the function to display the solutions of $n = 10$, $m = 10$, and $k = 3$ and of $n = 9$, $m = 12$, and $k = 3$. You can assume that the decoding function knows n, m, and k of the grid-coloring problem.