SAMPLING THEORY

Representing continuous signals with discrete numbers

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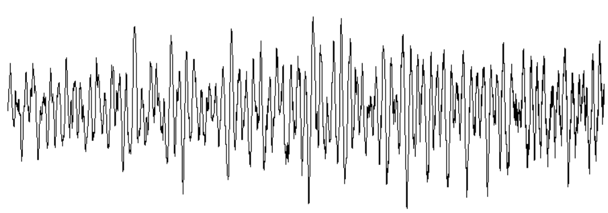
From analog to digital (and back)



010110
010100
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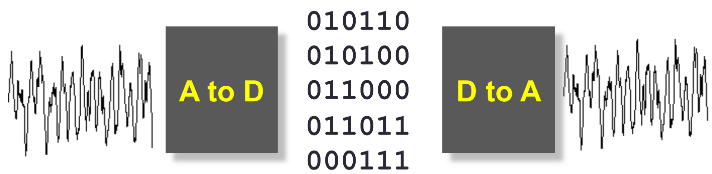
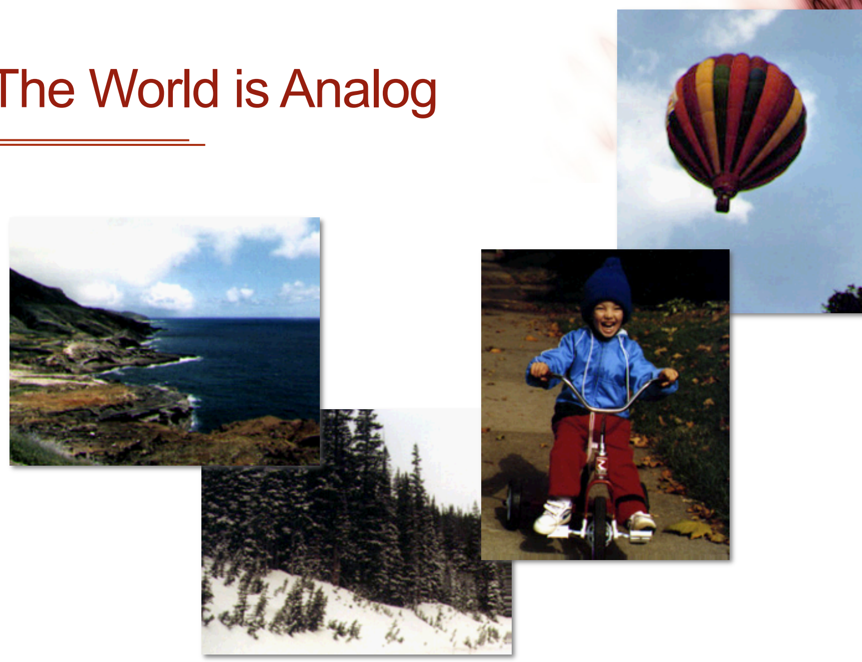
Analog to Digital Conversion
Digital to Analog Conversion



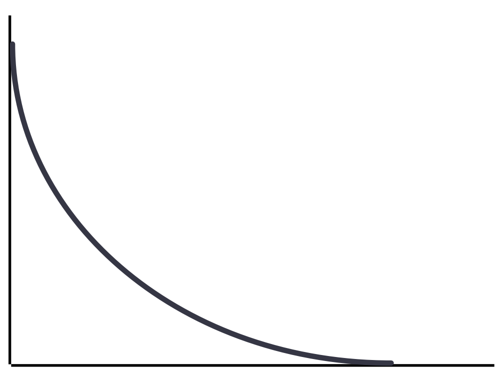
Approach

- Intuition
- Frequency Domain (Fourier Transform)
- Sampling Theory
- Practical Results

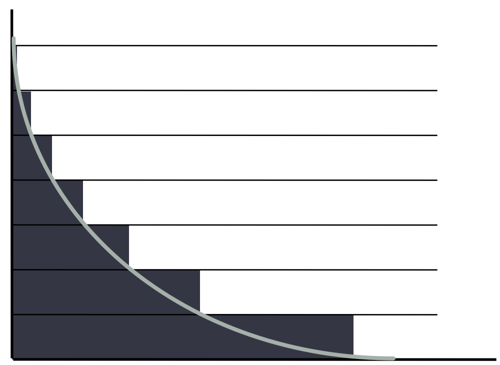
The World is Analog



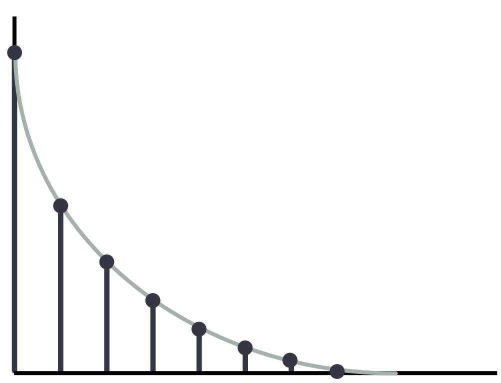
Continuous or Discrete?



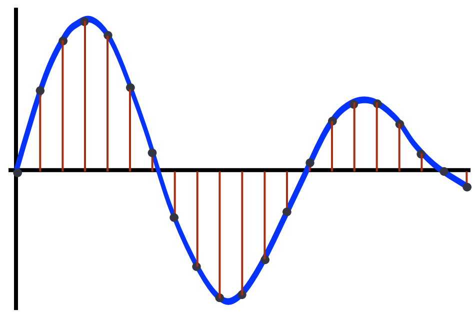
Discrete Amplitude (Y axis)



Discrete Time (X axis)



Digitizing a continuous function (or signal)

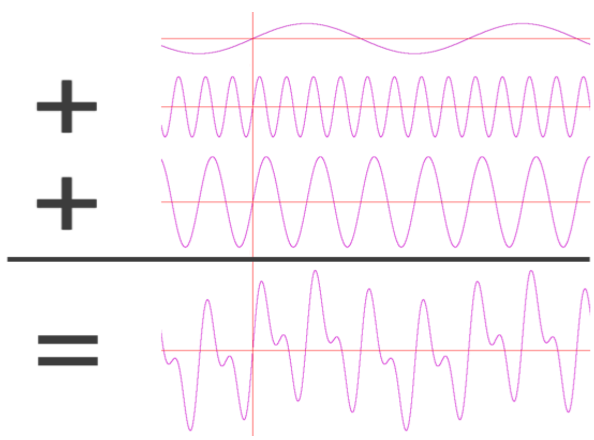


Questions

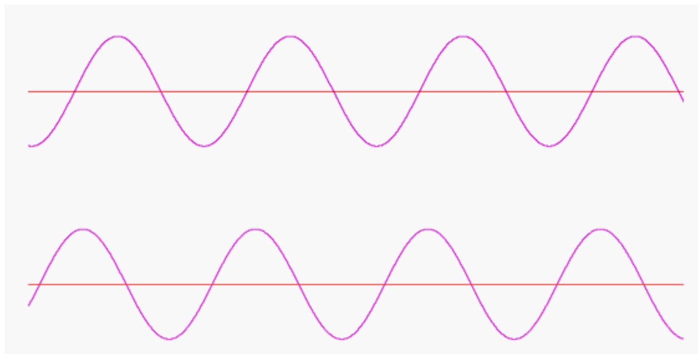
- What **sample rate** should we use? Why does it matter?
- How many **bits per sample** should we use? Why does it matter?
- **Interpolation**: How can we interpolate samples to recover the sampled signal?
- What's the effect of **rounding** to the nearest integer sample value?
- How do we **convert** analog to/from digital?

Introduction to the Spectrum

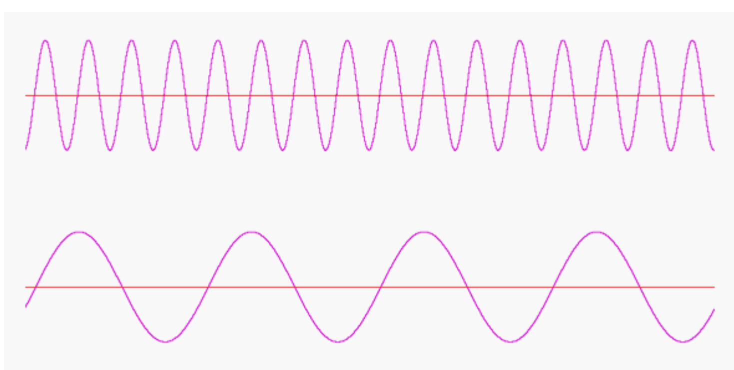
Introduction to the Spectrum (2)



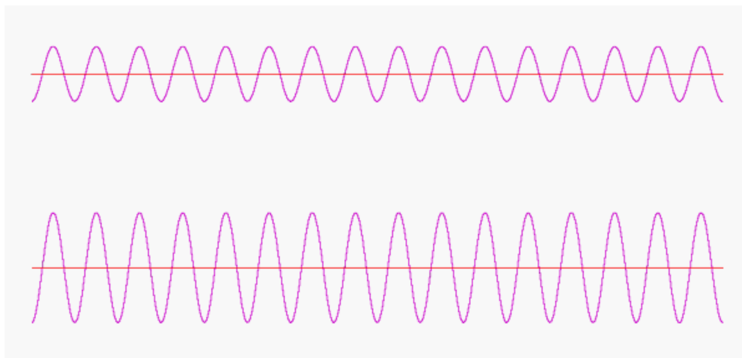
Phase



Frequency



Amplitude



Sinusoidal Partials

$$A \cdot \sin(\omega t + \phi)$$

Amplitude A

Frequency ω

Phase ϕ

Fourier Transform

- Our goal is to *transform* a function-of-time representation of a signal to a function-of-frequency representation
- Express the time function as an (infinite) sum of sinusoids.
- Express the infinite sum as a function from frequency to amplitude
- I.e. for each frequency, what is the amplitude of the sinusoid of that frequency within this infinite sum?

Fourier Transform: Cartesian Coordinates

Real part:

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

Imaginary part:

$$X(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

What About Phase?

- Remember at each frequency, we said there is *one* sinusoidal component: $A \cdot \sin(\omega t + \phi)$
  - *A* is amplitude
  - ω is frequency
  - ϕ is phase
- The Fourier analysis computes two amplitudes:
  - $R(\omega)$ and $X(\omega)$
  - Trig identities tell us there is no conflict:

$$A = \sqrt{R^2 + X^2} \qquad \phi = \arctan(X / R)$$

$$A(\omega) = \sqrt{R^2(\omega) + X^2(\omega)} \qquad \phi(\omega) = \arctan(X(\omega) / R(\omega))$$

From Cartesian to Complex

- *R* is "real" or cosine part
- *X* is "imaginary" or sine part
- Use $F(\omega) = R(\omega) + j \cdot X(\omega)$

Fourier Transform (Complex Form)

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t \, dt$$

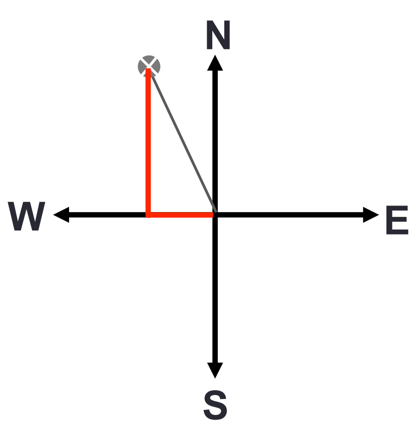
$+$

$$j \cdot X(\omega) = -j\int_{-\infty}^{\infty} f(t)\sin\omega t \, dt$$

$=$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

Orthogonal Basis Functions



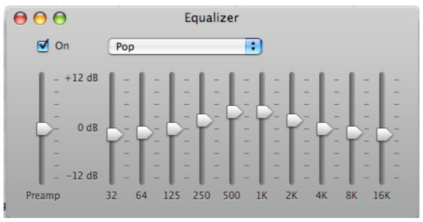
Horizontal and vertical axes are independent or orthogonal in the 2-dimensional plane, sinusoids are orthogonal in the infinite-dimensional space of continuous signals.

Just as every point in the plane is a *unique linear* combination of the unit E and N vectors, every signal is a *unique linear* combination of sinusoids.

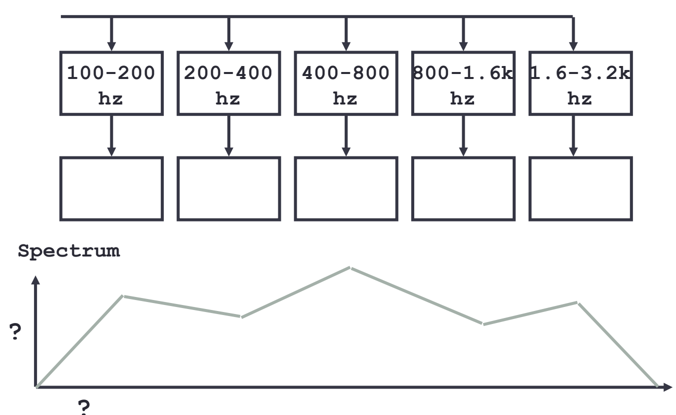
The Frequency Domain

Graphic Equalizer

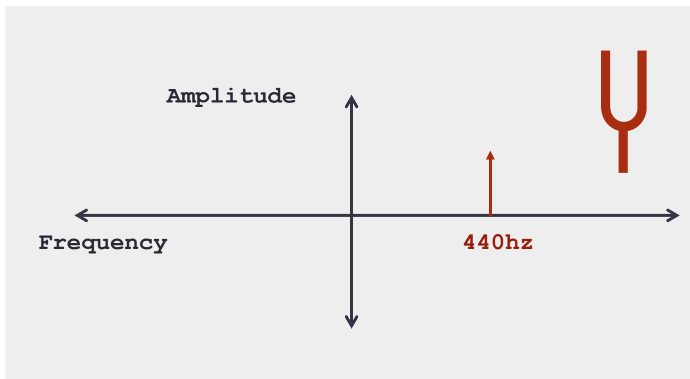
Spectral Analyzer



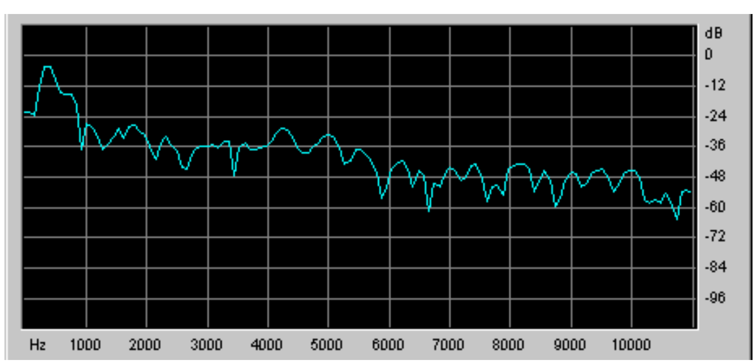
The Frequency Domain (2)



The Amplitude Spectrum



Amplitude Spectrum of a "Real" Signal



Representations

- (Real, Imaginary) or (Amplitude, Phase)?
  - Power ~ Amplitude^2
  - We generally cannot hear phase
  - Measure a stationary signal after Δ*t*: Amplitude spectrum is unchanged, but phase changes by $\Delta t \cdot \omega$
- Given (amplitude, phase)
  - It's hard to plot both
  - Usually, we ignore the phase

Time vs Frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

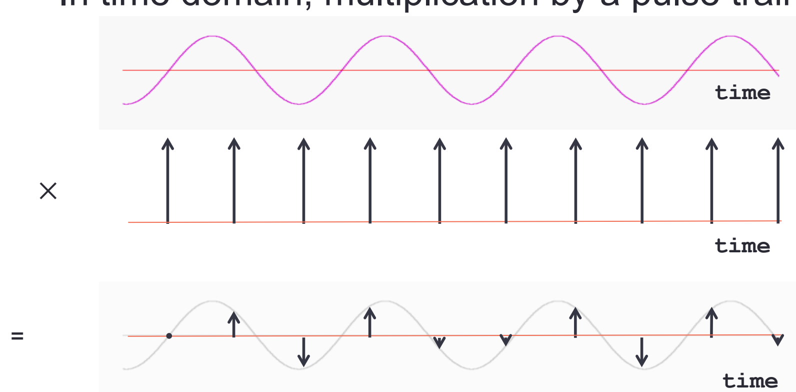
- What happens to time when you transform to the frequency domain?
- Note that time is "integrated out"
- NO TIME REMAINS
- The Fourier Transform of a signal *is not a function of time !!!!!*
- (Later, we'll look at *short-time transforms* – e.g. what you see on a time-varying spectral display – which *are* time varying.)

PERFECT SAMPLING

From continuous signals to discrete samples and back again

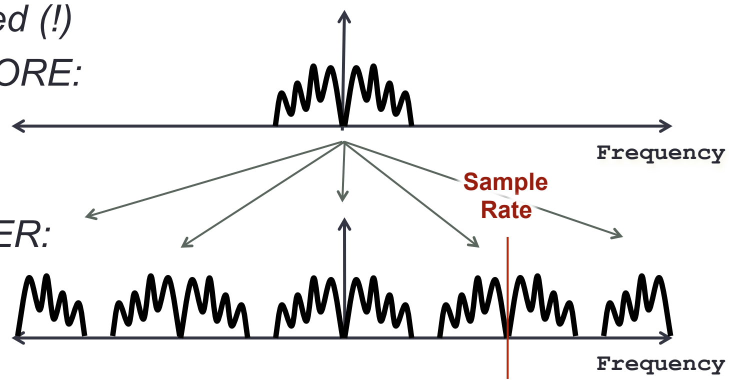
Sampling – Time Domain

- What happens when you sample a signal?
- In time domain, multiplication by a pulse train:



Sampling – Frequency Domain

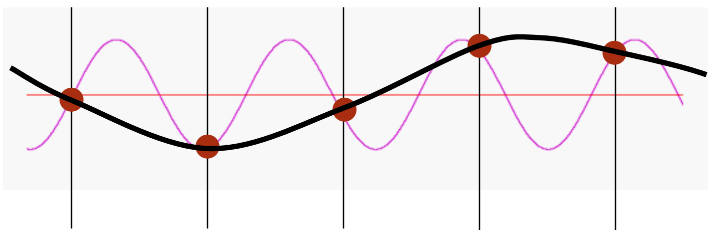
- What happens when you sample a signal?
- In frequency domain, the spectrum is *copied and shifted (!)*
- *BEFORE:*
- *AFTER:*



An Aside

- Why copied and shifted?
- We're glossing over some details …
- Multiplication in the time domain is equivalent to convolution in the frequency domain.
- The transform of a pulse train is a pulse train(!)
- Convolution with a pulse train copies and shifts the spectrum.
- See text for more detail.
- Take linear systems for derivation and proof.

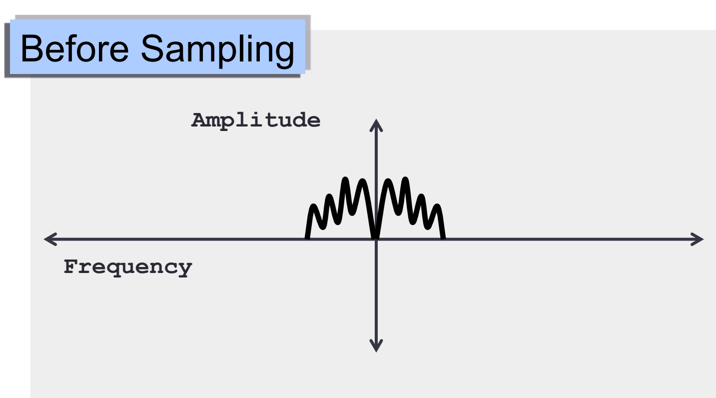
Aliasing: Time Domain View



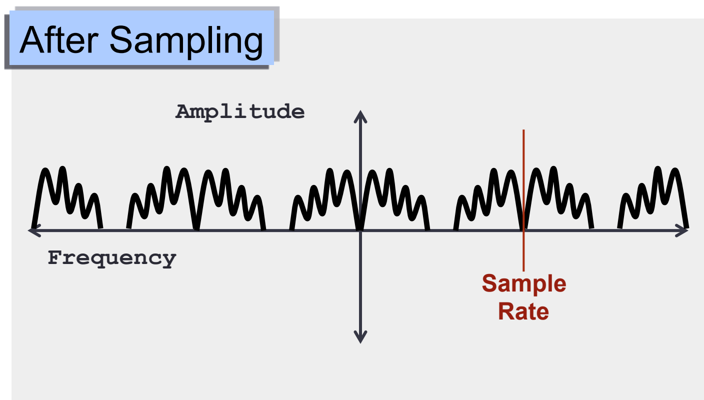
Are there other aliases?

At 16kHz SR,
Sine tones at:
1000 Hz
3010 Hz
5020 Hz
7030 Hz
9040 Hz
11060 Hz
13070 Hz
15080 Hz

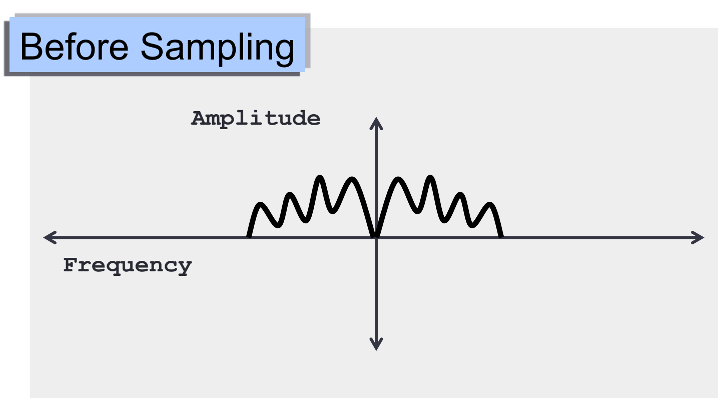
Aliasing: Frequency Domain View



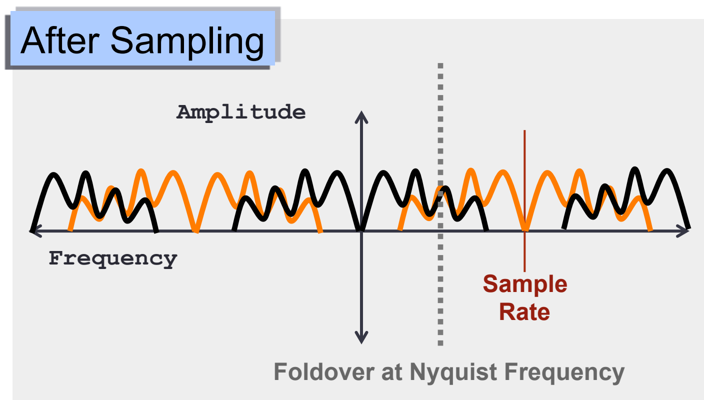
Frequency Domain View (2)



A Signal With Higher Frequency Components



A Signal With Higher Frequency Components



Bandwidth

What sample rate should we use?
Why does it matter?

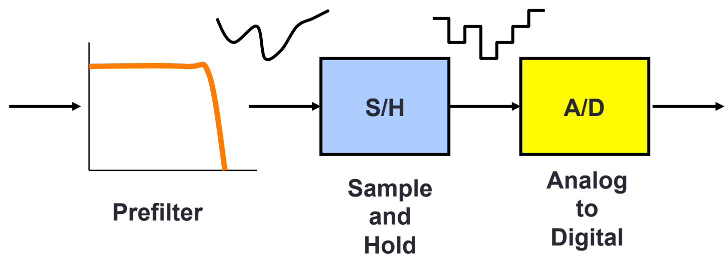
Bandwidth

The frequency range (bandwidth) is determined by the sample rate!

Sampling Without Aliasing

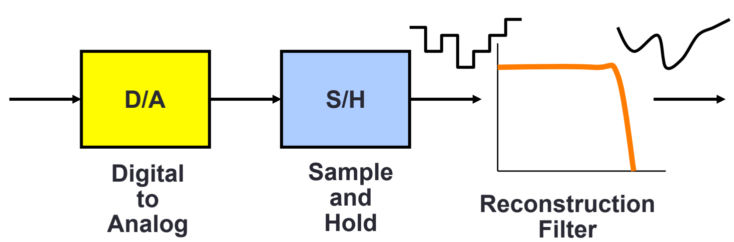
How do we **convert** analog to/from digital?

Sampling Without Aliasing



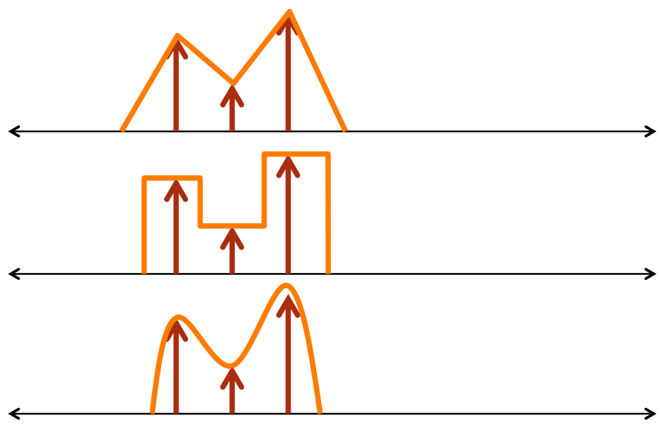
Prefilter removes all frequencies above 1/2 sampling rate (the Nyquist Frequency)

Conversion to Analog

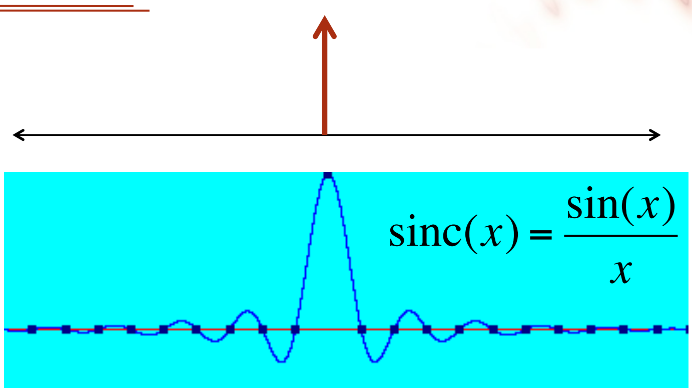


Reconstruction filter removes all frequencies above 1/2 sampling rate (the Nyquist Frequency)

What Does a Sample "Mean"?

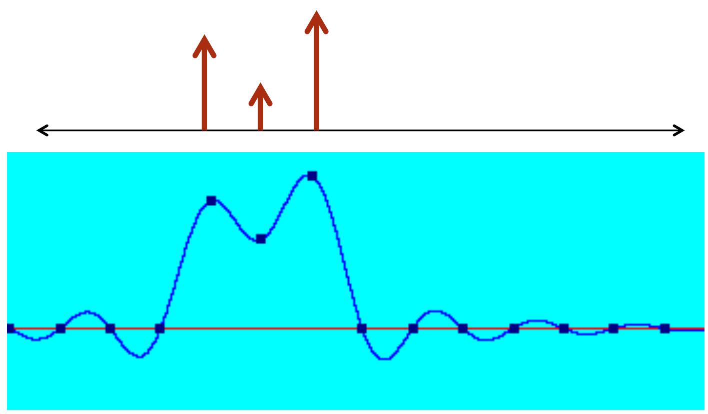


What Does a Sample "Mean"? (2)



Note: The time axis (x) is scaled so that the zeros of sinc(x) fall exactly on the times of other samples.

What Does a Sample "Mean"? (3)



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Why sinc function?

- An impulse has infinite bandwidth.
- If you perfectly cut the bandwidth down to half the sample rate (the Nyquist frequency), you get a sinc function!
- When you reconstruct the signal, replacing impulses with sinc functions, you get the entire continuous band limited signal.
- Samples uniquely determined by signal, signal uniquely determined by samples.
- Bijective (for Klaus ☺)
- AMAZING.

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Interpolation/Reconstruction

How can we interpolate samples to recover the sampled signal?

Interpolation/Reconstruction

- *Convolve* with a sinc function
- In other words, form the superposition of sinc functions shifted by the sample times and scaled by the sample values.
- Requires infinite lookahead and infinite computation!
- But sinc decays as 1/time, so good approximations are expensive but at least possible.

IMPERFECT SAMPLING

What is the impact of errors and rounding?

How to Describe Noise

- Since absolute levels rarely exist, measure RATIO of Signal to Noise.
- Since signal level is variable, measure MAXIMUM Signal to Noise.
- Units: dB = decibel
  10dB = ×10 power
  20dB = ×100 power = ×10 amplitude
  6dB = ×2 amplitude

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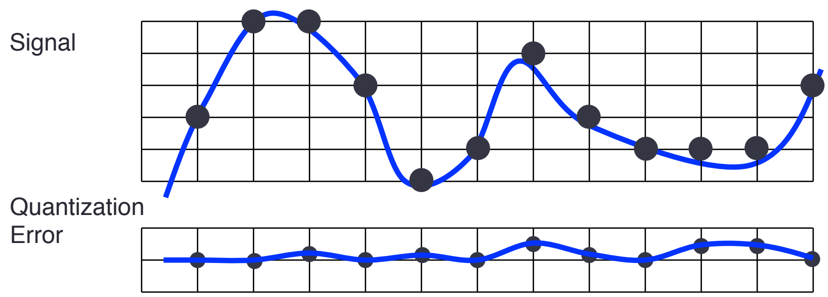
Quantization Noise



To simplify analysis, assume quantization error is uniformly randomly distributed in [-0.5, +0.5]

Quantization Examples

	Sine Tone	Cello
16-bit		
8-bit		
4-bit		
2-bit		

Quantization Noise, M bits/sample

What's the effect of **rounding** to the nearest integer sample value?

Quantization Noise, M bits/sample

- Rounding effects can be approximated by adding white noise (uniform random samples) of maximum amplitude of ½ least significant bit.

$$\mathbf{SNR(dB) = 6.02M + 1.76}$$

(about 6dB/bit)

Noise

How many **bits per sample** should we use? Why does it matter?

Noise

The signal-to-noise ratio is determined by the bits per sample!

Can Discrete Samples Really Capture a Continuous Signal?

DISCRETE SAMPLES CAN CAPTURE A CONTINUOUS BAND-LIMITED SIGNAL *WITHOUT LOSS*

- Band-limited signal ➡ no lost frequencies!
- To the extent you can do perfect sampling ➡ no noise!

Summary

- Theoretical result: discrete samples can capture *all* information in a band-limited signal!
- Practical result 1: sampling limits bandwidth to 1/2 sampling rate (the Nyquist frequency)
- Practical result 2: sampling adds quantization noise; SNR is about 6dB per bit
- What's a decibel?

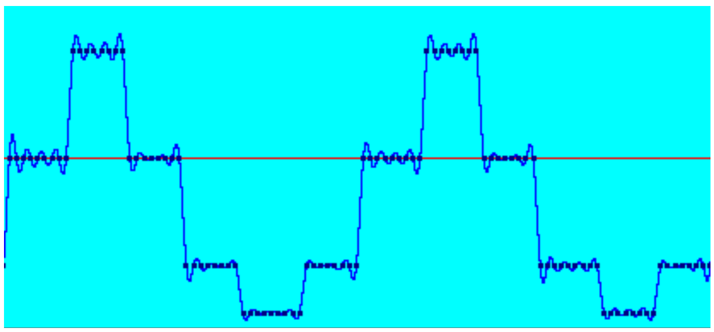
DITHER AND OVERSAMPLING

Additional techniques for practical digital audio

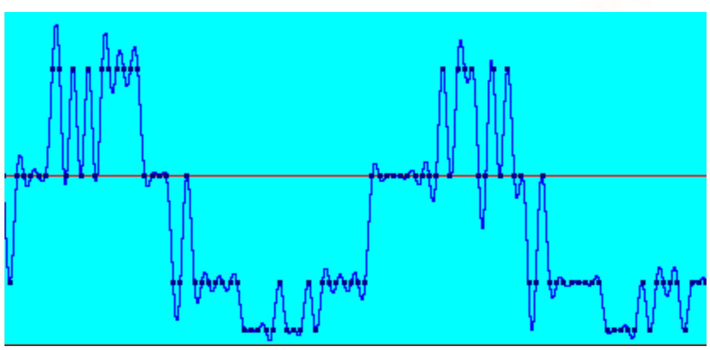
Dither

- Sometimes rounding error is correlated to signal.
- Add analog noise prior to quantization to decorrelate rounding.
- Typically, noise has peak-to-peak amplitude of one quantization step.

Heavily Quantized, Undithered Sinusoid



Sinusoid With Dithering

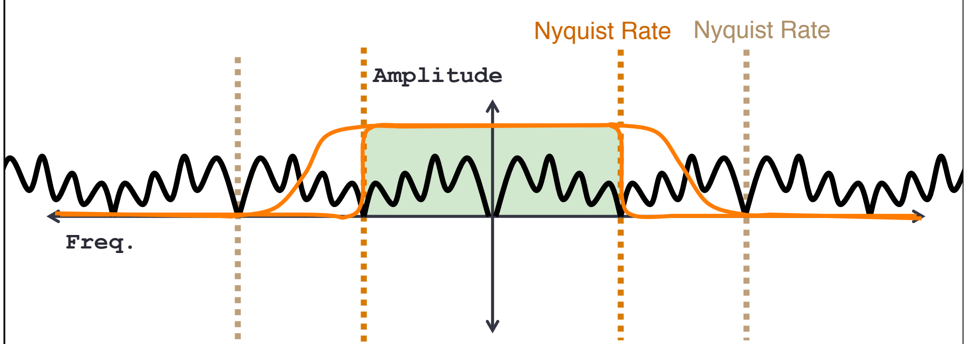


No dither   Dither

Oversampling

- Reconstruction filters are hard to build with analog components
- Idea: digitally reconstruct signal at high sample rate
- Result: simpler to build analog filter

Oversampling (2)



THE FREQUENCY DOMAIN

An alternative to waveforms (the time domain)

The Frequency Domain

- Examples of Simple Spectra
- Fourier Transform vs Short-Term Fourier Transform
- DFT – Discrete Fourier Transform
- FFT – Fast Fourier Transform
- Windowing

Formal Definition

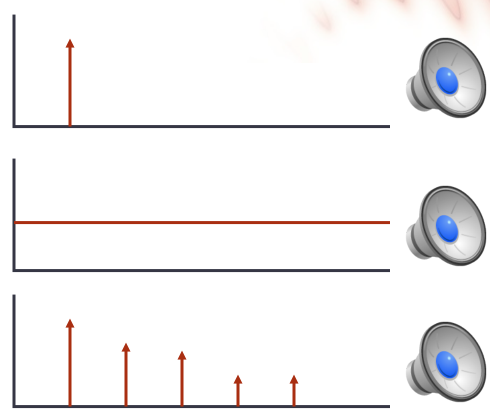
$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$X(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

Simple Spectra Examples

- Sinusoid
- Noise
- Tone with harmonics



More Examples

- Narrow Band Noise

- Impulse

Negative Frequencies

- Recall that FT is defined for negative as well as positive frequencies. What does this mean?
- $cos(\omega t) = cos(-\omega t)$, $sin(\omega t) = -sin(-\omega t)$
- For FT of *real* signals,
  - Imaginary part has odd symmetry: $X(\omega) = -X(-\omega)$
  - The real part has even symmetry: $R(\omega) = R(-\omega)$
- Therefore, the negative frequencies contain redundant information. That's why we've mostly ignored them.

Fourier Transform vs Short-Term Transform

- In practice, we can't do an infinite integral, so do a finite integral: the short term FT (STFT)

$$F(\omega) = \int_a^b f(t)e^{-j\omega t}\, dt$$

- In general, the interesting properties of true FT hold for STFT, but with annoying artifacts

Discrete Fourier Transform

- Since we work with samples rather than continuous data,
- We need a discrete version of FT: DFT
- DFT is essentially just like FT, except band limited and computable
- I'm glossing over many derivations, proofs, and details here.

Fast Fourier Transform

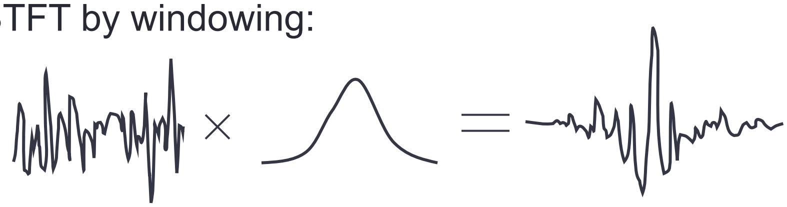
- Replacing integral with a sum, you would think computing $R(\omega)$ would be an $O(n^2)$ problem

$$F_k = \sum_{n=a}^{b} f_n e^{-j2\pi kn/N}$$

- Interestingly, there is an O(n log n) algorithm, the Fast Fourier Transform, or FFT

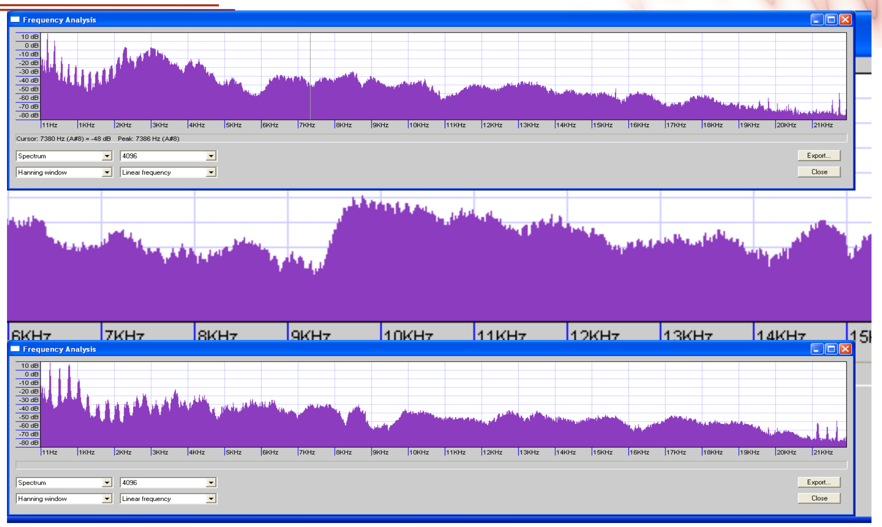
Windowing

- Typically, you can reduce the artifacts of the STFT by windowing:



- Different windows optimize different criteria: Hamming, Hanning, Blackman, etc.

More Examples Using Audacity



AMPLITUDE MODULATION

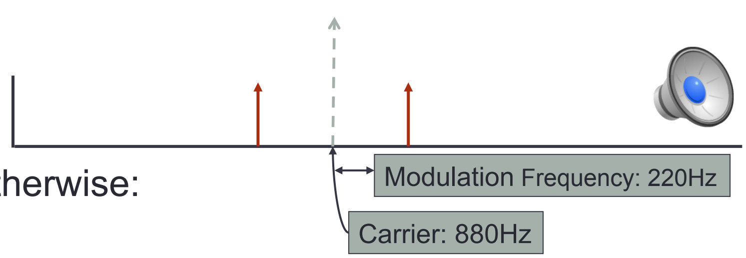
Synthesis techniques based on signal multiplication

Amplitude Modulation

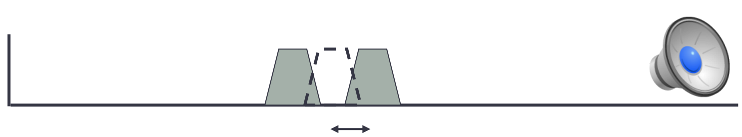
- Amplitude modulation is simply multiplication (MULT in Nyquist)
- Amplitude modulation (multiplication) in the time domain corresponds to convolution in the spectral domain (!)
- For each sinusoid in the modulator, the modulated signal is shifted up and down by the frequency of the sinusoid.

AM specta

- Assuming the modulated signal is a sinusoid:



- Otherwise:



Ring Modulation

- Ring Modulation is named after the "ring modulator," an analog approach to signal multiplication.
- See code_3.htm for AM examples

Constant Offset

- What is the difference between:
  lfo(6)
- And
  2 + lfo(6)
- ?

Summary

- Dithering sometimes used to avoid quantization artifacts
- Oversampling is standard technique to move (some) filtering to the digital domain
- Amplitude Modulation by a sinusoid shifts the spectrum up and down by the frequency of the modulator