SAMPLING THEORY

Representing continuous signals with discrete numbers

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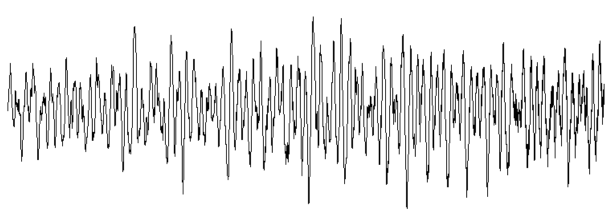
From analog to digital (and back)



010110
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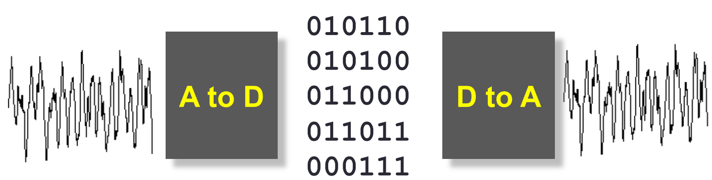
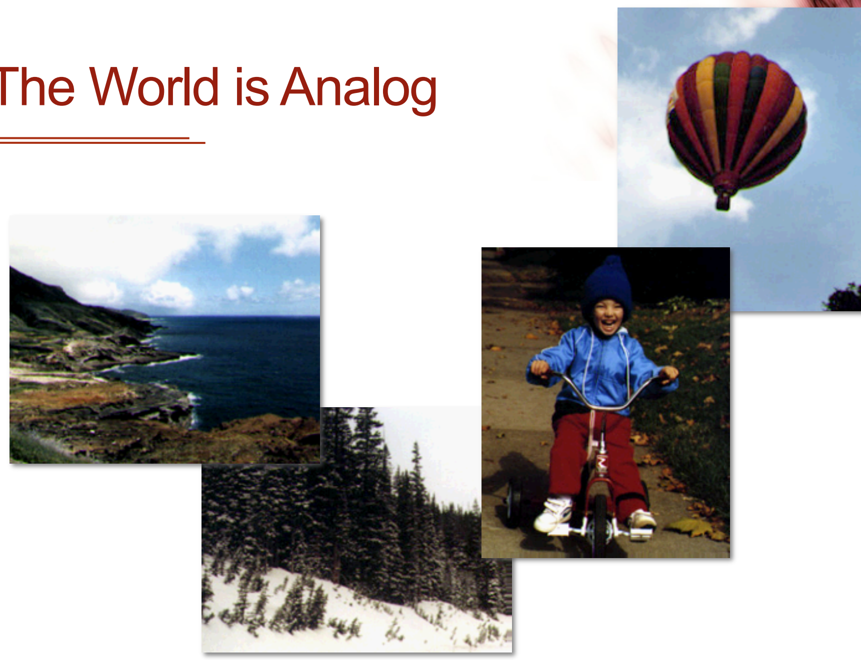
Analog to Digital Conversion
Digital to Analog Conversion



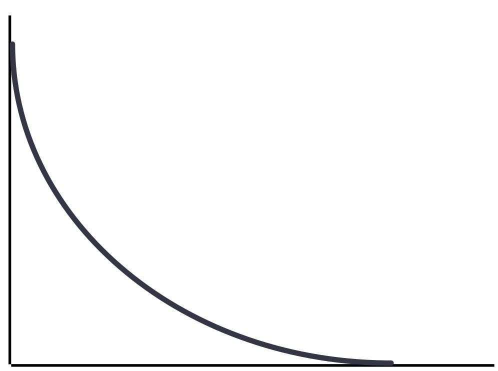
Approach

- Intuition
- Frequency Domain (Fourier Transform)
- Sampling Theory
- Practical Results

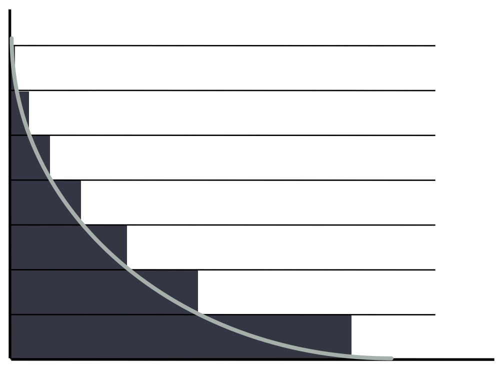
The World is Analog



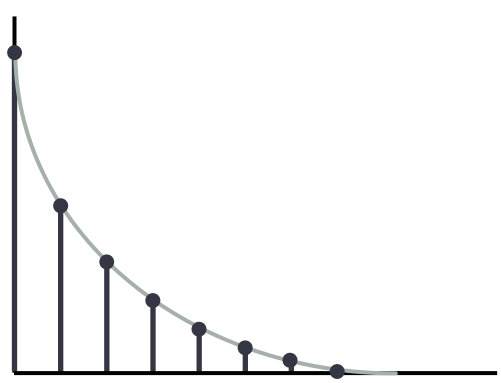
Continuous or Discrete?



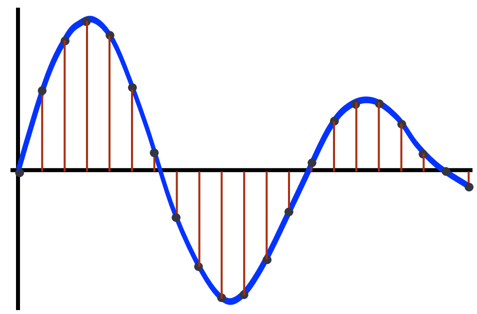
Discrete Amplitude (Y axis)



Discrete Time (X axis)



Digitizing a continuous function (or signal)

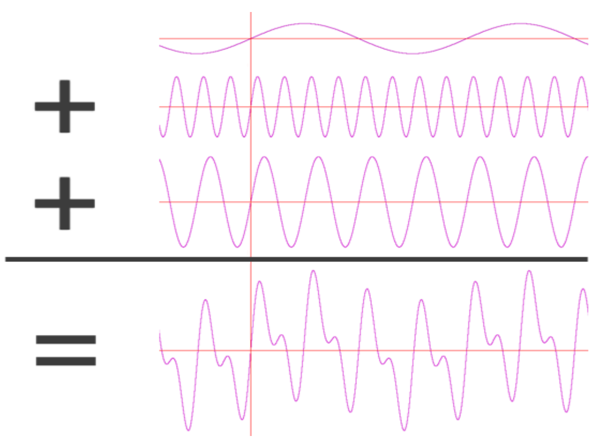


Questions

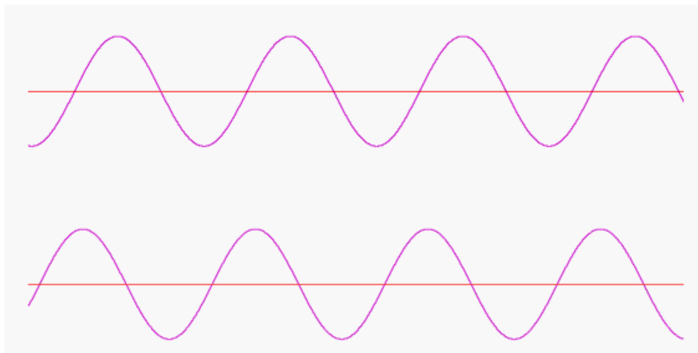
- What **sample rate** should we use? Why does it matter?
- How many **bits per sample** should we use? Why does it matter?
- **Interpolation**: How can we interpolate samples to recover the sampled signal?
- What's the effect of **rounding** to the nearest integer sample value?
- How do we **convert** analog to/from digital?

Introduction to the Spectrum

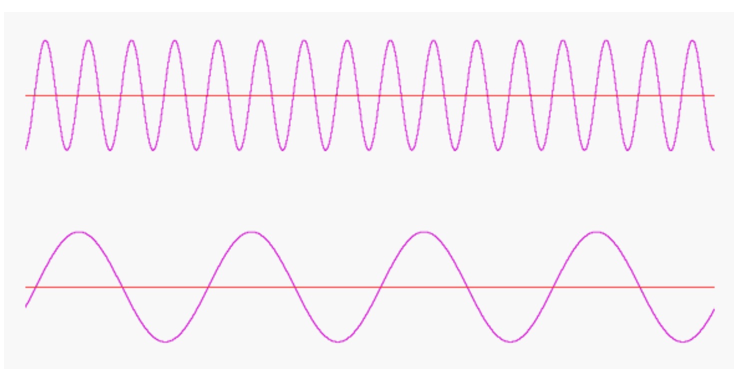
Introduction to the Spectrum (2)



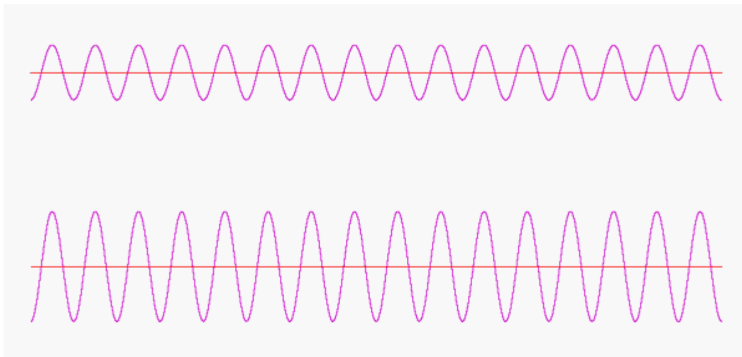
Phase



Frequency



Amplitude



Sinusoidal Partials

$$A \cdot \sin(\omega t + \phi)$$

Amplitude A

Frequency ω

Phase ϕ

Fourier Transform

- Our goal is to *transform* a function-of-time representation of a signal to a function-of-frequency representation
- Express the time function as an (infinite) sum of sinusoids.
- Express the infinite sum as a function from frequency to amplitude
- I.e. for each frequency, what is the amplitude of the sinusoid of that frequency within this infinite sum?

Fourier Transform: Cartesian Coordinates

Real part:

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

Imaginary part:

$$X(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

What About Phase?

- Remember at each frequency, we said there is *one* sinusoidal component: $A \cdot \sin(\omega t + \phi)$
  - *A* is amplitude
  - ω is frequency
  - ϕ is phase
- The Fourier analysis computes two amplitudes:
  - $R(\omega)$ and $X(\omega)$
  - Trig identities tell us there is no conflict:

$$A = \sqrt{R^2 + X^2} \qquad \phi = \arctan(X / R)$$

$$A(\omega) = \sqrt{R^2(\omega) + X^2(\omega)} \qquad \phi(\omega) = \arctan(X(\omega) / R(\omega))$$

From Cartesian to Complex

- *R* is "real" or cosine part
- *X* is "imaginary" or sine part
- Use $F(\omega) = R(\omega) + j \cdot X(\omega)$

Fourier Transform (Complex Form)

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

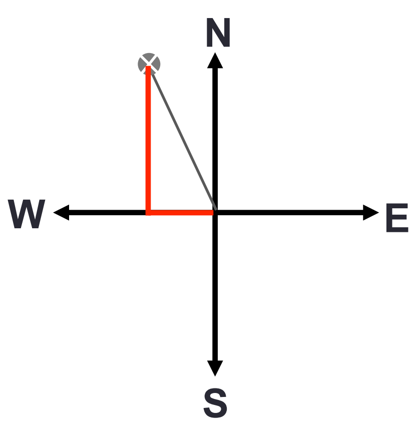
+

$$j \cdot X(\omega) = -j\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

=

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

Orthogonal Basis Functions

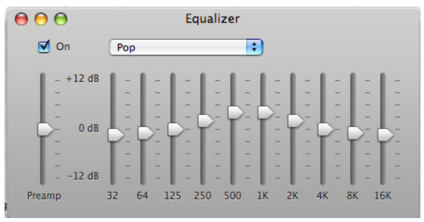


Horizontal and vertical axes are independent or orthogonal in the 2-dimensional plane, sinusoids are orthogonal in the infinite-dimensional space of continuous signals.

Just as every point in the plane is a *unique linear* combination of the unit E and N vectors, every signal is a *unique linear* combination of sinusoids.

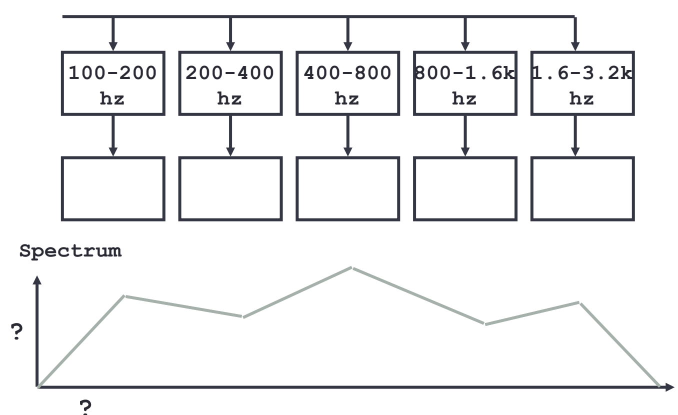
The Frequency Domain

Graphic Equalizer

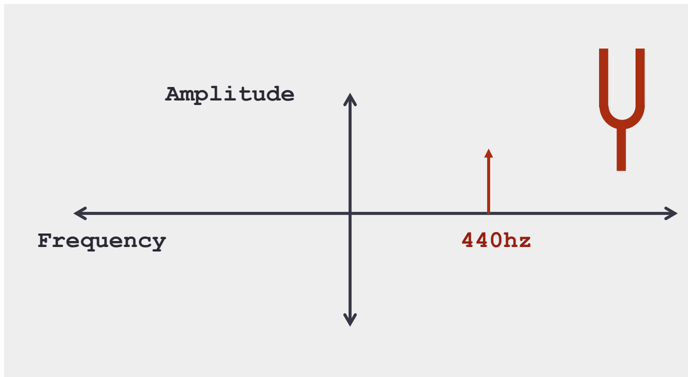


Spectral Analyzer

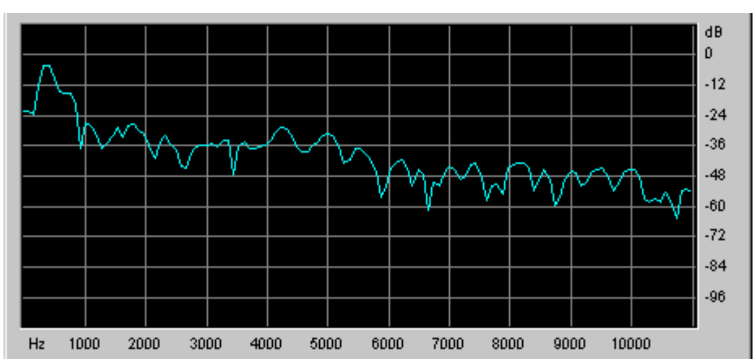
The Frequency Domain (2)



The Amplitude Spectrum



Amplitude Spectrum of a "Real" Signal



Representations

- (Real, Imaginary) or (Amplitude, Phase)?
  - Power ~ Amplitude^2
  - We generally cannot hear phase
  - Measure a stationary signal after Δt: Amplitude spectrum is unchanged, but phase changes by $\Delta t \cdot \omega$
- Given (amplitude, phase)
  - It's hard to plot both
  - Usually, we ignore the phase

Time vs Frequency

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

- What happens to time when you transform to the frequency domain?
- Note that time is "integrated out"
- NO TIME REMAINS
- The Fourier Transform of a signal *is not a function of time !!!!!*
- (Later, we'll look at *short-time transforms* – e.g. what you see on a time-varying spectral display – which *are* time varying.)

