Logic and Mechanized Reasoning
Propositional Logic

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50 Years of Successes in Computer-Aided Mathematics

1976 Four-Color Theorem

1998 Kepler Conjecture

2010 “God’s Number = 20”: Optimal Rubik’s cube strategy

2014 Boolean Erdős discrepancy problem

2016 Boolean Pythagorean triples problem

2018 Schur Number Five

2019 Keller’s Conjecture

2022 Packing Number of Square Grid

2023 Empty Hexagon in Every 30 Points

50 Years of Successes in Computer-Aided Mathematics

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2014 Boolean Erdős discrepancy problem (using a SAT solver)

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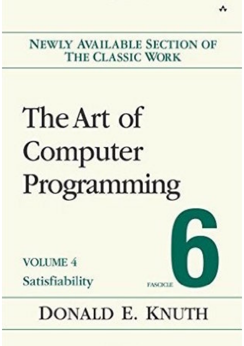
Breakthrough in SAT Solving in the Last 25 Years

Satisfiability problem: Can a propositional formula be satisfied?

mid '90s: formulas solvable with thousands of variables and clauses
now: formulas solvable with millions of variables and clauses

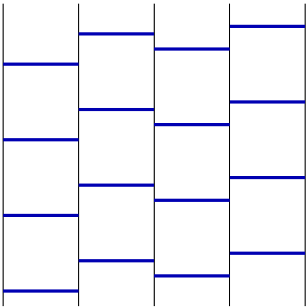


Edmund Clarke: *"a key technology of the 21st century"* [Biere, Heule, vanMaaren, and Walsh '09]

Donald Knuth: *"evidently a killer app, because it is key to the solution of so many other problems"* [Knuth '15]

Keller's Conjecture: A Tiling Problem

Consider tiling a floor with square tiles, all of the same size. Is it the case that any gap-free tiling results in at least two fully connected tiles, i.e., tiles that have an entire edge in common?



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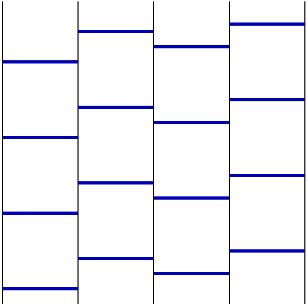
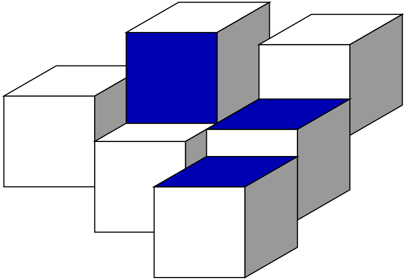


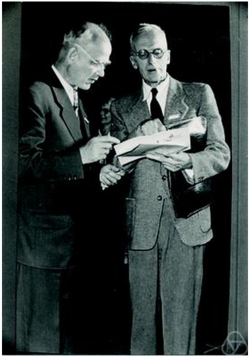


Keller's Conjecture: Resolved

[Brakensiek, Heule, Mackey, & Narvaez 2019]

In 1930, Ott-Heinrich Keller conjectured that this phenomenon holds in every dimension.

Keller's Conjecture.
For all $n \geq 1$, every tiling of the n-dimensional space with unit cubes has two which fully share a face.



[Wikipedia, CC BY-SA]

GEOMETRY

Computer Search Settles 90-Year-Old Math Problem

10

By translating Keller's conjecture into a computer-friendly search for a type of graph, researchers have finally resolved a problem about covering spaces with tiles.

Syntax

Semantics

Calculating with Propositions

Random Formulas

Syntax

Semantics

Calculating with Propositions

Random Formulas

Syntax: Definition

The set of propositional formulas is generated inductively:

- Each variable p_i is a formula.
- $\top$ and $\bot$ are formulas.
- If A is a formula, so is $\neg A$ ("not A").
- If A and B are formulas, so are
  - $A \wedge B$ ("A and B"),
  - $A \vee B$ ("A or B"),
  - $A \to B$ ("A implies B"), and
  - $A \leftrightarrow B$ ("A if and only if B").

Syntax: Complexity

Complexity: the number of connectives

$$
\begin{aligned}
complexity(p_i) &= 0 && \text{(C}p\text{)}\\
complexity(\top) &= 0 && \text{(C}\top\text{)}\\
complexity(\bot) &= 0 && \text{(C}\bot\text{)}\\
complexity(\neg A) &= complexity(A) + 1 && \text{(C}\neg\text{)}\\
complexity(A \wedge B) &= complexity(A) + complexity(B) + 1 && \text{(C}\wedge\text{)}\\
complexity(A \vee B) &= complexity(A) + complexity(B) + 1 && \text{(C}\vee\text{)}\\
complexity(A \to B) &= complexity(A) + complexity(B) + 1 && \text{(C}\to\text{)}\\
complexity(A \leftrightarrow B) &= complexity(A) + complexity(B) + 1 && \text{(C}\leftrightarrow\text{)}
\end{aligned}
$$

Syntax: Depth

Depth of the parse tree

$$
\begin{aligned}
depth(p_i) &= 0 && (\mathsf{D}p)\\
depth(\top) &= 0 && (\mathsf{D}\top)\\
depth(\bot) &= 0 && (\mathsf{D}\bot)\\
depth(\neg A) &= depth(A) + 1 && (\mathsf{D}\neg)\\
depth(A \wedge B) &= \max(depth(A), depth(B)) + 1 && (\mathsf{D}\wedge)\\
depth(A \vee B) &= \max(depth(A), depth(B)) + 1 && (\mathsf{D}\vee)\\
depth(A \to B) &= \max(depth(A), depth(B)) + 1 && (\mathsf{D}\to)\\
depth(A \leftrightarrow B) &= \max(depth(A), depth(B)) + 1 && (\mathsf{D}\leftrightarrow)
\end{aligned}
$$

Syntax: Complexity and Depth

Theorem

For every formula A, we have $complexity(A) \leq 2^{depth(A)} - 1$.

Proof.

Base case: $complexity(p_i) = 0 = 2^0 - 1 = 2^{depth(p_i)} - 1$,

Inductive case (first $\neg$, afterwards $\wedge$):

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Inductive case (first $\neg$, afterwards $\wedge$):

$$\begin{aligned} complexity(\neg A) &= complexity(A) + 1 && [\mathsf{C}\neg] \\ &\leq 2^{depth(A)} - 1 + 1 && [\mathsf{IH}] \\ &\leq 2^{depth(A)} + 2^{depth(A)} - 1 && [\mathsf{math}] \\ &\leq 2^{depth(A)+1} - 1 = 2^{depth(\neg A)} - 1 && [\mathsf{math},\ \mathsf{D}\neg] \end{aligned}$$

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Base case: $complexity(p_i) = 0 = 2^0 - 1 = 2^{depth(p_i)} - 1$,

Inductive case (first $\neg$, afterwards $\wedge$):

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&\leq 2^{depth(A)} - 1 + 1 && [\text{IH}] \\
&\leq 2^{depth(A)} + 2^{depth(A)} - 1 && [\text{math}] \\
&\leq 2^{depth(A)+1} - 1 = 2^{depth(\neg A)} - 1 && [\text{math, D}\neg]
\end{aligned}
$$

$$
\begin{aligned}
complexity(A \wedge B) &= complexity(A) + complexity(B) + 1 && [\text{C}\wedge] \\
&\leq 2^{depth(A)} - 1 + 2^{depth(B)} - 1 + 1 && [\text{IH}] \\
&\leq 2 \cdot 2^{\max(depth(A), depth(B))} - 1 && [\text{math}] \\
&= 2^{\max(depth(A), depth(B))+1} - 1 && [\text{math}] \\
&= 2^{depth(A \wedge B)} - 1 && [\text{D}\wedge]
\end{aligned}
$$

Syntax: Subformulas

$$
\begin{array}{rcl}
\mathit{subformulas}(A) & = & \{A\} \quad \text{if } A \text{ is atomic} \\
\mathit{subformulas}(\neg A) & = & \{\neg A\} \cup \mathit{subformulas}(A) \\
\mathit{subformulas}(A \star B) & = & \{A \star B\} \cup \mathit{subformulas}(A) \cup \\
 & & \mathit{subformulas}(B)
\end{array}
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Example

Consider the formula $(\neg A \wedge C) \rightarrow \neg(B \vee C)$.
The *subformulas* function returns

Syntax: Subformulas

$$\begin{array}{rcl} subformulas(A) & = & \{A\} \quad \text{if } A \text{ is atomic} \\ subformulas(\neg A) & = & \{\neg A\} \cup subformulas(A) \\ subformulas(A \star B) & = & \{A \star B\} \cup subformulas(A) \cup \\ & & subformulas(B) \end{array}$$

Example

Consider the formula $(\neg A \wedge C) \rightarrow \neg(B \vee C)$.
The *subformulas* function returns
$\{(\neg A \wedge C) \rightarrow \neg(B \vee C), \neg A \wedge C, \neg A, A, C, \neg(B \vee C), B \vee C, B)\}$

Syntax: Proposition

Proposition

For every pair of formulas A and B, if $B \in subformulas(A)$ and $A \in subformulas(B)$ then A and B are atomic.

True or false?

Syntax: Proposition

Proposition

For every pair of formulas A and B, if $B \in subformulas(A)$ and $A \in subformulas(B)$ then A and B are atomic.

True or false?

Proof.

False. A counterexample is $A = B = \neg p$. $\square$

Syntax: Substitution

Let A and B be formulas and p a propositional variable

$A[B/p]$ denotes the substitution of p by B in A

$$
\begin{aligned}
p_i[B/p] &= \begin{cases} B & \text{if } p \text{ is } p_i \\ p_i & \text{otherwise} \end{cases} \\
(\neg C)[B/p] &= \neg(C[B/p]) \\
C \star D[B/p] &= C[B/p] \star D[B/p]
\end{aligned}
$$

Syntax

Semantics

Calculating with Propositions

Random Formulas

Semantics: Introduction

Consider the formula $p \wedge (\neg q \vee r)$. Is it true?

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Once we specify which of p, q, and r are true and which are false, the truth value of $p \wedge (\neg q \vee r)$ is completely determined.

Semantics: Introduction

Consider the formula $p \wedge (\neg q \vee r)$. Is it true?

It depends on the truth of p, q, and r.

Once we specify which of p, q, and r are true and which are false, the truth value of $p \wedge (\neg q \vee r)$ is completely determined.

A truth assignment τ provides this specification by mapping propositional variables to the constants $\top$ and $\bot$.

Semantics: Evaluation

$$
\begin{array}{rcl}
[\![p_i]\!]_\tau & = & \tau(p_i) \\
[\![\top]\!]_\tau & = & \top \\
[\![\bot]\!]_\tau & = & \bot \\
[\![\neg A]\!]_\tau & = & \begin{cases} \top & \text{if } [\![A]\!]_\tau = \bot \\ \bot & \text{otherwise} \end{cases} \\
[\![A \wedge B]\!]_\tau & = & \begin{cases} \top & \text{if } [\![A]\!]_\tau = \top \text{ and } [\![B]\!]_\tau = \top \\ \bot & \text{otherwise} \end{cases} \\
[\![A \vee B]\!]_\tau & = & \begin{cases} \top & \text{if } [\![A]\!]_\tau = \top \text{ or } [\![B]\!]_\tau = \top \\ \bot & \text{otherwise} \end{cases} \\
[\![A \to B]\!]_\tau & = & \begin{cases} \top & \text{if } [\![A]\!]_\tau = \bot \text{ or } [\![B]\!]_\tau = \top \\ \bot & \text{otherwise} \end{cases} \\
[\![A \leftrightarrow B]\!]_\tau & = & \begin{cases} \top & \text{if } [\![A]\!]_\tau = [\![B]\!]_\tau \\ \bot & \text{otherwise} \end{cases}
\end{array}
$$

Semantics: Satisfiable, Unsatisfiable, and Valid

- If $[\![A]\!]_\tau = \top$, then A is satisfied by τ. In that case, τ is a satisfying assignment of A.
- A propositional formula A is satisfiable iff there exists an assignment τ that satisfies it and unsatisfiable otherwise.
- A propositional formula A is valid iff every assignment satisfies it.

Semantics: Satisfiable, Unsatisfiable, and Valid

- If $[\![A]\!]_\tau = \top$, then A is satisfied by τ. In that case, τ is a satisfying assignment of A.
- A propositional formula A is satisfiable iff there exists an assignment τ that satisfies it and unsatisfiable otherwise.
- A propositional formula A is valid iff every assignment satisfies it.

Example

Which one(s) of the formulas is satisfiable/unsatisfiable/valid?

- $(A \leftrightarrow B) \lor (\neg C)$
- $(A) \lor (\neg B) \lor (\neg A \land B)$
- $(A) \land (\neg B) \land (A \to B)$

Semantics: Relation Valid and Unsatisfiable

Theorem

A propositional formula A is valid if and only if $\neg A$ is unsatisfiable.

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Proof.

A is valid if and only if $[\![A]\!]_\tau = \top$ for every assignment τ.

Semantics: Relation Valid and Unsatisfiable

Theorem

A propositional formula A is valid if and only if $\neg A$ is unsatisfiable.

Proof.

A is valid if and only if $[\![A]\!]_\tau = \top$ for every assignment τ.

By the def of $[\![\neg A]\!]_\tau$, this happens iff $[\![\neg A]\!]_\tau = \bot$ for every τ.

Semantics: Relation Valid and Unsatisfiable

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A propositional formula A is valid if and only if $\neg A$ is unsatisfiable.

Proof.

A is valid if and only if $[\![A]\!]_\tau = \top$ for every assignment τ.

By the def of $[\![\neg A]\!]_\tau$, this happens iff $[\![\neg A]\!]_\tau = \bot$ for every τ.

This is the same as saying that $\neg A$ is unsatisfiable. $\square$

Semantics: Proposition 1

Proposition

For every pair of formulas A and B, $A \wedge B$ is valid if and only if A is valid and B is valid.

True or false?

Semantics: Proposition 1

Proposition

For every pair of formulas A and B, $A \wedge B$ is valid if and only if A is valid and B is valid.

True or false?

Proof.

True. $A \wedge B$ is valid means that for every assignment τ we have $[\![A \wedge B]\!]_\tau = \top$. By the definition of $[\![A \wedge B]\!]$, this happens if and only if $[\![A]\!]_\tau = \top$ and $[\![B]\!]_\tau = \top$ for every τ, i.e. if and only if A and B are both valid. $\square$

Semantics: Proposition 2

Proposition

For every pair of formulas A and B, $A \wedge B$ is satisfiable if and only if A is satisfiable and B is satisfiable.

True or false?

Semantics: Proposition 2

Proposition

For every pair of formulas A and B, $A \wedge B$ is satisfiable if and only if A is satisfiable and B is satisfiable.

True or false?

Proof.

False. Consider the formula $A \wedge B$ with $A = p$ and $B = \neg p$. Clearly both A and B are satisfiable, while $A \wedge B$ is unsatisfiable. $\square$

Semantics: Proposition 3

Proposition

For every pair of formulas A and B, $A \vee B$ is valid if and only if A is valid or B is valid.

True or false?

Semantics: Proposition 3

Proposition

For every pair of formulas A and B, $A \vee B$ is valid if and only if A is valid or B is valid.

True or false?

Proof.

False. Consider the formula $A \vee B$ with $A = p$ and $B = \neg p$. The formula $A \vee B$ is valid, while either A nor B is valid. $\square$

Semantics: Proposition 4

Proposition

For every pair of formulas A and B, $A \vee B$ is satisfiable if and only if A is satisfiable or B is satisfiable.

True or false?

Semantics: Proposition 4

Proposition

For every pair of formulas A and B, $A \vee B$ is satisfiable if and only if A is satisfiable or B is satisfiable.

True or false?

Proof.
True. Suppose $A \vee B$ is satisfied by τ. By definition it must be the case that $[\![A]\!]_\tau = \top$ or $[\![B]\!]_\tau = \top$, so τ satisfies A or B. Conversely, if an assignment τ satisfies either A or B, then $[\![A]\!]_\tau = \top$ or $[\![B]\!]_\tau = \top$. In either case, $[\![A \vee B]\!]_\tau = \top$. So if A is satisfiable or B is satisfiable, so is $A \vee B$. $\square$

Semantics: Entailment and Equivalence

- If every satisfying assignment of a formula A, also satisfies formula B, the A entails B, denoted by $A \models B$.
- If $A \models B$ and $B \models A$, then A and B are logically equivalent, denoted by $A \equiv B$.

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Example

Which formula entails which other formula?

- A
- $\neg A \to B$
- $\neg(\neg A \vee \neg B)$

Semantics: Proposition 5

Proposition

Suppose A and B are formulas and $A \models B$.
If A is valid, then B is valid.

True or false?

Semantics: Proposition 5

Proposition

Suppose A and B are formulas and $A \models B$.
If A is valid, then B is valid.

True or false?

Proof.

True. Suppose $A \models B$, and suppose A is valid. Let τ be any truth assignment. Since A is valid, $[\![A]\!]_\tau = \top$. Since $A \models B$, $[\![B]\!]_\tau = \top$. We have shown $[\![B]\!]_\tau = \top$ for every τ, i.e. B is valid. $\square$

Semantics: Proposition 6

Proposition

Suppose A and B are formulas and $A \models B$.
If B is satisfiable, then A is satisfiable.

True or false?

Semantics: Proposition 6

Proposition

Suppose A and B are formulas and $A \models B$.
If B is satisfiable, then A is satisfiable.

True or false?

Proof.
False. A counterexample is $A = p \wedge \neg p$ and $B = p$. $\square$

Semantics: Proposition 7

Proposition

For every triple of formulas A, B, and C, if $A \models B \models C \models A$ then $A \equiv B \equiv C$.

True or false?

Semantics: Proposition 7

Proposition

For every triple of formulas A, B, and C, if $A \models B \models C \models A$ then $A \equiv B \equiv C$.

True or false?

Proof.

True. Suppose $A \models B \models C \models A$. Let τ be any truth assignment. We need to show $[\![A]\!]_\tau = [\![B]\!]_\tau = [\![C]\!]_\tau$. Suppose $[\![A]\!]_\tau = \top$. Since $A \models B$,$[\![B]\!]_\tau = \top$, and since $B \models C$, we have $[\![C]\!]_\tau = \top$. So, in that case, $[\![A]\!]_\tau = [\![B]\!]_\tau = [\![C]\!]_\tau$.
The other possibility is $[\![A]\!]_\tau = \bot$. Since $C \models A$, we must have $[\![C]\!]_\tau = \bot$, and since $B \models C$, we have $[\![B]\!]_\tau = \bot$. So, in that case also, $[\![A]\!]_\tau = [\![B]\!]_\tau = [\![C]\!]_\tau$. $\square$

Semantics: Diplomacy Problem

"You are chief of protocol for the embassy ball. The crown prince instructs you either to invite *Peru* or to exclude *Qatar*. The queen asks you to invite either *Qatar* or *Romania* or both. The king, in a spiteful mood, wants to snub either *Romania* or *Peru* or both. Is there a guest list that will satisfy the whims of the entire royal family?"

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$$(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p)$$

Semantics: Truth Table

$$\Gamma = (p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p)$$

p	q	r	falsifies	$[\![\Gamma]\!]_\tau$
$\bot$	$\bot$	$\bot$	$(q \vee r)$	$\bot$
$\bot$	$\bot$	$\top$	—	$\top$
$\bot$	$\top$	$\bot$	$(p \vee \neg q)$	$\bot$
$\bot$	$\top$	$\top$	$(p \vee \neg q)$	$\bot$
$\top$	$\bot$	$\bot$	$(q \vee r)$	$\bot$
$\top$	$\bot$	$\top$	$(\neg r \vee \neg p)$	$\bot$
$\top$	$\top$	$\bot$	—	$\top$
$\top$	$\top$	$\top$	$(\neg r \vee \neg p)$	$\bot$

Syntax

Semantics

Calculating with Propositions

Random Formulas

Calculating with Propositions: Laws

Some propositional laws (more in the textbook):

$$
\begin{aligned}
A \vee \top &\equiv \top \\
A \wedge \top &\equiv A \\
A \vee B &\equiv B \vee A \\
(A \vee B) \vee C &\equiv A \vee (B \vee C) \\
A \wedge (B \vee C) &\equiv (A \wedge B) \vee (A \wedge C) \\
A \vee (B \wedge C) &\equiv (A \vee B) \wedge (A \vee C) \\
A \wedge (A \vee B) &\equiv A \\
A \to B &\equiv \neg A \vee B
\end{aligned}
$$

Calculating with Propositions: Laws

Some propositional laws (more in the textbook):

$$\begin{array}{rcl} A \vee \top & \equiv & \top \\ A \wedge \top & \equiv & A \\ A \vee B & \equiv & B \vee A \\ (A \vee B) \vee C & \equiv & A \vee (B \vee C) \\ A \wedge (B \vee C) & \equiv & (A \wedge B) \vee (A \wedge C) \\ A \vee (B \wedge C) & \equiv & (A \vee B) \wedge (A \vee C) \\ A \wedge (A \vee B) & \equiv & A \\ A \to B & \equiv & \neg A \vee B \end{array}$$

De Morgan's laws:

$$\begin{array}{rcl} \neg(A \wedge B) & \equiv & \neg A \vee \neg B \\ \neg(A \vee B) & \equiv & \neg A \wedge \neg B \end{array}$$

Calculating with Propositions: Example

Theorem
For any propositional formulas A and B, we have
$(A \wedge \neg B) \vee B \equiv A \vee B$.

Proof.

$$(A \wedge \neg B) \vee B \quad \equiv$$

Calculating with Propositions: Example

Theorem

For any propositional formulas A and B, we have $(A \wedge \neg B) \vee B \equiv A \vee B$.

Proof.

$$\begin{aligned}(A \wedge \neg B) \vee B &\equiv (A \vee B) \wedge (\neg B \vee B) \\ &\equiv\end{aligned}$$

Calculating with Propositions: Example

Theorem

For any propositional formulas A and B, we have $(A \wedge \neg B) \vee B \equiv A \vee B$.

Proof.

$$\begin{aligned}
(A \wedge \neg B) \vee B &\equiv (A \vee B) \wedge (\neg B \vee B) \\
&\equiv (A \vee B) \wedge \top \\
&\equiv
\end{aligned}$$

Calculating with Propositions: Example

Theorem

For any propositional formulas A and B, we have $(A \wedge \neg B) \vee B \equiv A \vee B$.

Proof.

$$\begin{aligned} (A \wedge \neg B) \vee B &\equiv (A \vee B) \wedge (\neg B \vee B) \\ &\equiv (A \vee B) \wedge \top \\ &\equiv (A \vee B). \end{aligned}$$

$\square$

Calculating with Propositions: A Harder Example

Theorem

For any propositional formulas A, B, and C, we have
$\neg((A \vee B) \wedge (B \to C)) \equiv (\neg A \vee B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$.

Proof.

$\neg((A \vee B) \wedge (B \to C)) \equiv$

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Theorem

For any propositional formulas A, B, and C, we have
$\neg((A \vee B) \wedge (B \to C)) \equiv (\neg A \vee B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$.

Proof.

$$\begin{aligned} \neg((A \vee B) \wedge (B \to C)) &\equiv \neg((A \vee B) \wedge (\neg B \vee C)) \\ &\equiv \end{aligned}$$

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Proof.

$$\begin{aligned}
\neg((A \vee B) \wedge (B \to C)) &\equiv \neg((A \vee B) \wedge (\neg B \vee C)) \\
&\equiv \neg(A \vee B) \vee \neg(\neg B \vee C) \\
&\equiv
\end{aligned}$$

Calculating with Propositions: A Harder Example

Theorem

For any propositional formulas A, B, and C, we have $\neg((A \vee B) \wedge (B \to C)) \equiv (\neg A \vee B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$.

Proof.

$$
\begin{aligned}
\neg((A \vee B) \wedge (B \to C)) &\equiv \neg((A \vee B) \wedge (\neg B \vee C)) \\
&\equiv \neg(A \vee B) \vee \neg(\neg B \vee C) \\
&\equiv (\neg A \wedge \neg B) \vee (B \wedge \neg C) \\
&\equiv
\end{aligned}
$$

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Theorem

For any propositional formulas A, B, and C, we have
$\neg((A \vee B) \wedge (B \to C)) \equiv (\neg A \vee B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$.

Proof.

$$
\begin{aligned}
\neg((A \vee B) \wedge (B \to C)) &\equiv \neg((A \vee B) \wedge (\neg B \vee C)) \\
&\equiv \neg(A \vee B) \vee \neg(\neg B \vee C) \\
&\equiv (\neg A \wedge \neg B) \vee (B \wedge \neg C) \\
&\equiv (\neg A \vee (B \wedge \neg C)) \wedge (\neg B \vee (B \wedge \neg C)) \\
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Calculating with Propositions: A Harder Example

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\end{aligned}
$$

$\square$

Syntax

Semantics

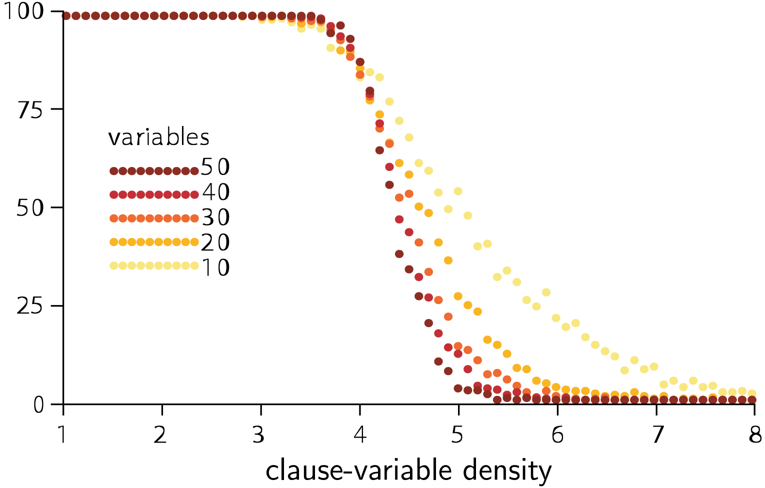
Calculating with Propositions

Random Formulas

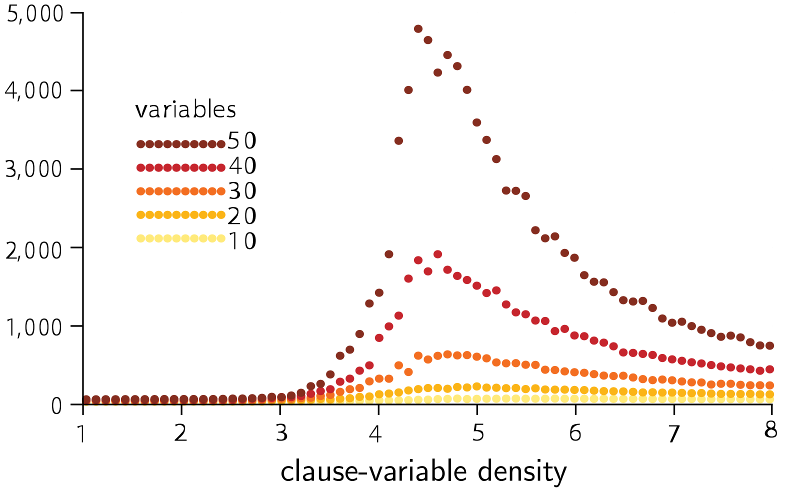
Random Formulas: Introduction

- Formulas in conjunctive normal form
- All clauses have length k
- Variables have the same probability to occur
- Each literal is negated with probability of 50%
- Density is ratio Clauses to Variables

Random Formulas: Phase Transition



Random Formulas: Exponential Runtime



SAT Game

by Olivier Roussel

`http://www.cs.utexas.edu/~marijn/game/`