Name: ______________________________

Logic and Mechanized Reasoning

Final Practice Exam

Spring 2024

Write your answers in the space provided, using the back of the page if necessary. You may use additional scratch paper. Justify your answers, and provide clear, readable explanations.

Problem	**Points**	**Score**
1	20	
2	20	
3	18	
4	12	
5	20	
6	20	
7	15	
8	25	
Total	**150**	

Good luck!

Problem 1 (20 points).

Consider the following first-order formula with equality:

$$f(a) = g^3(a) \;\wedge\; f(a) = g^5(a) \;\wedge\; f(a) \neq a \;\wedge\; f(a) \neq g(a).$$

Here $g^3(a)$ abbreviates $g(g(g(a)))$, and similarly for $g^5(a)$.

Compute the congruence closure and list the equivalence classes. In case the formula is unsatisfiable, list the conflict. In case the formula is satisfiable, construct a model.

Problem 2 (20 points).

Use the Fourier–Motzkin procedure (the decision procedure for linear arithmetic that you helped implement for homework) to determine whether the following set of inequalities is satisfiable:

1. $x + 2y - 3z < -8$
2. $2x - 4y + 2z < 7$
3. $-x + z < 2$

Solution 2:

Solving the first two inequalities for y, we get $y < -x/2 + 3z/2 - 4$ and $y > x/2 + z/2 - 7/4$. Combining these we get $x/2 + z/2 - 7/4 < -x/2 + 3z/2 - 4$, which simplifies to $x + 9/4 < z$.

The third inequality is equivalent to $z < 2 + x$. Combining it with the last inequality yields $x + 9/4 < x + 2$, which simplifies to $1/4 < 0$, a contradiction. Thus the inequalities are unsatisfiable.

Problem 3.

For this problem, we consider quantifier-free bitvector formulas (QF_BV) with unsigned bitvectors of length 4. We use the following notation: $|$ for logical or, & for logical and, $\sim$ for logical negation (all bits are negated), $>_u$, $\geq_u$, $<_u$, and $\leq_u$, for unsigned greater than, greater or equal, less than, and less or equal. Furthermore, $\ll k$ denotes left shift by k bits, while $\gg k$ denotes a right shift by k bits. Determine for each of them whether the formula is satisfiable or unsatisfiable. In case it is satisfiable, provide a satisfying assignment (a bitvector of length 4).

Part a) (6 points)

$$(a \mid {\sim}a) <_u a$$

Part b) (6 points)

$$(a \ll 2) >_u a$$

Part c) (6 points)

$$(a \mathbin{\&} (a \gg 1)) >_u a$$

Problem 4. (12 points)

Consider the following two universally quantified clauses:

$$\forall x, y.\, R(x, f(g(y))) \lor S(x, y)$$

$$\forall u, v.\, \neg R(g(u), f(v)) \lor T(u, v).$$

Carry out a resolution step, indicate the most general unifier, and write down the resolvent.

Problem 5.

The Green Bridge of Wales is a famous rock formation. However you need to be lucky to see it: *If it rains during a day, you can't see the bridge.* Unfortunately it rains a lot in Wales: *On any two consecutive days, it rains on at least one of them.* That is why the tourist guide states: *If you can see the bridge, then it will rain tomorrow.*

Part a) (10 points) Express in first-order logic the two axioms and the conclusion (the text shown in italics) using the predicates $\text{Rains}(x)$ and $\text{Visible}(x)$ and the function $\text{nextDay}(x)$. ($\text{Visible}(x)$ means that the bridge is visible on day x.) Note that the variables range over days; you do not need to refer to any other kinds of objects.

Part b) (10 points) Show that the conclusion follows from the two axioms using resolution for first-order logic.

From the hypotheses and negated conclusion, we get 1–4.

Problem 6. For these problems, use $S(x)$ for "x is a student," $H(x, y)$ for "x has y," $C(y)$ for "y is a car," and $D(y)$ for "y is a driver's license."

Part a) (4 points) Write down a formalization of the statement "every student that has a car has a driver's license."

Part b) (6 points) Skolemize the previous statement and transform it to an equisatisfiable set of (one or more) universally quantified clauses.

Part c) (4 points) Write down a formalization of the statement "not every student that has a driver's license has a car."

Part d) (6 points) Skolemize the previous statement and transform it to an equisatisfiable set of universally quantified clauses.

Problem 7. **(15 points)**

For this problem, consider first-order logic *without* the equality symbol. In class, we proved that if $\mathfrak{M}$ is a term model for a language L, t is any term, and σ is any substitution, we have

$$[\![t]\!]_{\mathfrak{M},\sigma} = \sigma t,$$

where the right-hand side is the result of applying the substitution t to σ.

Let Γ be a set of universal sentences, and let τ be a truth assignment to the atoms of L that satisfies every quantifier-free instance of a sentence in Γ. Let $\mathfrak{M}$ be the term model in which, for every relation symbol of $\mathfrak{L}$, $R^{\mathfrak{M}}(t_1, \ldots, t_n)$ holds if and only if $\models_\tau R(t_1, \ldots, t_n)$. Show that $\mathfrak{M}$ is a model of Γ.

Problem 8. For the following Lean proofs, these declarations are in the context:

```
variable {α β γ : Type} (P Q R : α → Prop) (S : α → α → Prop)
variable (f : α → β) (g : β → γ)
```

As best you can, try to write Lean proofs of the three theorems shown, using a tactic proof or a proof term, as indicated. For the remaining ones, use tactics. You can use the tactics `intro`, `apply`, `exact`, `left`, `right`, `constructor`, `rcases`, `rw`, and `use`. (You don't have to use all of them.) If you are not sure about the tactics in the tactic proofs, you can get partial credit by indicating the goal you expect at each stage.

Part a) (tactic proof) (5 points)

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x := by
```

Part b) (proof term) (5 points)

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x :=
```

Part c) (tactic proof) (5 points)

```
example (h1 : ∃ x, P x) (h2 : ∀ x, P x → Q x) : ∃ x, Q x := by
```

Part d) (tactic proof) (5 points)

```
example (h : ∃ x, ∀ y, S x y) : ∀ u, ∃ v, S v u := by
```

Part e) (tactic proof) (5 points).

```
example (Injf : ∀ x₁ x₂ : α, f x₁ = f x₂ → x₁ = x₂)
  (Injg : ∀ y₁ y₂ : β, g y₁ = g y₂ → y₁ = y₂) :
    ∀ x₁ x₂, g (f x₁) = g (f x₂) → x₁ = x₂ := by
```