Name: ____________________________

Logic and Mechanized Reasoning

Final Practice Exam

Spring 2024

Write your answers in the space provided, using the back of the page if necessary. You may use additional scratch paper. Justify your answers, and provide clear, readable explanations.

Problem	**Points**	**Score**
1	20	
2	20	
3	18	
4	12	
5	20	
6	20	
7	15	
8	25	
Total	**150**	

Good luck!

Problem 1 (20 points).

Consider the following first-order formula with equality:

$$f(a) = g^3(a) \;\wedge\; f(a) = g^5(a) \;\wedge\; f(a) \neq a \;\wedge\; f(a) \neq g(a).$$

Here $g^3(a)$ abbreviates $g(g(g(a)))$, and similarly for $g^5(a)$.

Compute the congruence closure and list the equivalence classes. In case the formula is unsatisfiable, list the conflict. In case the formula is satisfiable, construct a model.

Solution

The classes are as follows:

- $[a]$
- $[g(a)]$
- $[g^2(a)]$
- $[g^3(a)] = \{f(a), g^3(a), g^5(a)\}$
- $[g^4(a)]$

The formula is satisfiable. Add a new element b to the list of classes above, and define

$$\begin{aligned}
&a^{\mathfrak{M}} = [a],\\
&f^{\mathfrak{M}}([a]) = [g^3(a)], f^{\mathfrak{M}}([g(a)]) = b, f^{\mathfrak{M}}([g^2(a)]) = b\\
&f^{\mathfrak{M}}([g^3(a)]) = b, f^{\mathfrak{M}}([g^4(a)]) = b, f^{\mathfrak{M}}(b) = b,\\
&g^{\mathfrak{M}}([a]) = [g(a)], g^{\mathfrak{M}}([g(a)]) = [g^2(a)], g^{\mathfrak{M}}([g^2(a)]) = [g^3(a)],\\
&g^{\mathfrak{M}}([g^3(a)]) = [g^4(a)], g^{\mathfrak{M}}([g^4(a)]) = [g^3(a)], g^{\mathfrak{M}}(b) = b.
\end{aligned}$$

Problem 2 (20 points).

Use the Fourier–Motzkin procedure (the decision procedure for linear arithmetic that you helped implement for homework) to determine whether the following set of inequalities is satisfiable:

1. $x + 2y - 3z < -8$
2. $2x - 4y + 2z < 7$
3. $-x + z < 2$

Solution

There are alternative (equivalent) ways of describing Fourier-Motzkin, so we provide two solutions. The first uses the method you implemented for homework, and the second uses the one in the textbook.

Solution 1:

Adding the first and the third inequalities, we get

$$2y - 2z < -6.$$

Adding twice the third to the second we get

$$-4y + 4z < 11.$$

Adding twice the first of these to the second, we get

$$0 < -1.$$

This is a contradiction, so the inequalities are unsatisfiable.

Solution 2:

Solving the first two inequalities for y, we get $y < -x/2 + 3z/2 - 4$ and $y > x/2 + z/2 - 7/4$. Combining these we get $x/2 + z/2 - 7/4 < -x/2 + 3z/2 - 4$, which simplifies to $x + 9/4 < z$.

The third inequality is equivalent to $z < 2 + x$. Combining it with the last inequality yields $x + 9/4 < x + 2$, which simplifies to $1/4 < 0$, a contradiction. Thus the inequalities are unsatisfiable.

Problem 3.

For this problem, we consider quantifier-free bitvector formulas (QF_BV) with unsigned bitvectors of length 4. We use the following notation: $|$ for logical or, & for logical and, $\sim$ for logical negation (all bits are negated), $>_u$, $\geq_u$, $<_u$, and $\leq_u$, for unsigned greater than, greater or equal, less than, and less or equal. Furthermore, $\ll k$ denotes left shift by k bits, while $\gg k$ denotes a right shift by k bits. Determine for each of them whether the formula is satisfiable or unsatisfiable. In case it is satisfiable, provide a satisfying assignment (a bitvector of length 4).

Part a) (6 points)

$$(a \mid \sim a) <_u a$$

Solution This is unsatisfiable. A logical or can only set more bits to 1, which increases the value.

Part b) (6 points)

$$(a \ll 2) >_u a$$

Solution

Satisfiable: set $a = 0001$. Then $0100 >_u 0001$.

Part c) (6 points)

$$(a \ \& \ (a \gg 1)) >_u a$$

Solution

This is unsatisfiable. A logical and can only decrease the value.

Problem 4. (12 points)

Consider the following two universally quantified clauses:

$$\forall x, y.\, R(x, f(g(y))) \vee S(x, y)$$

$$\forall u, v.\, \neg R(g(u), f(v)) \vee T(u, v).$$

Carry out a resolution step, indicate the most general unifier, and write down the resolvent.

Solution

The map $\{x \mapsto g(u), v \mapsto g(y)\}$ is a most general unifier of $R(x, f(g(y)))$ and $R(g(u), f(w))$. The resolution step yields $\forall u, y.\, S(g(u), y) \vee T(u, g(y))$.

Problem 5.

The Green Bridge of Wales is a famous rock formation. However you need to be lucky to see it: *If it rains during a day, you can't see the bridge.* Unfortunately it rains a lot in Wales: *On any two consecutive days, it rains on at least one of them.* That is why the tourist guide states: *If you can see the bridge, then it will rain tomorrow.*

Part a) (10 points) Express in first-order logic the two axioms and the conclusion (the text shown in italics) using the predicates $\text{Rains}(x)$ and $\text{Visible}(x)$ and the function $\text{nextDay}(x)$. ($\text{Visible}(x)$ means that the bridge is visible on day x.) Note that the variables range over days; you do not need to refer to any other kinds of objects.

Solution

- $\forall x.\, \text{Rains}(x) \to \neg\text{Visible}(x)$
- $\forall x.\, \text{Rains}(x) \lor \text{Rains}(\text{nextDay}(x))$.
- $\forall x.\, \text{Visible}(x) \to \text{Rains}(\text{nextDay}(x))$.

Part b) (10 points) Show that the conclusion follows from the two axioms using resolution for first-order logic.

From the hypotheses and negated conclusion, we get 1–4.

Solution

1. $\forall x.\, \neg\text{Rains}(x) \lor \neg\text{Visible}(x)$
2. $\forall x.\, \text{Rains}(x) \lor \text{Rains}(\text{nextDay}(x))$
3. $\text{Visible}(a)$
4. $\neg\text{Rains}(\text{nextDay}(a))$
5. $\neg\text{Rains}(a)$, from 1 and 3
6. $\text{Rains}(a)$, from 2 and 4
7. $\bot$ from 5 and 6.

Problem 6. For these problems, use $S(x)$ for "x is a student," $H(x, y)$ for "x has y," $C(y)$ for "y is a car," and $D(y)$ for "y is a driver's license."

Part a) (4 points) Write down a formalization of the statement "every student that has a car has a driver's license."

Solution

$$\forall x, y.\, S(x) \wedge H(x, y) \wedge C(y) \rightarrow \exists z.\, H(x, z) \wedge D(z)$$

The y can also be moved into the antecedent and expressed as $\exists$, with the scope limited:

$$\forall x.\, S(x) \wedge (\exists y.\, H(x, y) \wedge C(y)) \rightarrow \exists z.\, H(x, z) \wedge D(z)$$

Part b) (6 points) Skolemize the previous statement and transform it to an equisatisfiable set of (one or more) universally quantified clauses.

Solution

$$\forall x, y.\, \neg S(x) \vee \neg H(x, y) \vee \neg C(y) \vee H(x, f(x, y))$$

and

$$\forall x, y.\, \neg S(x) \vee \neg H(x, y) \vee \neg C(y) \vee D(f(x, y)).$$

Part c) (4 points) Write down a formalization of the statement "not every student that has a driver's license has a car."

Solution

$$\exists x, y.\, S(x) \wedge H(x, y) \wedge D(y) \wedge \forall z.\, C(z) \rightarrow \neg H(x, z)$$

Part d) (6 points) Skolemize the previous statement and transform it to an equisatisfiable set of universally quantified clauses.

Solution

$$S(b), \quad H(b, c), \quad D(c), \quad \forall z.\, \neg C(z) \vee \neg H(b, z).$$

Problem 7. (15 points)

For this problem, consider first-order logic *without* the equality symbol. In class, we proved that if $\mathfrak{M}$ is a term model for a language L, t is any term, and σ is any substitution, we have

$$[\![t]\!]_{\mathfrak{M},\sigma} = \sigma t,$$

where the right-hand side is the result of applying the substitution t to σ.

Let Γ be a set of universal sentences, and let τ be a truth assignment to the atoms of L that satisfies every quantifier-free instance of a sentence in Γ. Let $\mathfrak{M}$ be the term model in which, for every relation symbol of $\mathfrak{L}$, $R^{\mathfrak{M}}(t_1, \ldots, t_n)$ holds if and only if $\models_\tau R(t_1, \ldots, t_n)$. Show that $\mathfrak{M}$ is a model of Γ.

Solution

Let $\forall \vec{x}.\, A(x_1, \ldots, x_n)$ be any universal sentence in Γ, where A is quantifier free. By the definition of the semantics for the universal quantifier (and the definition of "term model," this sentence is true in $\mathfrak{M}$ if and only if $\models_{\mathfrak{M}} A(t_1, \ldots, t_n)$. We are assuming $\models_\tau A(t_1, \ldots, t_n)$. So it suffices to show that for every quantifier-free sentence A, $\models_{\mathfrak{M}} A$ if and only if $\models_\tau A$.

Do this by induction on formulas. In the base case, A is an atomic formula $R(t_1, \ldots, t_n)$. By the definition of the semantics for first-order logic, this is true if and only if $R^{\mathfrak{M}}([\![t_1]\!]_{\mathfrak{M}}, \ldots, [\![t_n]\!]_{\mathfrak{M}})$ holds. By the definition of $\mathfrak{M}$ and the fact proved in class, this hold if and only if $\models_\tau R(t_1, \ldots, t_n)$, as required. (In the base case where A is $\bot$, $[\![\bot]\!]_{\mathfrak{M}} = [\![\bot]\!]_\tau = \bot$.)

In the inductive step, A is of the form $B \wedge C$, $B \vee C$, or $B \to C$. In those cases, the claim follows immediately from the inductive hypotheses, given that the definitions of $\models_{\mathfrak{M}} A$ and $\models_\tau A$ agree on the propositional connectives.

Problem 8. For the following Lean proofs, these declarations are in the context:

```
variable {α β γ : Type} (P Q R : α → Prop) (S : α → α → Prop)
variable (f : α → β) (g : β → γ)
```

As best you can, try to write Lean proofs of the three theorems shown, using a tactic proof or a proof term, as indicated. For the remaining ones, use tactics. You can use the tactics `intro`, `apply`, `exact`, `left`, `right`, `constructor`, `rcases`, `rw`, and `use`. (You don't have to use all of them.) If you are not sure about the tactics in the tactic proofs, you can get partial credit by indicating the goal you expect at each stage.

Part a) (tactic proof) (5 points)

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x := by
```

Solution:

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x := by
  intro x
  apply h1
  apply h2
```

Part b) (proof term) (5 points)

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x :=
```

Solution:

```
example (h1 : ∀ x, P x → Q x) (h2 : ∀ x, P x) : ∀ x, Q x :=
  fun x => h1 x (h2 x)
```

Part c) (tactic proof) (5 points)

```
example (h1 : ∃ x, P x) (h2 : ∀ x, P x → Q x) : ∃ x, Q x := by
```

Solution:

```
example (h1 : ∃ x, P x) (h2 : ∀ x, P x → Q x) : ∃ x, Q x := by
```

```
  rcases h1 with ⟨x, px⟩
  use x
  apply h2
  exact px
```

Part d) (tactic proof) (5 points)

```
example (h : ∃ x, ∀ y, S x y) : ∀ u, ∃ v, S v u := by
```

Solution:

```
example (h : ∃ x, ∀ y, S x y) : ∀ u, ∃ v, S v u := by
  rcases h with ⟨x, hx⟩
  intro u
  use x
  apply hx
```

Part e) (tactic proof) (5 points).

```
example (Injf : ∀ x₁ x₂ : α, f x₁ = f x₂ → x₁ = x₂)
  (Injg : ∀ y₁ y₂ : β, g y₁ = g y₂ → y₁ = y₂) :
    ∀ x₁ x₂, g (f x₁) = g (f x₂) → x₁ = x₂ := by
```

Solution:

```
example (Injf : ∀ x₁ x₂ : α, f x₁ = f x₂ → x₁ = x₂)
  (Injg : ∀ y₁ y₂ : β, g y₁ = g y₂ → y₁ = y₂) :
    ∀ x₁ x₂, g (f x₁) = g (f x₂) → x₁ = x₂ := by
  intro x₁ x₂ h
  have h1 : f x₁ = f x₂ := by
    apply Injg
    exact h
  apply Injf
  exact h1
```

Alternatively:

```
example (Injf : ∀ x₁ x₂ : α, f x₁ = f x₂ → x₁ = x₂)
```

```
(Injg : ∀ y₁ y₂ : β, g y₁ = g y₂ → y₁ = y₂) :
  ∀ x₁ x₂, g (f x₁) = g (f x₂) → x₁ = x₂ := by
intro x₁ x₂ h
apply Injf
apply Injg
exact h
```