Logic and Mechanized Reasoning
Computer-Generated Propositional Proofs

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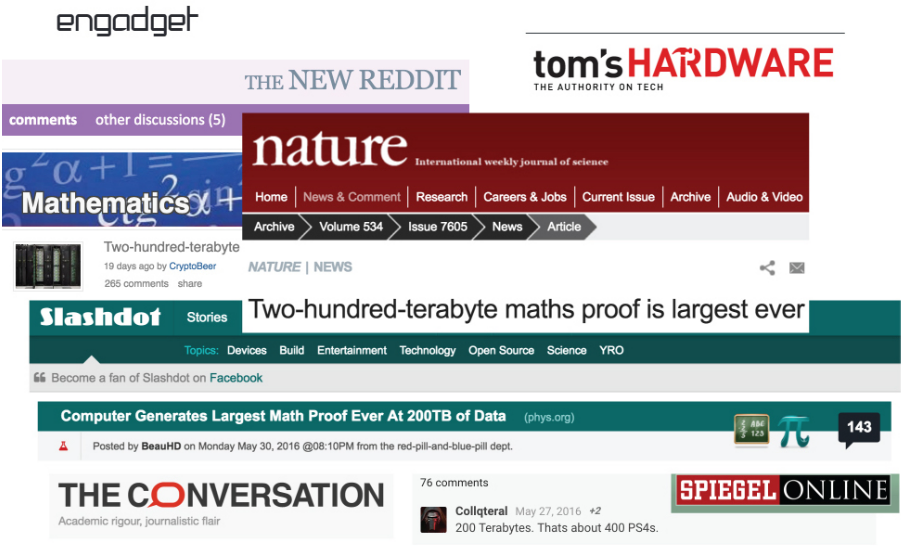
"The Largest Math Proof Ever"



Motivation

SAT solvers can efficiently solve many application problems

However, for various small problems the runtime is exponential

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- What are the limitations of this kind of reasoning?

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Research motivated by advancing the techniques and verification

Proofs of Unsatisfiability

Beyond Resolution

Strong Extension-Free Proof Systems

Beyond Symmetry Breaking

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- Certifying unsatisfiability is not so easy:
  - If a formula has n variables, there are 2^n possible assignments.
  - ➥ Checking whether every assignment falsifies the formula is costly.
  - More compact certificates of unsatisfiability are desirable.
    - ➥ Proofs

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    ... but can be of exponential size with respect to a formula.
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- Example: Resolution proofs
  - A resolution proof is a sequence $C_1, \ldots, C_m$ of clauses.
  - Every clause is either contained in the formula or derived from two earlier clauses via the resolution rule:

$$\frac{C \lor p \qquad \neg p \lor D}{C \lor D}$$

  - C_m is the empty clause (containing no literals), denoted by $\bot$.
  - There exists a resolution proof for every unsatisfiable formula.

Resolution Proofs

Example

$\Gamma := (\neg p \lor \neg q \lor r) \land (\neg r) \land (p \lor \neg q) \land (\neg s \lor q) \land (s)$

Resolution proof: $(\neg p \lor \neg q \lor r)$, $(\neg r)$, $(\neg p \lor \neg q)$, $(p \lor \neg q)$, $(\neg q)$, $(\neg s \lor q)$, $(\neg s)$, (s), $\bot$

$$\dfrac{\dfrac{\neg s \lor q \qquad \dfrac{\dfrac{\neg p \lor \neg q \lor r \qquad \neg r}{\neg p \lor \neg q} \qquad p \lor \neg q}{\neg q}}{\neg s} \qquad s}{\bot}$$

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Drawbacks of resolution:

- For many seemingly simple formulas, there are only resolution proofs of exponential size.
- State-of-the-art techniques are not succinctly expressible.

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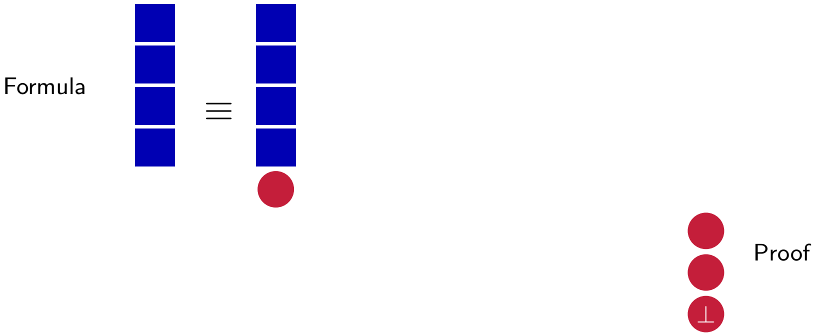

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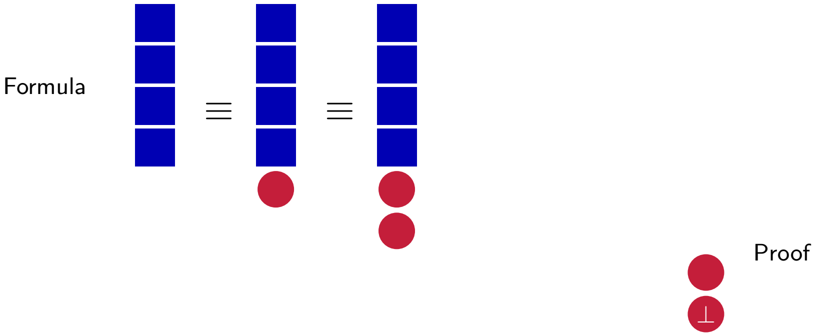

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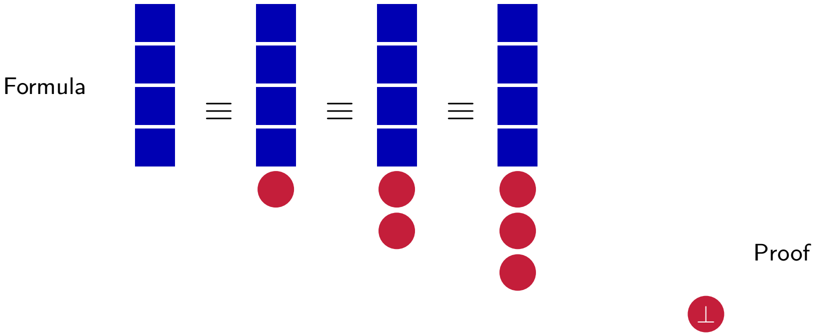

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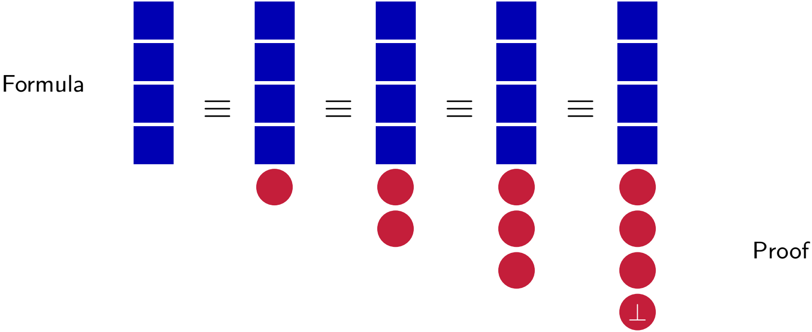

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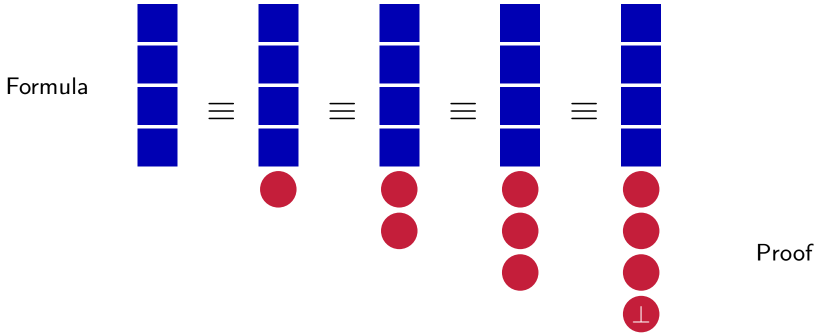

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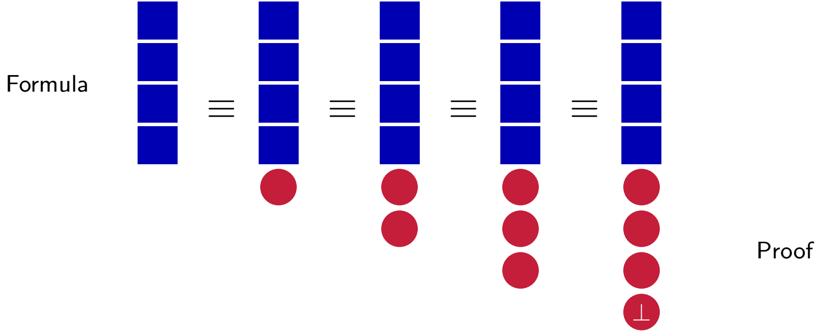

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- Checking redundancy should be efficient.

➥ Idea: Only add clauses that fulfill an efficiently checkable redundancy criterion.

Proofs of Unsatisfiability

Beyond Resolution

Strong Extension-Free Proof Systems

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Traditional Proofs vs. Interference-Based Proofs

- In traditional proof systems, everything that is inferred, is logically implied by the premises.

$$\frac{C \lor p \qquad \neg p \lor D}{C \lor D}\ (\text{RES}) \qquad \frac{A \qquad A \rightarrow B}{B}\ (\text{MP})$$

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➥ Inference rules reason about the presence of facts.

- If certain premises are present, infer the conclusion.

- Different approach: Allow not only implied conclusions.
  - Require only that the addition of facts preserves satisfiability.
  - Reason also about the absence of facts.

  ➥ This leads to interference-based proof systems.

Early work on reasoning beyond resolution

The early SAT decision procedures used the Pure Literal rule [Davis and Putnam 1960; Davis, Logemann and Loveland 1962]:

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Extended Resolution (ER) [Tseitin 1966]

- Combines resolution with the Extension rule:

$$\frac{p \notin \Gamma \qquad \neg p \notin \Gamma}{(p \vee \neg a \vee \neg b) \wedge (\neg p \vee a) \wedge (\neg p \vee b)} \text{ (ER)}$$

- Equivalently, adds the definition $p := \mathrm{AND}(a, b)$
- Can be considered the first interference-based proof system
- Is very powerful: Only modest lower bounds results are known

Short Proofs of Pigeon Hole Formulas [Cook 1967]

Can $n+1$ pigeons be in n holes (at-most-one pigeon per hole)?

$$PHP_n := \bigwedge_{1 \le i \le n+1} (p_{1,i} \vee \cdots \vee p_{n,i}) \wedge \bigwedge_{1 \le h \le n,\ 1 \le i < j \le n+1} (\neg p_{h,i} \vee \neg p_{h,j})$$

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However, these proofs require introducing new variables:

- Hard to find such proofs automatically
- Existing ER approaches produce exponentially large proofs
- How to get rid of this hurdle? First approach: blocked clauses...

Blocked Clauses [Kullmann 1999]

Definition (Blocked Clause)

A clause $(C \vee p)$ is a blocked on p w.r.t. a CNF formula Γ if for every clause $(D \vee \neg p) \in \Gamma$, resolvent $C \vee D$ is a tautology.

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Example

Consider the formula $(p \vee q) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee r)$.
Second clause is blocked by both p *and* $\neg r$.
Third clause is blocked by p

Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

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Theorem

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Proof sketch: Given a formula Γ and a clause $C \vee p$ that is blocked on p w.r.t. Γ. Let assignment τ satisfy Γ, but falsify $C \vee p$. Note that all clauses $D \vee \neg p$ are doubly satisfied by τ. Flipping p to true in τ satisfies both Γ and $C \vee p$.

Blocked Clause Addition and Blocked Clause Elimination

The Blocked Clause proof system (BC) combines the resolution rule with the addition of blocked clauses.

- BC generalizes ER [Kullmann 1999]
- Recall

$$\frac{p \notin \Gamma \qquad \neg p \notin \Gamma}{(p \vee \neg a \vee \neg b) \wedge (\neg p \vee a) \wedge (\neg p \vee b)} \text{(ER)}$$

- The ER clauses are blocked on the literals p and $\neg p$ w.r.t. Γ

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Blocked clause elimination used in preprocessing and inprocessing

- Simulates many circuit optimization techniques
- Removes redundant Pythagorean Triples

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF Γ, remove C from Γ.

Example

Consider $(p \vee q) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee r)$.
After removing either $(p \vee \neg q \vee \neg r)$ *or* $(\neg p \vee r)$*, the clause* $(p \vee q)$ *becomes blocked (no clause with either* $\neg q$ *or* $\neg p$*).*

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Proposition

BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

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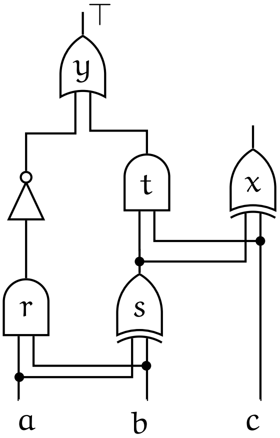

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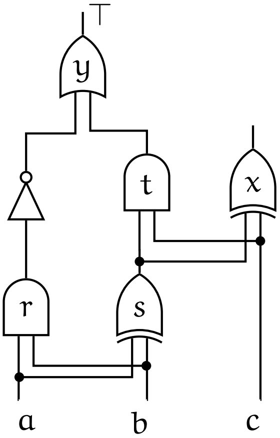

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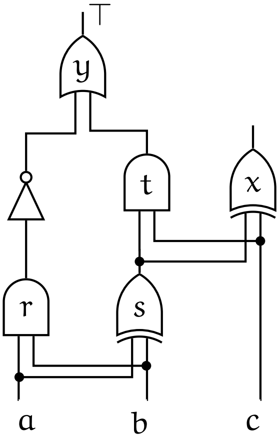

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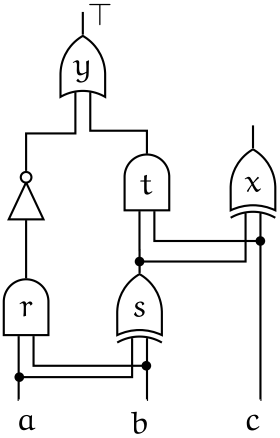

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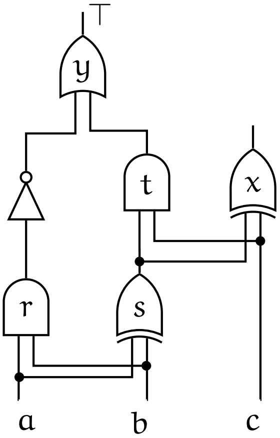

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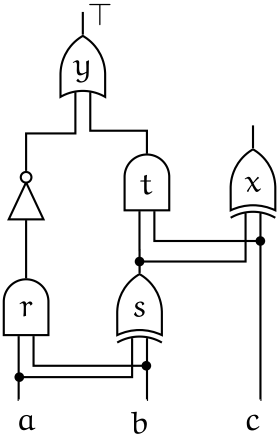

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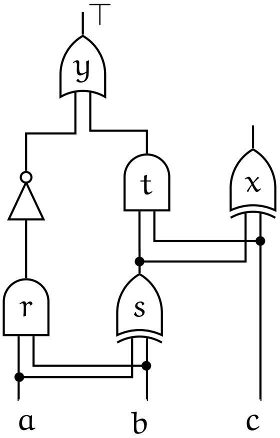

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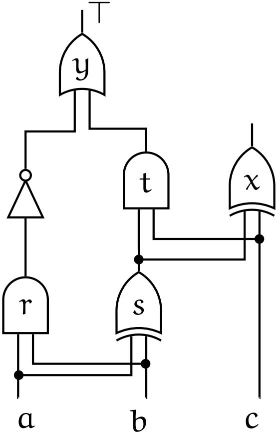

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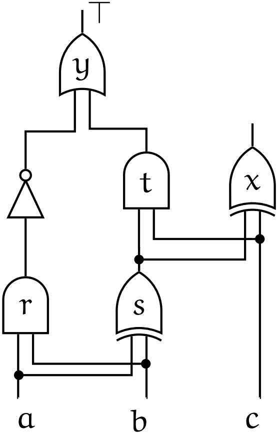

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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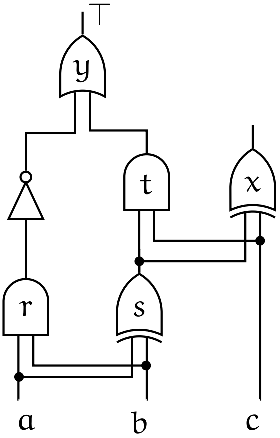

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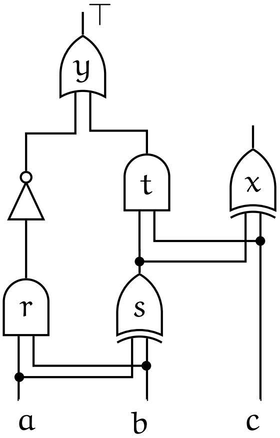


Solution Reconstruction

Input:

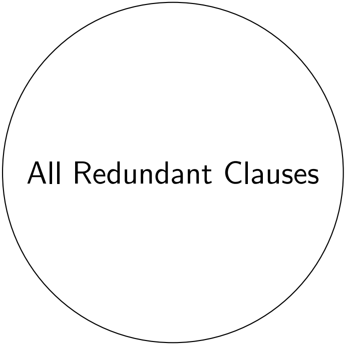
- stack S of eliminated blocked clauses
- formula Γ (without the blocked clauses)
- assignment τ that satisfies Γ

Output: an assignment that satisfies $\Gamma \wedge S$

while S.size () **do**
    $\langle C, l \rangle :=$ S.pop ()
    if τ falsifies C **then** $\tau := \tau \cup \{l = \top\}$
end while
return τ

Redundant Clauses

- Strong proof systems allow adding many redundant clauses.



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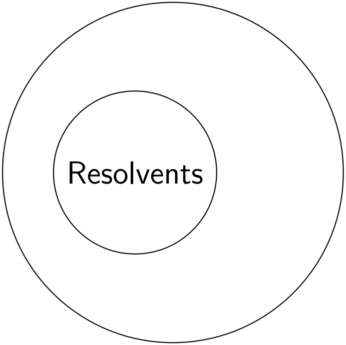

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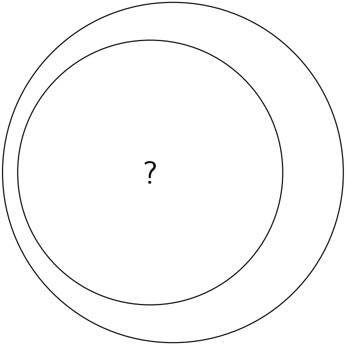

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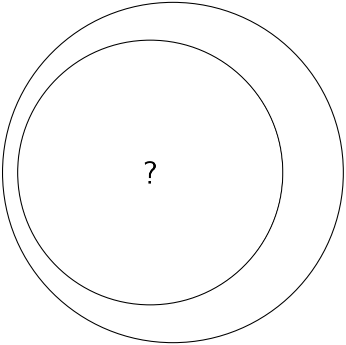

- The new proof systems can give short proofs of formulas that are considered hard.
- Are stronger derivability notions still efficiently checkable?

Redundant clauses: Intuition

A proof is a sequence of redundant clauses ending with $\bot$

- (Informal): A clause C is redundant with respect to a formula Γ if adding C to Γ preserves satisfiability
- Redundancy should be checkable in polynomial time

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When is a clause C redundant for a formula Γ (again informal)?

- Can assignments that satisfy Γ, but falsify C be repaired?
- Let τ be a (partial) assignment such that every assignment σ that satisfies Γ, but falsifies C will be turned into an assignment $\sigma \circ \tau$ that satisfies $\Gamma \land C$
- For the example, a valid repair is $\tau := \{p = \top\}$

Redundant Clauses: One Proof Rule

(Informal): A clause C is redundant w.r.t. a formula Γ if adding C to Γ preserves satisfiability

Example $\quad \Gamma = (\neg p \vee q) \wedge (p \vee q) \quad\quad C = (p \vee \neg q)$

Applying an assignment τ to a formula Γ removes satisfied clauses and falsified literals and is denoted by $[\![\Gamma]\!]_\tau$

Definition (Clause Redundancy)

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Example

$$(\neg p \vee q) \wedge (p \vee q) \wedge \neg p \wedge q \models [\![(\neg p \vee q) \wedge (p \vee q) \wedge (p \vee \neg q)]\!]_p$$
$$\models q$$

Different Proof Systems Based on Restrictions on τ

To ensure polynomial time computation, we restrict entailment ($\models$) by unit propagation ($\vdash_1$).

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Proof systems differ based on the restrictions to τ:

- Reverse unit propagation (RUP): τ is empty
- Resolution asymmetric tautology (RAT): $|\tau| \leq 1$
- Propagation redundancy (PR): τ assigns any var to $\bot$ or $\top$
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focus on variants without new variables

Hand-crafted PR Proofs of Pigeon Hole Formulas

We manually constructed PR proofs of the famous pigeon hole formulas and the two-pigeons-per-hole family.

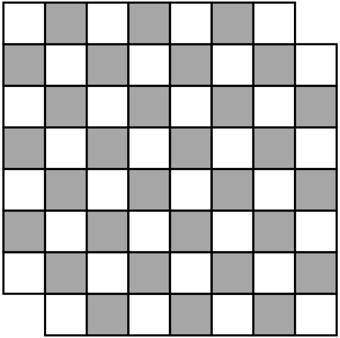
- The proofs consist only of binary and unit clauses.
- All proofs are linear in the size of the formula.
- Only original variables appear in the proof.

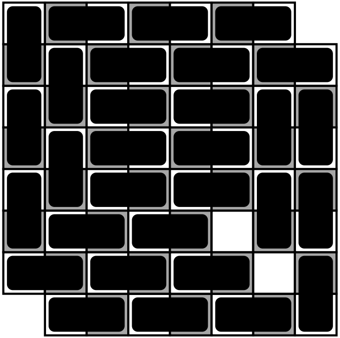
➥ The PR proofs are smaller than Cook's ER proofs.

- All resolution proofs must be exponential in size.
- Similar proofs can also be computed automatically.

Mutilated Chessboards: “A Tough Nut to Crack” [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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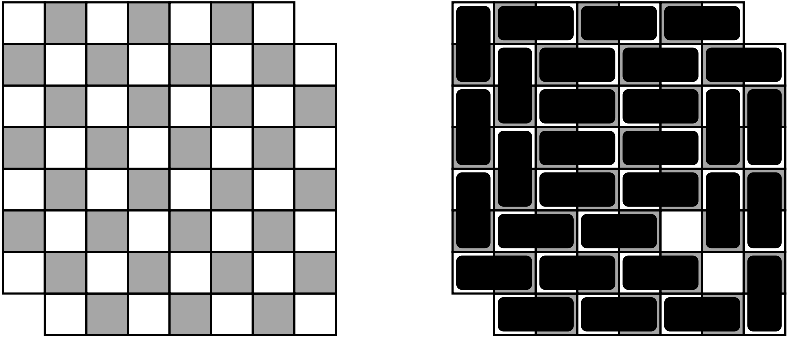
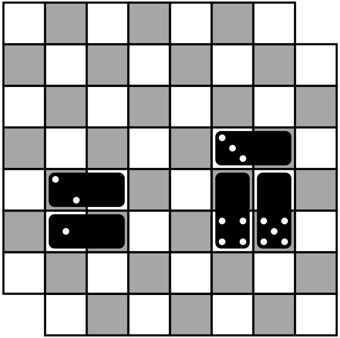


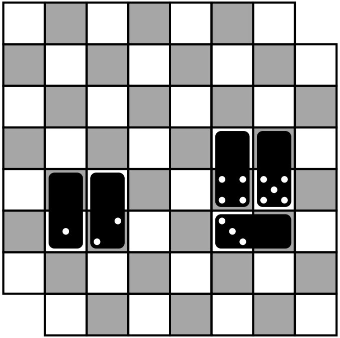
Easy to refute based on the following two observations:

- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Mutilated Chessboards: A Computer-Generated Proof

Modern SAT solvers produce proofs that can be very different compared to human-made proofs for the same problem.





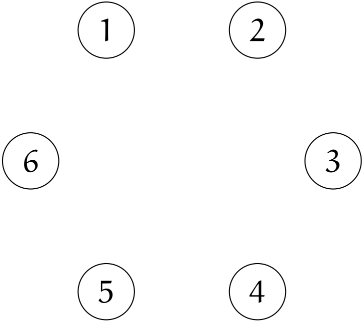
The two patterns can be automatically detected and blocked

- This reduces the number of explored states exponentially
- The PR proofs are linear in the formula size

Ramsey Numbers

Ramsey Number $R(k)$: What is the smallest n such that any graph with n vertices has either a clique or a co-clique of size k?

$$\begin{aligned} R(3) &= 6 \\ R(4) &= 18 \\ 43 \leq\ R(5) &\leq 48 \end{aligned}$$



SAT solvers can determine that $R(4) = 18$ in 1 second using symmetry breaking; w/o symmetry breaking it requires weeks.

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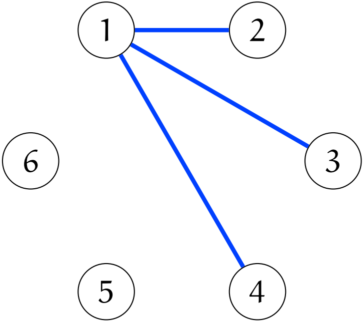

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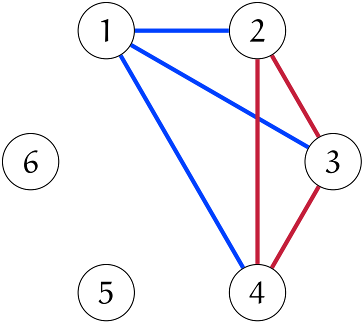

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SR Proof of Ramsey Number Three (I)

Variable $e_{i,j}$: edge (i, j) is colored red ($\top$) or blue ($\bot$)

Step one: sort the edges adjacent to vertex v_1 (blue first).

- $(\neg e_{1,2} \vee e_{1,3})$
- τ applies the mapping $(v_2, v_3)(v_3, v_2)$ or specially
- $\tau := \{e_{1,2} = \bot, e_{1,3} = \top, e_{2,4} = e_{3,4}, e_{3,4} = e_{2,4}, e_{2,5} = e_{3,5}, e_{3,5} = e_{2,5}\}$

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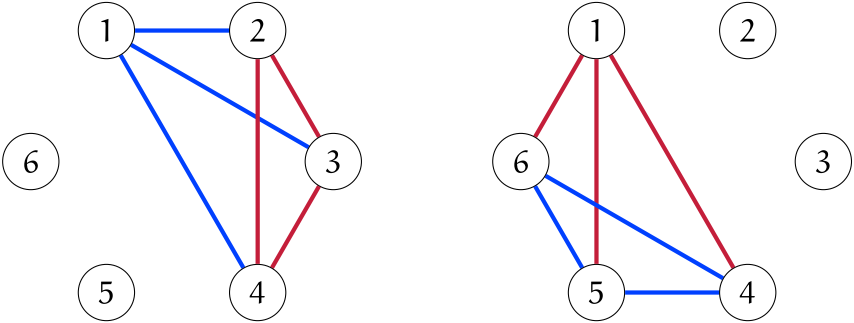
- $(\neg e_{1,5} \vee e_{1,6})$
- τ applies the mapping $(v_2, v_3, v_4, v_5, v_6)(v_3, v_4, v_5, v_6, v_2)$

SR Proof of Ramsey Number Three (II)

With sorted edges you can learn $(\neg e_{1,4})$ and $(e_{1,4})$ directly



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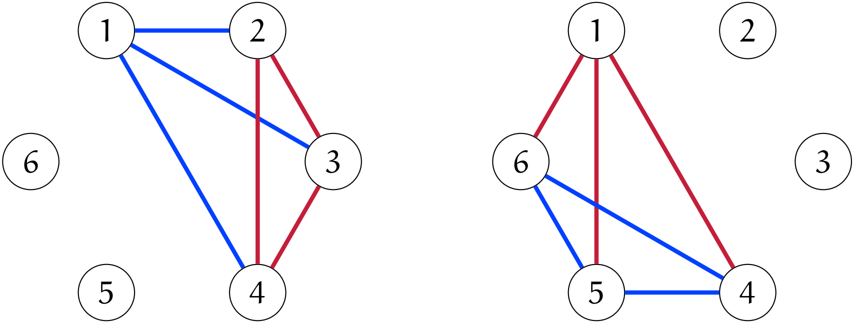

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Similarly, a 38-clause SR proof can be constructed for $R(4) \leq 18$.

Only Useful for Symmetry Breaking?

The hardness of the example formulas were due symmetries

- Global symmetries (e.g. Ramsey)
- Local symmetries (e.g. mutilated chessboards)

The WLOS proof systems can break symmetries

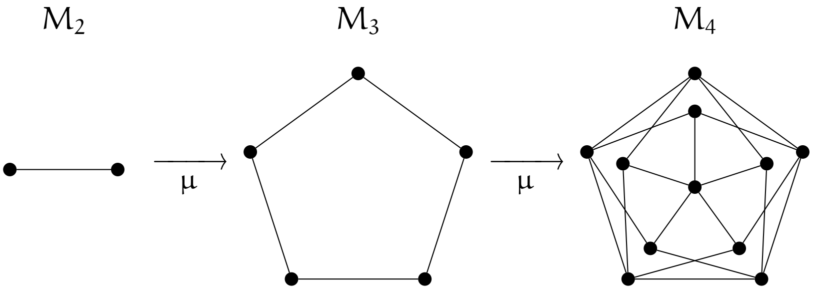
- without new variables
- ... although sometimes it requires the strength of SR

Can they also provide strong non-symmetry reasoning?

Mycielski Graphs

M_k is a triangle-free graph with chromatic number k

- M_2: a path of length 2
- M_3: a cycle of length 5
- M_i can be constructed from M_{i+1} using operation μ
- These graph coloring problems are extremely hard



Mycielski Operation

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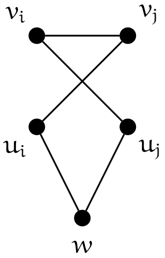
Operation μ works as follows on graph $G = (V, E)$:

1. Start with G
2. For $v_i \in V$, create a copy vertex u_i
3. For $(v_i, v_j) \in E$, add edges (v_i, u_j), (u_i, v_j)
4. Add a vertex w and all edges (w, u_i)



Short PR Proofs of Mycielski Graphs

Short PR proofs can be constructed using this observation:

- If no neighbor of u_i has color c, then u_i can have color c
- If a vertex v_i has color c and its corresponding w doesn't, then no neighbor of the corresponding u_i has color c.
- So the above is enough to color u_i with color c
- The PR clauses $(v_{i,c} \vee \neg w_{i,c} \vee u_{i,c})$ enable a linear-size proof
- Repair: $u_{i,c} = \top$ and $u_{i,d} = \bot$ for all $d \neq c$